The 30th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2013 — official reference solutions

Translated from the official reference answers (参考解答) printed after the problems in the booklet "第30届决赛试题及答案". Equation numbers (1), (2), . . . within each problem follow the numbers of the Chinese original. Where the original repeats the same numbers inside an alternative method ("另解二" = "Method 2"), the repeated numbers are primed here, (4′), (5′), ..., so that every tag in a problem is unique. The per-problem marking schemes (评分标准) printed by the original are translated in full. Obvious misprints of the source are transcribed as printed, with a translator's note giving the correct form. The wording and the symbols of the problem statements are those of the companion file of problems.

Problem 1 (bouncing ball: total impulse from the ground)

Solution

Let the magnitude of the vertical velocity after each bounce be λ times the magnitude just before that bounce. The vertical speed at the first landing is

$$v_0 = \sqrt{2gh} \tag{1}$$

The vertical speed just after the first bounce is

$$v_1 = \lambda\sqrt{2gh}, \qquad 0 < \lambda < 1. \tag{2}$$

The height of the first rebound is

$$h_1 = \frac{v_1^2}{2g} = \lambda^2 h. \tag{3}$$

The time of flight after the first bounce is

$$t_1 = 2\,\frac{v_1}{g} = 2\lambda\sqrt{\frac{2h}{g}}. \tag{4}$$

The horizontal displacement from the first bounce to the second bounce is

$$x_1 = \sqrt{2gh}\,t_1 = 4\lambda h. \tag{5}$$

The area enclosed between the trajectory of the ball from the first bounce to the second bounce and the projection of that trajectory on the ground is

$$S_1 = \frac{2}{3}h_1 x_1 = \frac{8}{3}\lambda^3 h^2. \tag{6}$$

Let v_n and h_n be the magnitude of the largest vertical velocity and the largest height reached between the n-th bounce and the $(n+1)$-th bounce. From the statement of the problem and from the argument above,

$$v_{n+1} = \lambda v_n, \tag{7}$$
$$h_{n+1} = \lambda^2 h_n, \tag{8}$$
$$t_{n+1} = \lambda t_n, \tag{9}$$
$$x_{n+1} = \lambda x_n, \tag{10}$$
$$S_{n+1} = \lambda^3 S_n. \tag{11}$$

Thus $S_1, S_2, \ldots$ form an infinite decreasing geometric series, and its sum is

$$S = \sum_{n=1}^{\infty} S_n = S_1\left(1 + \lambda^3 + \lambda^6 + \cdots\right) = \frac{S_1}{1-\lambda^3} = \frac{8}{21}h^2 \tag{12}$$

From (6) and (12),

$$\lambda = \frac{1}{2} \tag{13}$$

Let I_n be the impulse of the force on the ball during the n-th ($n \geq 1$) collision. The impulse–momentum theorem gives

$$I_n = mv_n - m(-v_{n-1}) = m(1+\lambda)v_{n-1} \tag{14}$$

Since the horizontal component of the velocity of the ball does not change at a bounce, the horizontal impulse received in each collision is zero. The total impulse received by the ball in all of the collisions with the ground is

$$I = \sum_{n=1}^{\infty} I_n = (mv_0)(1+\lambda)\left(1 + \lambda + \lambda^2 + \cdots\right) = mv_0\frac{1+\lambda}{1-\lambda} \tag{15}$$

and its direction is upward. Substituting (13) into (15),

$$I = 3mv_0 = 3m\sqrt{2gh} \tag{16}$$

Marking scheme.
This problem is worth 15 points. Equations (1) to (6): 1 point each; equation (11): 4 points; equations (12) to (16): 1 point each.

Problem 2 (rod with two balls in a corner: when does the upper ball leave the wall?)

Solution

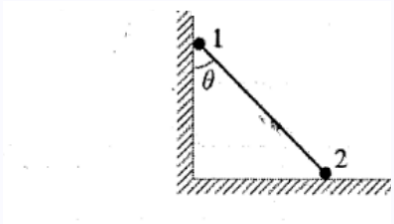
[Translator's note: On the printed page the blocks of this solution appear out of order (equations (9)–(13) are printed before equations (3)–(8)). They are restored here to the numerical order, which is also the logical order of the argument.]

As in the figure, before ball 1 leaves the vertical wall, when the angle between the rod and the vertical wall is θ the coordinates of ball 1 are

$$x_1 = 0, \qquad y_1 = l\cos\theta, \tag{1}$$

and the coordinates of ball 2 are

$$x_2 = l\sin\theta, \qquad y_2 = 0. \tag{2}$$



The velocity of ball 1 is

$$v_{1x} = \frac{\mathrm{d}x_1}{\mathrm{d}t} = 0, \qquad v_{1y} = \frac{\mathrm{d}y_1}{\mathrm{d}t} = -\omega l \sin\theta, \tag{3}$$

where $\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ is the angular velocity of rotation of the rod. The velocity of ball 2 is

$$v_{2x} = \frac{\mathrm{d}x_2}{\mathrm{d}t} = \omega l \cos\theta, \qquad v_{2y} = \frac{\mathrm{d}y_2}{\mathrm{d}t} = 0. \tag{4}$$

Conservation of mechanical energy gives

$$mgl = \frac{1}{2}m\left(v_{1x}^2 + v_{1y}^2\right) + \frac{1}{2}m\left(v_{2x}^2 + v_{2y}^2\right) + mgl\cos\theta = \frac{1}{2}ml^2\omega^2 + mgl\cos\theta. \tag{5}$$

From this,

$$\omega = \sqrt{\frac{2g(1-\cos\theta)}{l}}. \tag{6}$$

Here it has been taken into account that θ grows as the time t increases, so that $\omega > 0$, and the negative root has been discarded.
The x coordinate of the centre of mass c of the system is

$$x_c = \frac{mx_1 + mx_2}{2m} = \frac{1}{2}l\sin\theta. \tag{7}$$

The x component of the velocity of the centre of mass is

$$v_{cx} = \frac{\mathrm{d}x_c}{\mathrm{d}t} = \frac{1}{2}\omega l\cos\theta. \tag{8}$$

The x component of the acceleration of the centre of mass is

$$a_{cx} = \frac{\mathrm{d}v_{cx}}{\mathrm{d}t} = -\frac{1}{2}\omega^2 l\sin\theta + \frac{1}{2}l\cos\theta\,\frac{\mathrm{d}\omega}{\mathrm{d}t}. \tag{9}$$

From (6),

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{2}\omega\sin\theta\sqrt{\frac{2g}{l(1-\cos\theta)}} = \frac{g}{l}\sin\theta. \tag{10}$$

In obtaining this result (6) has been used once more. Substituting (6) and (10) into (9),

$$a_{cx} = \frac{\mathrm{d}v_{cx}}{\mathrm{d}t} = -g\sin\theta(1-\cos\theta) + \frac{1}{2}g\sin\theta\cos\theta = g\sin\theta\left(\frac{3}{2}\cos\theta - 1\right). \tag{11}$$

Let T be the normal force of the vertical wall on ball 1. The motion of the centre of mass c in the x direction satisfies

$$T = 2ma_{cx}. \tag{12}$$

From (12), when $a_{cx} = 0$,

$$T = 0, \tag{13}$$

and ball 1 begins to leave the vertical wall. From (11), when ball 1 begins to leave the vertical wall the angle θ satisfies the equation

$$\sin\theta\Big(\frac{3}{2}\cos\theta - 1\Big) = 0. \tag{14}$$

One solution of equation (13) satisfies $\sin\theta = 0$, so that $\theta = 0$; this angle corresponds to the initial position. The other solution of equation (13) satisfies $\frac{3}{2}\cos\theta - 1 = 0$, so [a]

$$\theta = \arccos\frac{2}{3}. \tag{15}$$

This is the angle between the rod and the vertical wall at the moment when ball 1 begins to leave the wall.

Marking scheme.
This problem is worth 15 points. Equations (1) to (15): 1 point each.

[a] *Translator's note:* The source writes "equation (13)" twice in this sentence; the equation that is meant is (14).

Problem 3 (vertical transport tube from the equator to a hovering spacecraft)

Solution

1. When the goods (including the pallet; the same below) of mass m are at the distance r from the centre of the Earth, the gravitational potential energy U_G of the system "goods m + Earth" and the inertial centrifugal force F_L acting on the goods m because of the spin of the Earth are

$$U_G = -\frac{GM_e m}{r}, \qquad F_L = m\omega_e^2 r. \tag{1}$$

Here the direction outward from the centre of the Earth has been taken as the positive direction of a force. First calculate the work done by the Earth's gravity on the goods m. When the goods m are at the supply station and at the spacecraft, the gravitational potential energies of the system "goods + Earth" are

$$U_G(R_e) = -\frac{GM_e m}{R_e}, \qquad U_G(L+R_e) = -\frac{GM_e m}{L+R_e} = -\frac{GM_e m}{(\alpha+1)R_e}. \tag{2}$$

By the work–energy principle the work done by the Earth's gravity on the goods m is

$$W_G = U_G(R_e) - U_G(L+R_e) = -\frac{GM_e m}{R_e}\Big(1 - \frac{1}{\alpha+1}\Big) = -\frac{\alpha GM_e m}{(\alpha+1)R_e}. \tag{3}$$

Next calculate the work W_L done by the inertial centrifugal force on the goods m. Since the inertial centrifugal force is proportional to r, its work can be computed with the average of the inertial centrifugal force at the supply station and at the spacecraft. Thus

$$W_L = \frac{1}{2}\big[F_L(R_e) + F_L(L+R_e)\big]L = \frac{1}{2}m\omega_e^2\big[R_e + (L+R_e)\big]L = \frac{1}{2}\alpha(\alpha+2)m\omega_e^2 R_e^2. \tag{4}$$

2. After the goods leave the supply station, the attraction of the Earth becomes smaller and the inertial centrifugal force becomes larger. Once the inertial centrifugal force exceeds the attraction of the Earth, no more work has to be done to push the goods. Therefore the part of the path on which the external force must do positive work runs from the starting point to the place where the attraction of the Earth equals the inertial centrifugal force. Let r_0 be the distance from the centre of the Earth to the place where the attraction of the Earth equals the inertial centrifugal force; then

$$\frac{GM_e m}{r_0^2} = m\omega_e^2 r_0,$$

or

$$r_0 = \left(\frac{GM_e}{\omega_e^2}\right)^{1/3}. \tag{5}$$

When the goods reach r_0, the work done by the Earth's gravity is

$$W_G(r_0) = U_G(R_e) - U_G(r_0) = -\frac{GM_e m}{R_e} + \frac{GM_e m}{r_0} = -\frac{GM_e m}{R_e}\Big(1 - \frac{R_e}{r_0}\Big), \tag{6}$$

and the work done by the inertial centrifugal force is

$$W_L(r_0) = \frac{1}{2}\big[F_L(r_0) + F_L(R_e)\big](r_0 - R_e) = \frac{1}{2}m\omega_e^2\big(r_0^2 - R_e^2\big). \tag{7}$$

Therefore, in order to push the goods from the supply station to the spacecraft, the minimum positive work of the external pushing force is

$$\begin{aligned} W_{\min} &= -\big[W_G(r_0) + W_L(r_0)\big] = \frac{GM_e m}{R_e}\Big(1 - \frac{R_e}{r_0}\Big) - \frac{1}{2}m\omega_e^2\big(r_0^2 - R_e^2\big) \\ &= \frac{GM_e m}{R_e}\left[1 - \frac{3}{2}\frac{R_e}{r_0} + \frac{1}{2}\Big(\frac{R_e}{r_0}\Big)^3\right] > 0. \end{aligned}$$

Here (5) has been used. Substituting (5) into the last expression,

$$W_{\min} = \frac{GM_e m}{R_e} + \frac{1}{2}m\omega_e^2 R_e^2 - \frac{3}{2}m\big(GM_e\omega_e\big)^{2/3}. \tag{8}$$

3. Setting the sum of (3) and (4) equal to zero,

$$-\frac{\alpha GM_e m}{(\alpha+1)R_e} + \frac{1}{2}\alpha(\alpha+2)m\omega_e^2 R_e^2 = 0,$$

that is,

$$\alpha^2 + 3\alpha + 2\left(1 - \frac{GM_e}{\omega_e^2 R_e^3}\right) = 0, \tag{9}$$

whose solutions are

$$\alpha_\pm = \frac{1}{2}\left[-3 \pm \left(1 + \frac{8GM_e}{\omega_e^2 R_e^3}\right)^{1/2}\right]. \tag{10}$$

Discarding the negative root for α and writing

$$\alpha_0 \equiv \alpha_+ = \frac{1}{2}\left[-3 + \left(1 + \frac{8GM_e}{\omega_e^2 R_e^3}\right)^{1/2}\right], \tag{11}$$

the distance of the spacecraft from the ground is

$$L_0 = \alpha_0 R_e = \frac{R_e}{2}\left[-3 + \left(1 + \frac{8GM_e}{\omega_e^2 R_e^3}\right)^{1/2}\right]. \tag{12}$$

4. To find the force of the supply station on the transport tube, first calculate the work done by the Earth's gravity, by the inertial centrifugal force and by the force of the ground supply station on the tube during a process in which the tube moves down by an infinitesimal distance Δl. Since the tube is in equilibrium, the sum of the works of these three forces for any infinitesimal displacement is zero. In order to compute the work of the Earth's gravity and of the inertial centrifugal force, the downward motion of the tube by the infinitesimal distance Δl may be imagined as the infinitesimal topmost segment Δl (of mass $\Delta m = m_p \Delta l/L$) being carried infinitely slowly to the very bottom. In this process the work done by the Earth's gravity and by the inertial centrifugal force can be obtained from results similar to (3) and (4) above. When this infinitesimal segment Δl is at the distance r from the centre of the Earth, its gravitational potential energy and the inertial centrifugal force on it are

$$U_G = -\frac{GM_e \Delta m}{r} = -\frac{GM_e m_p \Delta l}{Lr}, \qquad F_L = \Delta m\, \omega_e^2 r = \frac{1}{L} m_p \omega_e^2 r\, \Delta l. \tag{13}$$

In the process in which the infinitesimal topmost segment Δl is carried infinitely slowly to the bottom, the work done by the Earth's gravity is

$$\Delta W_G = U_G(L+R_e) - U_G(R_e) = \frac{GM_e m_p}{LR_e}\Big(1 - \frac{1}{\alpha+1}\Big)\Delta l = \frac{GM_e m_p}{(\alpha+1)R_e^2}\,\Delta l, \tag{14}$$

and the work done by the inertial centrifugal force is

$$\Delta W_L = -\frac{1}{2}\big[F_L(R_e) + F_L(L+R_e)\big] L = -\frac{1}{2}(\alpha+2) m_p \omega_e^2 R_e\, \Delta l. \tag{15}$$

Take the downward direction as positive for the force F_{base} of the supply station on the tube. In the process in which the tube moves down by the infinitesimal distance Δl, the work done by F_{base} is

$$\Delta W_{\text{base}} = F_{\text{base}} \Delta l. \tag{16}$$

From $\Delta W_G + \Delta W_L + \Delta W_{\text{base}} = 0$,

$$\frac{GM_e m_p}{(\alpha+1)R_e^2}\Delta l - \frac{1}{2}(\alpha+2) m_p \omega_e^2 R_e \Delta l + F_{\text{base}}\Delta l = 0. \tag{17}$$

Hence

$$F_{\text{base}} = -\frac{GM_e m_p}{(\alpha+1)R_e^2} + \frac{1}{2}(\alpha+2) m_p \omega_e^2 R_e. \tag{18}$$

If $F_{\text{base}} < 0$, the force of the supply station on the tube is directed upward.
For the discussion of the three cases $L > L_0$, $L = L_0$ and $\alpha_m R_e < L < L_0$, rewrite (18) as

$$\begin{aligned} F_{\text{base}} &= -\frac{GM_e m_p}{(\alpha+1)R_e^2} + \frac{1}{2}(\alpha+2) m_p \omega_e^2 R_e = \frac{m_p \omega_e^2 R_e}{2(\alpha+1)}\left[(\alpha+1)(\alpha+2) - \frac{2GM_e}{\omega_e^2 R_e^3}\right] \\ &= \frac{m_p \omega_e^2 R_e}{2(\alpha+1)}(\alpha - \alpha_+)(\alpha - \alpha_-) = \frac{m_p \omega_e^2 R_e}{2(\alpha+1)}(\alpha - \alpha_0)(\alpha - \alpha_-), \end{aligned} \tag{19}$$

where $\alpha_\pm$ are given by (10) and α_0 by (11).
For $L > L_0$, that is $\alpha > \alpha_0$, the magnitude of the force of the supply station on the tube is given by (18) or (19), and its direction is downward. For $L = L_0$, that is $\alpha = \alpha_0$, (19) gives

$$F_{\text{base}} = 0 \qquad (L = L_0). \tag{20}$$

For $\alpha_m R_e < L < L_0$, that is $\alpha_m < \alpha < \alpha_0$, (19) shows that $F_{\text{base}} < 0$. In this case the magnitude of the force of the supply station on the tube is

$$\left|F_{\text{base}}\right| = \frac{GM_e m_p}{(\alpha+1)R_e^2} - \frac{1}{2}(\alpha+2)m_p\omega_e^2 R_e, \tag{21}$$

and its direction is upward.

Marking scheme.
This problem is worth 25 points.

- Part 1: 6 points. Equations (1), (3), (4): 2 points each.
- Part 2: 6 points. Equation (5): 2 points; equations (6), (7): 1 point each; equation (8): 2 points.
- Part 3: 4 points. Equation (9): 2 points; equation (10): 1 point; equation (11) or (12): 1 point.
- Part 4: 9 points. Equations (13) to (16): 1 point each; equation (18): 1 point; equation (19): 2 points; equations (20), (21): 1 point each.

Problem 4 (chain of shielded conducting spheres in a dc circuit)

Solution

A long time after the switch has been closed, a steady direct current flows in the circuit, and the charge distribution on the conducting spheres is also steady. Let the charge on the i-th conducting sphere be Q_i; then the conducting shell that surrounds it carries the charge $-Q_i$, and these charges must be distributed uniformly over the two spherical surfaces. The potential difference between the conducting sphere and the outer shell is therefore

$$\Delta U_i = kQ_i\left(\frac{1}{r} - \frac{1}{r_0}\right), \tag{1}$$

and, since the outer conducting shell is grounded, the potential of the i-th conducting sphere itself is

$$U_i = kQ_i\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{2}$$

Summing both sides of (2) over all the conducting spheres gives

$$k\left(\frac{1}{r} - \frac{1}{r_0}\right)Q = \sum_{i=1}^{N+1} U_i, \tag{3}$$

where

$$Q = \sum_{i=1}^{N+1} Q_i \tag{4}$$

is the total charge carried by the conducting spheres. At the same time, the direct current in the circuit is

$$I = \frac{E}{NR + R_E}. \tag{5}$$

Hence the potential of the i-th conducting sphere is

$$U_i = (i-1)IR = \frac{(i-1)ER}{NR + R_E}. \tag{6}$$

Substituting (6) into (3),

$$k\left(\frac{1}{r}-\frac{1}{r_0}\right)Q=\frac{ER}{NR+R_E}\sum_{i=1}^{N+1}(i-1)=\frac{1}{2}N(N+1)\,\frac{ER}{NR+R_E}. \tag{7}$$

Equation (7) can be solved for the radius:

$$r=\left(\frac{N(N+1)ER}{2kQ(NR+R_E)}+\frac{1}{r_0}\right)^{-1}. \tag{8}$$

Marking scheme.
This problem is worth 20 points. Equation (1): 4 points; equations (2), (3): 2 points each; equation (4): 1 point; equations (5), (6), (7): 3 points each; equation (8): 2 points.

Problem 5 (superconducting ring falling above a cylindrical magnet)

Solution

1. When the current in the thin superconducting ring is I, the magnetic flux through the ring is

$$\Phi=\pi r_0^2 B_z+LI \tag{1}$$

Substituting $B_z=B_0(1-2\alpha z)$ into the above,

$$\Phi=\pi r_0^2 B_0(1-2\alpha z)+LI. \tag{2}$$

Since the magnetic flux through the superconducting ring does not change, Φ is a constant. From the initial conditions — at $t=0$: $z=0$, $r=0$, $I=I_0$ — one has

$$\Phi_0=\pi r_0^2 B_0+LI_0. \tag{3}$$

Therefore

$$I=\frac{2}{L}\pi r_0^2 B_0\alpha z+I_0\equiv I(z). \tag{4}$$

Now consider the Ampère force on the ring. Because the magnetic field is non-uniform, the Ampère force on the ring is not zero. By the axial symmetry the Ampère force is directed along the axis. The Ampère force on the ring is

$$\begin{aligned}F_{\mathrm{A}}(z)&=-B_r(z)I(z)\,2\pi r_0\\&=-B_0\alpha r_0\left(\frac{2}{L}\pi r_0^2 B_0\alpha z+I_0\right)2\pi r_0\\&=-kz-F_0,\end{aligned} \tag{5}$$

where

$$k=\frac{4}{L}\pi^2\alpha^2B_0^2r_0^4,\qquad F_0=2\pi\alpha B_0r_0^2I_0. \tag{6}$$

Taking also the gravity $F_g(z)=-mg$ on the ring into account, the resultant force on the ring is

$$F(z)=F_{\mathrm{A}}(z)+F_g(z)=-kz-(mg+F_0). \tag{7}$$

At the equilibrium position z_0 one has $F(z_0) = 0$. From the last equation the equilibrium position is

$$z_0 = -\frac{mg + F_0}{k} = -\frac{(mg + 2\pi\alpha B_0 r_0^2 I_0)L}{4\pi^2\alpha^2 B_0^2 r_0^4}. \tag{8}$$

Equation (7) shows that the motion of the ring is a simple harmonic oscillation about its equilibrium position, with the angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi\alpha B_0 r_0^2}{\sqrt{mL}}, \tag{9}$$

The general form of the z coordinate of the ring at time t is

$$z(t) = z_0 + A\cos(\omega t + \varphi_0), \tag{10}$$

where the amplitude A and the initial phase φ_0 are fixed by the initial conditions

$$z(t=0) = 0, \qquad v_z(t=0) = \left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} = 0. \tag{11}$$

From (10) and (11),

$$z_0 + A\cos\varphi_0 = 0, \qquad \omega A\sin\varphi_0 = 0,$$

so that

$$A = -z_0, \qquad \varphi_0 = 0. \tag{12}$$

Therefore

$$\begin{aligned} z(t) &= z_0 - z_0\cos(\omega t) = -z_0\big[\cos(\omega t) - 1\big] \\ &= \frac{(mg + 2\pi\alpha B_0 r_0^2 I_0)L}{4\pi^2\alpha^2 B_0^2 r_0^4}\left[\cos\left(\frac{2\pi\alpha B_0 r_0^2}{\sqrt{mL}}\,t\right) - 1\right]. \end{aligned} \tag{13}$$

2. Substituting (13) into (4) gives the current in the ring at time t:

$$\begin{aligned} I(t) &= \left(\frac{mg}{2\pi\alpha B_0 r_0^2} + I_0\right)\left[\cos\left(\frac{2\pi\alpha B_0 r_0^2}{\sqrt{mL}}\,t\right) - 1\right] + I_0 \\ &= \left(\frac{mg}{2\pi\alpha B_0 r_0^2} + I_0\right)\cos\left(\frac{2\pi\alpha B_0 r_0^2}{\sqrt{mL}}\,t\right) - \frac{mg}{2\pi\alpha B_0 r_0^2}. \end{aligned} \tag{14}$$

Marking scheme.
This problem is worth 20 points.

- Part 1: 18 points. Equation (1): 2 points; equations (2) to (4): 1 point each; equation (5): 2 points; equations (6), (7), (8): 1 point each; equations (9), (10): 2 points each; equation (11): 1 point; equation (12): 2 points; equation (13): 1 point.
- Part 2: 2 points. Equation (14): 2 points.

Problem 6 (metal disc in sunlight: effect of doubling the thickness)

Solution

Let the temperatures of the upper and the lower surface of the metal disc be T_t and T_b, and let the temperature of the air be T_0. According to the statement of the problem, the energies that the upper and the lower surface of the disc lose to the air in unit time are

$$cA(T_t - T_0), \qquad cA(T_b - T_0), \tag{1}$$

where c is a constant and A is the area of one surface of the disc. Let the energy of the sunlight absorbed by the upper surface in unit time be P_a. Conservation of energy gives

$$P_a = cA(T_t - T_0) + cA(T_b - T_0). \tag{2}$$

Therefore

$$T_t + T_b = \frac{P_a + 2cAT_0}{cA} = \frac{P_a}{cA} + 2T_0. \tag{3}$$

One sees that the sum of the temperatures of the upper and the lower surface of the disc is a constant. With the data given in the problem,

$$T_t + T_b = 360.0\,\mathrm{K} + 340.0\,\mathrm{K} = 700.0\,\mathrm{K}. \tag{4}$$

When the thickness of the disc is t, the law of heat conduction gives the heat conducted in unit time from the upper surface of the disc to the lower surface as

$$P_c = kA\,\frac{T_t - T_b}{t}, \tag{5}$$

where k is the coefficient of heat conduction. The heat conducted from the upper surface to the lower surface is lost from the lower surface to the air, so that

$$kA\,\frac{T_t - T_b}{t} = cA(T_b - T_0). \tag{6}$$

When the thickness of the disc is $2t$, let the temperatures of its upper and lower surfaces be T'_t and T'_b; in the same way,

$$kA\,\frac{T'_t - T'_b}{2t} = cA(T'_b - T_0). \tag{7}$$

Dividing (7) by (6), and using (4) together with

$$T'_t + T'_b = 700.0\,\mathrm{K}, \tag{8}$$

one obtains

$$\frac{700.0\,\mathrm{K} - 2T'_b}{2 \times (700.0\,\mathrm{K} - 2T_b)} = \frac{T'_b - T_0}{T_b - T_0}. \tag{9}$$

From the last equation,

$$T'_b = \frac{700.0\,\mathrm{K}\,(T_b + T_0) - 4T_bT_0}{1400.0\,\mathrm{K} - 2T_b - 2T_0} \approx 333.3\,\mathrm{K}. \tag{10}$$

From (8),

$$T'_t = 700\,\mathrm{K} - T'_b \approx 366.7\,\mathrm{K}. \tag{11}$$

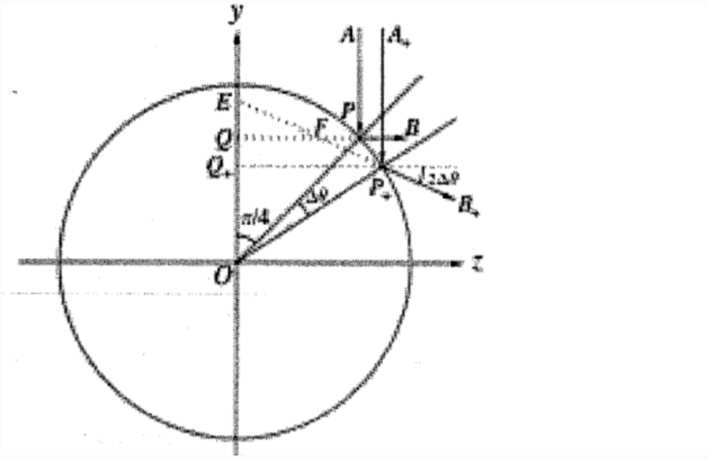
Marking scheme.
This problem is worth 15 points. Equations (1), (2): 2 points each; equations (3), (4): 1 point each; equations (5), (6): 2 points each; equations (7) to (11): 1 point each.

Problem 7 (diameter of a thin wire by two-beam interference)

Solution

1. Consider the thin pencil AP that falls on the outer surface of the wire from the upper side at the angle of incidence 45°. After the reflection at the outer surface of the wire it becomes the reflected ray PB, whose direction is parallel to the positive z direction; the backward extension of PB meets the y axis at the point Q, as shown in the figure. From the geometry, the distance of the ray PB from the xz plane is

$$OQ = +\frac{\sqrt{2}}{4}d = y_+ \tag{1}$$



Now consider the thin pencil A_+P_+ that falls on the outer surface of the wire from the upper side at the angle $45° + \Delta\theta$ ($\Delta\theta$ very small). After the reflection at the outer surface of the wire it becomes the reflected ray P_+B_+, whose backward extension meets the y axis at the point E, as shown in the figure. The law of reflection gives

$$\angle A_+P_+B_+ = 2 \times (45° + \Delta\theta) \tag{2}$$

Hence the angle between the reflected ray P_+B_+ and the z axis is

$$2\Delta\theta \tag{3}$$

Consider the triangle EOP_+; from the geometry,

$$\angle EP_+O = 45° + \Delta\theta = \angle EOP_+,$$

so that

$$OE = P_+E \tag{4}$$

Since $OP_+ = d/2$,

$$\frac{d}{4} = OE\cos(45° + \Delta\theta).$$

Using

$$\cos(45° + \Delta\theta) = \cos 45° \cos\Delta\theta - \sin 45° \sin\Delta\theta$$

and

$$\cos\Delta\theta \sim 1, \quad \sin\Delta\theta \sim \Delta\theta, \qquad \text{when } \Delta\theta \sim 0,$$

one gets

$$\cos(45^\circ + \Delta\theta) = \frac{\sqrt{2}}{2}(1 - \Delta\theta),$$

and therefore

$$OE = \frac{\sqrt{2}d}{4(1 - \Delta\theta)} \approx \frac{\sqrt{2}d}{4}(1 + \Delta\theta) \tag{5}$$

Hence (1) and (5) give

$$QE = OE - OQ = \frac{\sqrt{2}d}{4}\Delta\theta \tag{6}$$

Further, let EB_+ intersect QB at F (let the z coordinate of F be z_+); then in the right triangle EQF

$$z_+ = QF = \frac{QE}{\tan(2\Delta\theta)} = \frac{\sqrt{2}}{4}d\,\frac{\Delta\theta}{2\Delta\theta} = \frac{\sqrt{2}}{8}d \tag{7}$$

Therefore the position of the virtual image formed by the rays incident on the upper side with angle of incidence near $\pi/4$ is

$$y_+ = \frac{\sqrt{2}}{4}d, \qquad z_+ = \frac{\sqrt{2}}{8}d. \tag{8}$$

By symmetry, the position of the virtual image formed by the rays incident on the lower side with angle of incidence near $\pi/4$ is

$$y_- = -\frac{\sqrt{2}}{4}d, \qquad z_- = \frac{\sqrt{2}}{8}d. \tag{9}$$

[**Method 2:** (everything in the solution above, from the beginning down to "the angle is $2\Delta\theta$", is kept).

$$PP_+ = r\Delta\theta, \qquad PQ = OQ = \frac{\sqrt{2}}{2}r, \tag{4$'$}$$

Further, since the angle between PP_+ and QB is 45°,

$$Q_+P_+ = \frac{\sqrt{2}}{2}r + \frac{\sqrt{2}}{2}r\Delta\theta.$$

Extend P_+B_+ backwards; it meets the y axis at E. In the right triangle EQ_+P_+,

$$\begin{aligned} Q_+E &= Q_+P_+ \cdot \tan(2\Delta\theta) \\ &\approx \frac{\sqrt{2}}{2}r(1 + \Delta\theta) \cdot 2\Delta\theta \\ &\approx \sqrt{2}\,r\Delta\theta, \end{aligned} \tag{5$'$}$$

and since $QQ_+ = \frac{\sqrt{2}}{2}r\Delta\theta$, one has, to first order in $\Delta\theta$,

$$Q_+E = 2Q_+Q \tag{6$'$}$$

From $\triangle EQF \sim \triangle EQ_+P_+$,

$$z_+ \equiv QF = \frac{1}{2}P_+Q_+ = \frac{\sqrt{2}}{4}r = \frac{\sqrt{2}}{8}d. \tag{7$'$}$$

F is the intersection of the backward extensions of the two reflected rays PB and P_+B_+, that is, the virtual image.

For the case $\Delta\theta < 0$ the position of the virtual image comes out unchanged. By symmetry the position of the lower virtual image is symmetric to the upper one with respect to the z axis:

$$\text{upper virtual image:} \quad y_+ = +\frac{\sqrt{2}}{4}d, \quad z_+ = \frac{\sqrt{2}}{8}d, \tag{8'}$$

$$\text{lower virtual image:} \quad y_- = -\frac{\sqrt{2}}{4}d, \quad z_- = \frac{\sqrt{2}}{8}d. \tag{9'}$$

]

2. This problem can be regarded as the interference between the light emitted by the two virtual sources that the reflected beams from the upper and the lower side form; it is equivalent to a double-slit interference with the slits at

$$y_+ = \frac{\sqrt{2}}{4}d,\ z_+ = \frac{\sqrt{2}}{8}d \qquad \text{and} \qquad y_- = -\frac{\sqrt{2}}{4}d,\ z_- = \frac{\sqrt{2}}{8}d.$$

The distance from the plane of the two slits to the observation screen is $D - \dfrac{\sqrt{2}d}{8} \approx D$, and the separation of the two slits is $\dfrac{\sqrt{2}}{2}d$. Therefore the fringe spacing is

$$a = \lambda\,\frac{D}{(\sqrt{2}/2)d} = \frac{\sqrt{2}\lambda D}{d} \tag{10}$$

From this the diameter d of the wire is

$$d = \frac{\sqrt{2}\lambda D}{a} \tag{11}$$

Marking scheme.
This problem is worth 15 points.

- Part 1: 10 points. Equations (1), (2): 1 point each; equation (3): 2 points; equations (4) to (9): 1 point each.
- Part 2: 5 points. Equation (10): 3 points; equation (11): 2 points.

Problem 8 (relativistic exchange of claps between two moving students)

Solution

1. In the ground reference frame S, the time interval from the meeting of Zhang and Wang to the moment when the distance between them is L is

$$\Delta t = \frac{L}{2v}. \tag{1}$$

In the reference frame S′ in which Zhang is at rest, the two events at which Zhang reads the clock he carries happen at the same place; therefore the reading T of the clock carried by Zhang when he claps is the proper time. The relativistic effect for time gives

$$T = \Delta t\sqrt{1-\frac{v^2}{c^2}} = \Delta t\sqrt{1-\beta^2} = \frac{L}{2v}\sqrt{1-\beta^2}. \tag{2}$$

2. In the reference frame S$'$, the instant and the position of event 1, "Zhang claps for the first time", are $t'_1 = T$ and $x'_1 = 0$. It has been assumed that Zhang and Wang are at the origins of the rectangular coordinate systems of their own reference frames, that the origins of the two coordinate systems coincide at $t' = t'' = 0$, and that the x' and x'' axes coincide and point in the same direction. In the reference frame S$''$ in which Wang is at rest, Zhang's clock runs slow. Using the relativistic effect for time, the instant t''_1 at which event 1 happens in the reference frame S$''$ (this is the reading of the clock carried by Wang) is

$$t''_1 = \gamma t'_1 = \gamma T, \tag{3}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v_r^2/c^2}} = \frac{1+\beta^2}{1-\beta^2}, \tag{4}$$

and where the expression for the speed v_r of the uniform rectilinear motion of Zhang relative to Wang has been used in the derivation. Note: in the reference frame S$''$ the position at which event 1 happens is not at Wang himself, that is not at the origin, but at $x''_2 \neq 0$. [a]

According to the statement of the problem, Wang claps at the instant

$$t''_2 = t''_1 = \gamma T \tag{5}$$

This clap happens, in the reference frame S$''$, at the instant $t''_2 = t''_1$ and at the origin $x''_2 = 0$; it is an event different from event 1, and it is called event 2. In the reference frame S$'$ in which Zhang is at rest, Wang's clock runs slow. Using the relativistic effect for time, the instant t'_2 at which event 2 happens in the reference frame S$'$ is

$$t'_2 = \gamma t''_2 = \gamma(\gamma T) = \gamma^2 T, \tag{6}$$

According to the statement of the problem, Zhang claps for the second time (event 3) at the instant

$$t'_3 = t'_2 = \gamma^2 T \tag{7}$$

and t'_3 is the reading of the clock that he carries.

Continuing in this way: when Zhang claps for the n-th time, the instant t'_{2n-1} at which the $(2n-1)$-th event happens in the reference frame S$'$ (in other words, the reading of the clock carried by Zhang) is

$$t'_{2n-1} = \gamma^{2(n-1)} T. \tag{8}$$

Since in each time interval T observed by Zhang the distance between Zhang and Wang in the ground reference frame increases by L, the distance between Zhang and Wang in the ground reference frame when Zhang claps for the n-th time is

$$d_n = \gamma^{2(n-1)} L = \left(\frac{1+\beta^2}{1-\beta^2}\right)^{2(n-1)} L. \tag{9}$$

3. Using the relativistic effect for time, the instants at which Zhang and Wang clap one after another, seen from the reference frame in which Li is at rest, are

$$t_1 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} t'_1 = \left(1-\beta^2\right)^{-1/2} T = \frac{L}{2v}, \tag{10}$$

$$t_2 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} t''_2 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} (\gamma T) = \gamma \frac{L}{2v} = \frac{1+\beta^2}{1-\beta^2}\frac{L}{2v}, \tag{11}$$

$$t_3 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} t'_3 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} (\gamma^2 T) = \gamma^2 \frac{L}{2v} = \left(\frac{1+\beta^2}{1-\beta^2}\right)^2 \frac{L}{2v}, \tag{12}$$

$$\vdots$$

$$t_{2n-1} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} t'_{2n-1} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \left(\gamma^{2(n-1)} T\right)$$
$$= \gamma^{2(n-1)} \frac{L}{2v} = \left(\frac{1+\beta^2}{1-\beta^2}\right)^{2(n-1)} \frac{L}{2v}. \tag{13}$$

From (10) to (13) one sees directly that

$$t_1 < t_2 < t_3 < \cdots < t_{2n-1} \tag{14}$$

Marking scheme.
This problem is worth 15 points.

- Part 1: 3 points. Equation (1): 1 point; equation (2): 2 points.
- Part 2: 8 points. Equations (3) to (6): 1 point each; equations (8), (9): 2 points each.
- Part 3: 4 points. Equations (10), (11), (13), (14): 1 point each.

[a] *Translator's note:* As printed. The position of event 1 in S$''$ should be written $x''_1 \neq 0$.