The 31st Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2014 — official reference solutions

Equation numbers (1), (2), … in each problem follow the circled numbers of the Chinese original. Primed numbers mark the equations that change in an alternative case or method.

Problem 1 (speed-of-rotation control device)

Solution

(1) Let ω_A be the angular speed of the tube when the ball is at A, in radial equilibrium relative to the tube. The spring is stretched by $L - \frac{L}{4} = \frac{3}{4}L$, and the Coulomb force of the charge Q on the ball is repulsive, so

$$k_0\,\frac{3}{4}L - k\,\frac{qQ}{L^2} = Lm\omega_A^2. \tag{1}$$

When the ball is at B ($OB = L/2$), the charge at O is $-Q$, the Coulomb force is attractive, the spring is stretched by $\frac{L}{4}$, and with the angular speed ω_B of the tube

$$k_0\,\frac{1}{4}L + k\,\frac{qQ}{L^2/4} = \frac{L}{2}\,m\omega_B^2. \tag{2}$$

After the switch is triggered, the external torque is zero, and the forces on the ball are radial, so the angular momentum of the ball is conserved:

$$mL^2\omega_A = m\,\frac{L^2}{4}\,\omega_B, \tag{3}$$

so that

$$\omega_B = 4\omega_A.$$

Substitution into (2) gives

$$k_0\,\frac{1}{4}L + k\,\frac{qQ}{L^2/4} = 8Lm\omega_A^2. \tag{4}$$

From (1) and (4),

$$k_0 = k\,\frac{48qQ}{23L^3}. \tag{5}$$

Substitution into (2) gives the angular speed of the tube when the ball is at B:

$$\omega_B = 4\sqrt{\frac{13kqQ}{23mL^3}}. \tag{6}$$

(2) Put $r = r_b + x$ with $|x| \ll r_b$, where $r_b = L/2$ is the radius of the equilibrium point B. Relative to the tube, the radial resultant force on the ball near the equilibrium point is

$$F(x) = -k_0\Big(\frac{L}{4} + x\Big) - k\,\frac{qQ}{\big(\frac{L}{2} + x\big)^2} + \Big(\frac{L}{2} + x\Big)m\omega_x^2, \tag{7}$$

where ω_x is the angular speed of the tube at the position x. By the conservation of angular momentum,

$$m\Big(\frac{L}{2}+x\Big)^2\omega_x = m\Big(\frac{L}{2}\Big)^2\omega_B. \tag{8}$$

An expansion of (7) around the equilibrium point, to first order in x, gives

$$F(x) = -\Big(k_0 + 3m\omega_B^2 - k\,\frac{16qQ}{L^3}\Big)x = -\frac{304}{23}\,\frac{kqQ}{L^3}\,x. \tag{9}$$

The force is a linear restoring force. Therefore the ball performs small radial oscillations, relative to the tube, about the equilibrium point B. The period of this radial oscillation is

$$T = 2\pi\sqrt{\frac{23mL^3}{304\,kqQ}} = \frac{\pi}{2}\sqrt{\frac{23}{19}\,\frac{mL^3}{kqQ}}. \tag{10}$$

Problem 2 (multiple-warhead attack)

Solution

Solution 1. Set up the coordinate system of figure 1, with the origin O at the midpoint of the line WN: the x axis lies along the coastline, the y axis is vertical, and the z axis points from the coastline toward the sea. Let the position of the submarine at the moment of launch be $(x, 0, d)$; let θ be the elevation angle of the launch direction (the angle with the xOz plane), and let α be the angle, inside the xOz plane, between the projection of the launch direction and the x axis.

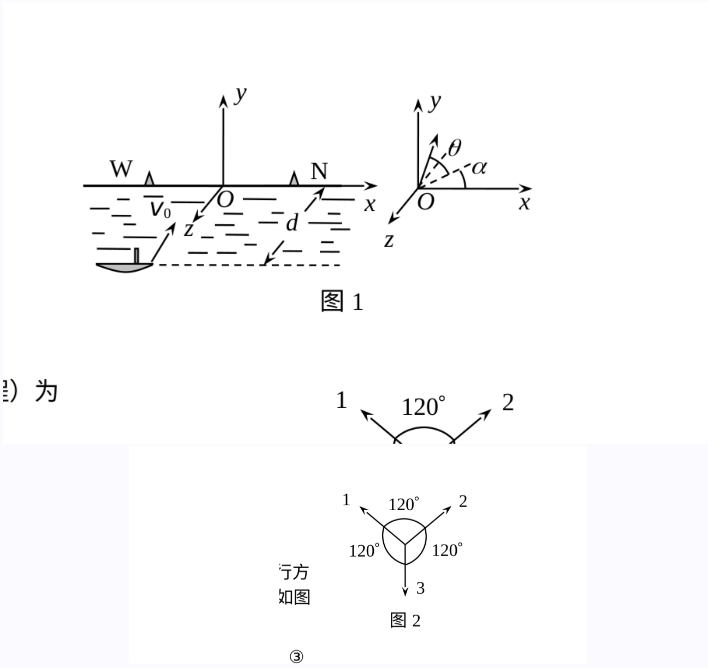


图 *1 = figure 1;* 图 *2 = figure 2.*

By the given data, the missile reaches its highest point at the time

$$t = \frac{v_0\sin\theta}{g}. \tag{1}$$

The distance of the center-of-mass motion along the launch direction, measured on the sea surface (the range), is

$$s = \frac{v_0^2 \sin 2\theta}{g}. \tag{2}$$

By momentum conservation, in the reference frame of the missile the flight directions of the three warheads lie in a horizontal plane and are symmetric, 120° apart. Assume the flight direction of warhead 3 is perpendicular to the coastline, pointing away from it (figure 2). The perpendicular component of the relative velocity of warheads 1 and 2 is then $v \cos 60° = v/2$, toward the coast. The distance that warheads 1 and 2 travel in the direction perpendicular to the coastline is

$$d = \left(\frac{v}{2} + v_0 \cos\theta \, \sin\alpha\right) t + v_0 \cos\theta \, \sin\alpha \cdot t. \tag{3}$$

The distance that warhead 1 travels parallel to the coastline satisfies

$$\left(\frac{\sqrt{3}}{2} v - v_0 \cos\theta \, \cos\alpha\right) t = \frac{L}{2} - |x| + \frac{s}{2} \cos\alpha, \tag{4}$$

and the distance that warhead 2 travels parallel to the coastline satisfies

$$\left(\frac{\sqrt{3}}{2} v + v_0 \cos\theta \, \cos\alpha\right) t = |x| + \frac{L}{2} - \frac{s}{2} \cos\alpha. \tag{5}$$

The sum of (4) and (5) with (1) gives $\sqrt{3}\, v \, \dfrac{v_0 \sin\theta}{g} = L$, so

$$\sin\theta = \frac{gL}{\sqrt{3}\, v_0 v}, \qquad \cos\theta = \sqrt{1 - \frac{g^2 L^2}{3 v_0^2 v^2}}. \tag{6}$$

Substitution of (1) and (6) into (3) gives, for a missile that separates above the sea,

$$d = \left(\frac{v}{2} + 2 v_0 \cos\theta \, \sin\alpha\right) \frac{v_0 \sin\theta}{g},$$

$$\sin\alpha = \pm \frac{(2\sqrt{3} d - L) v}{4 L v_0 \sqrt{1 - \dfrac{g^2 L^2}{3 v_0^2 v^2}}}, \qquad \cos\alpha = \sqrt{1 - \frac{9 v^4 \left(d - \dfrac{L}{2\sqrt{3}}\right)^2}{L^2 \left(3 v_0^2 v^2 - g^2 L^2\right)}}. \tag{7}$$

In (7) the sign $+$ means that the horizontal component of the launch velocity tilts toward the coast, and the sign $-$ means that it tilts away from the coast. Substitution of these results into (4) or (5) gives

$$x = \pm \sqrt{\frac{4 L^2}{9 v^4} \left(3 v_0^2 v^2 - g^2 L^2\right) - \left(d - \frac{L}{2\sqrt{3}}\right)^2}. \tag{8}$$

For the operation to be possible, the conditions

$$0 < \frac{gL}{\sqrt{3} v_0 v} \le 1, \qquad \left| \frac{(2\sqrt{3} d - L) v}{4 L v_0 \sqrt{1 - \dfrac{g^2 L^2}{3 v_0^2 v^2}}} \right| \le 1, \qquad d > 0 \tag{9}$$

must hold; the first two conditions come from (7), and the last one is the restriction given in the problem.
If the missile separates above the land, only (7), (8), (9) change; the changed results are

$$x = \pm\sqrt{\frac{4L^2}{9v^4}\left(3v_0^2v^2 - g^2L^2\right) - \left(d + \frac{L}{2\sqrt{3}}\right)^2}, \tag{7$'$}$$

$$\sin\alpha = \frac{(2\sqrt{3}d + L)v}{4Lv_0\sqrt{1 - \dfrac{g^2L^2}{3v_0^2v^2}}}, \qquad \cos\alpha = \sqrt{1 - \frac{9v^4\left(d + \dfrac{L}{2\sqrt{3}}\right)^2}{L^2\left(3v_0^2v^2 - g^2L^2\right)}}, \tag{8$'$}$$

$$0 < \frac{gL}{\sqrt{3}v_0v} \le 1, \qquad \frac{(2\sqrt{3}d + L)v}{4Lv_0\sqrt{1 - \dfrac{g^2L^2}{3v_0^2v^2}}} \le 1, \qquad d > 0. \tag{9$'$}$$

Solution 2. Use the same coordinate system (figure 1). The motion of the missile separates into the motion of the center of mass and the motion of the warheads relative to the center of mass. Equations (1) and (2) hold as before. If warheads 1 and 2 must hit the two targets (by the problem, they must hit them at the same time), the center of mass must land on the perpendicular bisector of WN (on the sea surface or on the land); the top view is shown in figure 3 (drawn for a center of mass that lands on the sea). The analysis gives

$$\frac{v}{2}t = \frac{L}{2\sqrt{3}}, \tag{3}$$

$$x^2 + \left(d \pm \frac{L}{2\sqrt{3}}\right)^2 = s^2, \tag{4}$$

where the signs $-$ and $+$ on the left side of (4) correspond to a center of mass that lands on the sea and on the land, respectively.

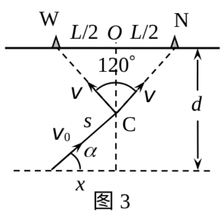


图 *3 = figure 3 (top view).*

Substitution of (1) into (3) gives

$$\sin\theta = \frac{gL}{\sqrt{3}\,v_0v}, \qquad \cos\theta = \sqrt{1 - \frac{g^2L^2}{3v_0^2v^2}}. \tag{5}$$

Substitution of (1) and (5) into (2) gives, for a center of mass that lands on the sea,

$$s = \frac{2L}{3v^2}\sqrt{3v_0^2v^2 - g^2L^2}, \qquad x = \pm\sqrt{\frac{4L^2}{9v^4}\left(3v_0^2v^2 - g^2L^2\right) - \left(d - \frac{L}{2\sqrt{3}}\right)^2}. \tag{6}$$

Also

$$|x| = s\cos\alpha, \tag{7}$$

$$\cos\alpha = \sqrt{1 - \frac{9v^4\left(d - \frac{L}{2\sqrt{3}}\right)^2}{L^2\left(3v_0^2v^2 - g^2L^2\right)}}, \qquad \sin\alpha = \pm\frac{(2\sqrt{3}d - L)v}{4Lv_0\sqrt{1 - \frac{g^2L^2}{3v_0^2v^2}}}, \tag{8}$$

where in (8) the sign + means that the horizontal component of the launch velocity tilts toward the coast and − that it tilts away from it. The conditions for the operation are

$$0 < \frac{gL}{\sqrt{3}v_0v} \le 1, \qquad \left|\frac{(2\sqrt{3}d - L)v}{4Lv_0\sqrt{1 - \frac{g^2L^2}{3v_0^2v^2}}}\right| \le 1, \qquad d > 0; \tag{9}$$

the first two conditions come from (8), the last is given by the problem. For a center of mass that lands on the land, only (6), (8), (9) change, with $d + \frac{L}{2\sqrt{3}}$ in place of $d - \frac{L}{2\sqrt{3}}$ (and the sign of $\sin\alpha$ fixed to +), exactly as in Solution 1.

Problem 3 (adiabatic container with three chambers)

Solution

(1) The free expansion of an ideal gas is not quasi-static; the surroundings do no work on the gas, and there is no heat exchange. By the first law of thermodynamics the internal energy does not change, $\Delta U = 0$ ($\Delta T = 0$), so the temperature does not change; only the volume doubles, so the pressure falls to one half:

$$T_1 = T_0, \qquad P_1 = \frac{1}{2}P_0. \tag{1}$$

After this, the gas goes from $\left(\frac{1}{2}P_0,\, T_1,\, 2V_0\right)$ through a quasi-static adiabatic compression to a new equilibrium state $\left(P',\, T',\, 1.7V_0\right)$. The adiabatic process satisfies

$$\frac{1}{2}P_0\,(2V_0)^\gamma = P'\,(1.70V_0)^\gamma,$$

where $\gamma = \frac{5}{3}$ is the heat-capacity ratio of a monatomic ideal gas. So

$$P' = 0.85^{-\frac{5}{3}} \times \frac{P_0}{2}. \tag{2}$$

From this equation and the equations of state

$$P_0V_0 = \nu_1RT_0, \qquad P' \times 1.70V_0 = \frac{1}{2} \times 0.85^{-\frac{5}{3}}P_0 \times 1.70V_0 = \nu_1RT',$$

we get

$$T' = \frac{P_0 V_0}{\nu_1 R} \times 0.85 \times 0.85^{-\frac{5}{3}} = 0.85^{-\frac{2}{3}}\, T_0. \tag{3}$$

(2) When the two gases in the container mix and together reach the equilibrium state $\big(2.7V_0,\, p'',\, T''\big)$, energy conservation gives

$$\nu_1 \frac{3}{2} RT' + \nu_2 \frac{5}{2} RT_c = \Big(\nu_1 \frac{3}{2} R + \nu_2 \frac{5}{2} R\Big) T'',$$

$$T'' = \frac{3\nu_1 T' + 5\nu_2 T_c}{3\nu_1 + 5\nu_2} = \frac{3T' + 10T_c}{13} = \frac{3 \times 0.85^{-\frac{2}{3}} + 5}{13}\, T_0 = 0.64\, T_0. \tag{4}$$

By the ideal-gas equation,

$$p_1'' = n_1 k T'', \qquad p_2'' = n_2 k T''; \qquad p'' = nkT'', \tag{5}$$

where k is the Boltzmann constant, p_1'', p_2'' are the pressures of the monatomic and the diatomic gas, n_1, n_2 are the number densities of the two kinds of molecules, and n is the number density of the mixture. With $n = n_1 + n_2$ and (5),

$$p'' = nkT'' = (n_1 + n_2)kT'' = p_1'' + p_2'' = \frac{(\nu_1 + \nu_2)}{2.7V_0} RT''. \tag{6}$$

The data give

$$p'' = \frac{3}{2.70V_0} R\, \frac{3 \times 0.85^{-\frac{2}{3}} + 5}{13} T_0 = \frac{9 \times 0.85^{-\frac{2}{3}} + 15}{35.1}\, p_0, \tag{7}$$

where

$$p_1'' = \frac{\nu_1}{2.7V_0} RT'', \qquad p_2'' = \frac{\nu_2}{2.7V_0} RT'', \qquad P_0 V_0 = \nu_1 R T_0$$

were used.

(3) Finally the gas mixture goes from $\big(P'', T'', 2.7V_0\big)$ through a quasi-static adiabatic expansion to the equilibrium state $\big(P''', T''', 3V_0\big)$. This adiabatic process satisfies

$$p''\,(2.70V_0)^{\gamma_m} = P'''\,(3V_0)^{\gamma_m}, \tag{8}$$

where γ_m, the heat-capacity ratio of the mixture of the two gases, is given by (the proof is at the end)

$$\gamma_m = \frac{\nu_1 C_{p1} + \nu_2 C_{p2}}{\nu_1 C_{V1} + \nu_2 C_{V2}}. \tag{9}$$

Here C_{pi}, C_{Vi} are the molar heat capacities of the i-th ideal gas at constant pressure and at constant volume. By the problem,

$$C_{V1} = \frac{3}{2}R, \quad C_{p1} = C_{V1} + R = \frac{5}{2}R; \qquad C_{V2} = \frac{5}{2}R, \quad C_{p2} = \frac{7}{2}R,$$

so (9) gives the heat-capacity ratio of the mixture:

$$\gamma_m = \frac{\nu_1 C_{p1} + \nu_2 C_{p2}}{\nu_1 C_{V1} + \nu_2 C_{V2}} = \frac{5 + 14}{3 + 10} = \frac{19}{13}.$$

Substitution into (8), with (7), gives

$$P''' = 0.90^{\gamma_m} P'' = 0.90^{\frac{19}{13}} \times \frac{9 \times 0.85^{-\frac{2}{3}} + 15}{35.1}\, p_0. \tag{10}$$

With the equation of state

$$\frac{P''(2.70V_0)}{T''} = \frac{P'''(3V_0)}{T'''},$$

equation (8) can be rewritten as

$$T''\,(2.70V_0)^{\gamma_m-1} = T'''\,(3V_0)^{\gamma_m-1}, \tag{11}$$

which together with (4) gives

$$T''' = 0.90^{(\gamma_m-1)}\,T'' = 0.90^{\frac{6}{13}} \times \frac{3\times 0.85^{-\frac{2}{3}}+5}{13.0}\,T_0. \tag{12}$$

We now prove that the heat-capacity ratio of the mixture of the two gases is given by (9).

Proof (a). By the hint, an adiabatic reversible process of a thermodynamic system that consists of two ideal gases satisfies

$$(\Delta S_1)_{if} + (\Delta S_2)_{if} = 0,$$

that is,

$$\nu_1 C_{V_1}\ln\left(\frac{T_f}{T_i}\right) + \nu_1 R\ln\left(\frac{V_f}{V_i}\right) + \nu_2 C_{V_2}\ln\left(\frac{T_f}{T_i}\right) + \nu_2 R\ln\left(\frac{V_f}{V_i}\right) = 0.$$

This can be rewritten as

$$\big(\nu_1 C_{V_1} + \nu_2 C_{V_2}\big)\ln\left(\frac{T_f}{T_i}\right) = -\big(\nu_1 R + \nu_2 R\big)\ln\left(\frac{V_f}{V_i}\right) = \big(\nu_1 R + \nu_2 R\big)\ln\left(\frac{V_i}{V_f}\right),$$

that is,

$$\left(\frac{T_f}{T_i}\right) = \left(\frac{V_i}{V_f}\right)^{\frac{\nu_1 R+\nu_2 R}{\nu_1 C_{V_1}+\nu_2 C_{V_2}}},$$

or

$$T_f V_f^{\,g} = T_i V_i^{\,g}, \qquad g = \frac{\nu_1 R+\nu_2 R}{\nu_1 C_{V_1}+\nu_2 C_{V_2}}. \tag{a1}$$

By the ideal-gas equation of state,

$$p_i V_i = (p_{i1}+p_{i2})V_i = (\nu_1+\nu_2)RT_i, \qquad p_f V_f = (p_{f1}+p_{f2})V_f = (\nu_1+\nu_2)RT_f,$$

where p_{i1}, p_{i2} and p_{f1}, p_{f2} are the partial pressures of the two components in the initial and in the final state. With (a1),

$$\frac{p_f}{p_i} = \frac{T_f}{T_i}\frac{V_i}{V_f} = \left(\frac{V_i}{V_f}\right)^g \frac{V_i}{V_f} = \left(\frac{V_i}{V_f}\right)^{g+1},$$

that is,

$$p_f V_f^{\gamma_m} = p_i V_i^{\gamma_m}, \tag{a2}$$

where

$$\gamma_m = g+1 = \frac{\nu_1 R+\nu_2 R}{\nu_1 C_{V_1}+\nu_2 C_{V_2}} + 1 = \frac{\nu_1 C_{p_1}+\nu_2 C_{p_2}}{\nu_1 C_{V_1}+\nu_2 C_{V_2}},$$

with $C_{p_1} = C_{V_1}+R$, $C_{p_2} = C_{V_2}+R$. □

Proof (b) (without the hint; needs the definite integral $\int_i^f \alpha\,\frac{\mathrm{d}x}{x} = \alpha\ln\left(\frac{x_f}{x_i}\right) = \ln\left(\frac{x_f}{x_i}\right)^{\alpha}$). From the adiabatic condition $\mathrm{d}Q = 0$ and the first law,

$$\begin{aligned}\mathrm{d}Q &= \mathrm{d}Q_1 + \mathrm{d}Q_2 = \big(p_1\mathrm{d}V + \nu_1 C_{V1}\mathrm{d}T\big) + \big(p_2\mathrm{d}V + \nu_2 C_{V2}\mathrm{d}T\big)\\ &= (p_1+p_2)\mathrm{d}V + \big(\nu_1 C_{V1}+\nu_2 C_{V2}\big)\mathrm{d}T = p\,\mathrm{d}V + \big(\nu_1 C_{V1}+\nu_2 C_{V2}\big)\mathrm{d}T = 0,\end{aligned} \tag{b1}$$

where $\mathrm{d}Q_i$ is the heat absorbed by the i-th ideal gas in the process. By the ideal-gas equation of state,

$$pV = \nu RT = (\nu_1 + \nu_2)RT. \tag{b2}$$

Differentiation of both sides gives

$$p\,\mathrm{d}V + V\mathrm{d}p = (\nu_1 + \nu_2)R\,\mathrm{d}T. \tag{b3}$$

Elimination of the term $p\,\mathrm{d}V$ from (b1) and (b3) gives

$$V\mathrm{d}p = (\nu_1 + \nu_2)R\,\mathrm{d}T + \big(\nu_1 C_{V1} + \nu_2 C_{V2}\big)\mathrm{d}T = \big(\nu_1 C_{p1} + \nu_2 C_{p2}\big)\mathrm{d}T, \tag{b4}$$

where $C_{p1} = C_{V1} + R$, $C_{p2} = C_{V2} + R$. With $V = (\nu_1 + \nu_2)RT/p$ from (b2), substitution into (b4) gives

$$(\nu_1 + \nu_2)R\,\frac{\mathrm{d}p}{p} = \big(\nu_1 C_{p1} + \nu_2 C_{p2}\big)\frac{\mathrm{d}T}{T},$$

that is,

$$\frac{\mathrm{d}p}{p} = \frac{\nu_1 C_{p1} + \nu_2 C_{p2}}{(\nu_1 + \nu_2)R}\,\frac{\mathrm{d}T}{T} = \frac{\gamma_m}{\gamma_m - 1}\,\frac{\mathrm{d}T}{T}. \tag{b5}$$

Integration of (b5) from the initial state (p_i, T_i, V_i) to the final state (p_f, T_f, V_f) gives

$$p_f\,T_f^{\gamma_m/(1-\gamma_m)} = p_i\,T_i^{\gamma_m/(1-\gamma_m)}. \tag{b6}$$

Then, with (b2) and (b6), elimination of T or of p gives

$$p_f V_f^{\gamma_m} = p_i V_i^{\gamma_m}, \qquad T_f V_f^{\gamma_m - 1} = T_i V_i^{\gamma_m - 1}. \tag{b7}$$

□

Problem 4 (fiber grating)

Solution

First analyse the back-and-forth propagation of the light wave in the first layer. When the light goes from the fiber (base refractive index n_1) onto the surface of the first layer, of refractive index n_2, let the incident field (electric field strength) be a; the reflected field is then

$$a_{r1} = r_1 a, \tag{1}$$

where $r_1 = \dfrac{n_1 - n_2}{n_1 + n_2}$. The transmitted field is

$$a_{t1} = t_1 a, \tag{2}$$

where $t_1 = \dfrac{2n_1}{n_1 + n_2}$. When this wave reaches the surface of the second layer, of refractive index n_1, then — without regard yet to the phase change of the propagation inside the first layer — the reflected field is

$$a_{r2} = r_2 t_1 a = \frac{n_2 - n_1}{n_1 + n_2}\,t_1 a = -r_1 t_1 a. \tag{3}$$

Because $t_1 > 0$, the factor (-1) on the right side of (3) shows a 180° phase difference between a_{r2} and a_{r1}.

Now include the phase change of the propagation in the first layer. If the minimum thickness of the first layer (smaller than the wavelength of the light in this medium) is

$$d_2 = \frac{\lambda}{4n_2} = \frac{1060\,\mathrm{nm}}{4 \times 1.55} \approx 171 \text{ nm}, \tag{4}$$

then the reflected field a_{r2} carries an extra 180° phase difference back at the entrance interface, and it interferes constructively with the reflected field a_{r1}. At the entrance interface this reflected field is

$$a'_{r2} = t_1 r_1 a; \tag{5}$$

after it passes out through the first layer it becomes

$$a''_{r2} = t_2 t_1 r_1 a, \tag{6}$$

where

$$r_1 = -r_2 = \frac{n_1 - n_2}{n_1 + n_2}, \qquad t_1 = \frac{2n_1}{n_1 + n_2}, \qquad t_2 = \frac{2n_2}{n_1 + n_2}. \tag{7}$$

Next analyse the propagation through the first and the second layer. When the light reaches the front surface of the second layer (refractive index n_1), then — again without the propagation phases — the transmitted field is

$$a_{t2} = t_2 a_{t1} = t_2 t_1 a. \tag{8}$$

By an analysis like the one above, the field of a_{t2} that passes through the second layer, reflects from the interface between the second and the third layer, and returns to the entrance interface, is

$$a'_{r3} = (t_1 t_2)^2 r_1 a. \tag{9}$$

Now include the propagation phases in the first and the second layer. If the second layer has the minimum thickness

$$d_1 = \frac{\lambda}{4n_1} = \frac{1060\,\mathrm{nm}}{4 \times 1.51} \approx 175 \text{ nm}, \tag{10}$$

then the phase change of the round trip through the first and the second layer is a 360° phase difference. The field reflected at the third interface and returned to the entrance interface is therefore

$$a''_{r3} = a'_{r3} = (t_1 t_2)^2 r_1 a, \tag{11}$$

and it interferes constructively with the reflected fields a_{r1} and a''_{r2}.
In this way — multiple reflections at each interface not considered — the total reflected field at the entrance of the device is

$$E = r_1 a + r_1 t_1 t_2 a + r_1 t_1^2 t_2^2 a + \cdots + r_1 t_1^N t_2^N a = r_1 a \, \frac{1 - (t_1 t_2)^{N+1}}{1 - t_1 t_2}. \tag{12}$$

With the required reflectivity $R_{\text{refl}} = E^2/a^2 = 8\%$, equation (12) gives

$$N = \frac{\ln\left[1 - \dfrac{\sqrt{R_{\text{refl}}}\,(1 - t_1 t_2)}{|r_1|}\right]}{\ln(t_1 t_2)} - 1 = 20.68; \tag{13}$$

the total number of reflecting layers is at least $N = 21$.

[Another method: because n_1 and n_2 are very close, $t_1 t_2 \approx 1$, and (12) can be approximated by

$$E = r_1 a \, \frac{1 - (t_1 t_2)^{N+1}}{1 - t_1 t_2} \approx (N+1)\, r_1 a.$$

If the reflectivity must reach 8%, then

$$E^2 = \big[(N+1)r_1 a\big]^2 = (N+1)^2 r_1^2 a^2 = 0.08\,a^2.$$

The given data give

$$r_1 = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.51 - 1.55}{1.51 + 1.55} \approx -0.01307,$$

so

$$N = \sqrt{\frac{0.08}{|r_1|^2}} - 1 \approx 20.64; \tag{13'}$$

the total number of reflecting layers is at least 21.]

Problem 5 (neutral-particle analyser)

Solution

(1) For a microscopic particle the gravity of the Earth is negligible, so outside the plates the ion moves in a straight line with constant velocity. Take the entry hole as the coordinate origin, the Y axis perpendicular to the plates and the X axis parallel to the plates. The ion enters the region between the plates at the point $(h_1 \cot\theta,\ h_1)$, and inside this region it moves like a projectile, with initial speed

$$v_0 = \sqrt{2E/m}. \tag{1}$$

The magnitude of the acceleration is $a_y = -qV/(md)$. Where the Y velocity becomes zero, the coordinates are

$$\Big(h_1 \cot\theta + \frac{Ed\sin 2\theta}{qV},\quad h_1 + \frac{Ed\sin^2\theta}{qV}\Big). \tag{2}$$

When the ion leaves the plate region its speed is again $v_0 = \sqrt{2E/m}$, and it then moves in a straight line with constant velocity. The X coordinate of the exit hole is therefore

$$l = (h_1 + h_2)\cot\theta + \frac{2Ed\sin(2\theta)}{qV}. \tag{3}$$

The relation between the ion energy and l is

$$E = \frac{qV\big[l - (h_1 + h_2)\cot\theta\big]}{2d\sin 2\theta}. \tag{4}$$

The maximum flight distance, perpendicular to the plates, of an ion of energy E inside the plates is

$$H = \frac{Ed\sin^2\theta}{qV}. \tag{5}$$

(2) When $\Delta\alpha = \theta$, the requirement is

$$\begin{aligned} l &= (h_1 + h_2)\cot(\theta + \Delta\alpha) + \frac{2Ed\sin[2(\theta + \Delta\alpha)]}{qV} \\ &= (h_1 + h_2)\cot\theta + \frac{2Ed\sin(2\theta)}{qV}. \end{aligned} \tag{6}$$

Also

$$\cot(\theta+\Delta\alpha) = \frac{\cos\theta\cos\Delta\alpha - \sin\theta\sin\Delta\alpha}{\sin\theta\cos\Delta\alpha + \cos\theta\sin\Delta\alpha} \approx \frac{\cos\theta - \Delta\alpha\sin\theta}{\sin\theta + \Delta\alpha\cos\theta} = \frac{\cot\theta - \Delta\alpha}{1+\Delta\alpha\cot\theta}$$
$$\approx \cot\theta - \Delta\alpha\,(1+\cot^2\theta) = \cot\theta - \frac{\Delta\alpha}{\sin^2\theta},$$

where $\sin\Delta\alpha \approx \Delta\alpha$, $\cos\Delta\alpha \approx 1$ were used; in the same way

$$\sin[2(\theta+\Delta\alpha)] = \sin(2\theta)\cos(2\Delta\alpha) + \cos(2\theta)\sin(2\Delta\alpha) \approx \sin(2\theta) + 2\Delta\alpha\cos(2\theta).$$

Substitution into (6) [with (4)] gives

$$\frac{\Delta\alpha\,(h_1+h_2)}{\sin^2\theta} - \frac{4\Delta\alpha\,Ed\cos(2\theta)}{qV} = 0,$$

that is,

$$h_2 = \frac{4Ed\sin^2\theta\cos(2\theta)}{qV} - h_1. \tag{7}$$

The corresponding energy expression is

$$E = \frac{qVl}{2d\sin(2\theta)\big[1+\cos(2\theta)\big]}. \tag{8}$$

(3) So that no ion of the whole measuring range touches the upper plate, the plate separation must be larger than H of (5); this gives

$$E_{\max} \le \frac{qV}{\sin^2\theta} \tag{9}$$

(in other words, the applied voltage must satisfy $V \ge E_{\max}\sin^2\theta/q$). From (8) and the requirement of the problem, the relation between $E_{\max}$ and $l_{\max}$ is

$$E_{\max} = \frac{qVl_{\max}}{2d\sin(2\theta)\big[1+\cos(2\theta)\big]}. \tag{10}$$

Substitution into (9) gives the minimum plate separation

$$d = \frac{l_{\max}\sin^2\theta}{2\sin(2\theta)\big[1+\cos(2\theta)\big]}. \tag{11}$$

For $\theta = 30^\circ$ the requirement is

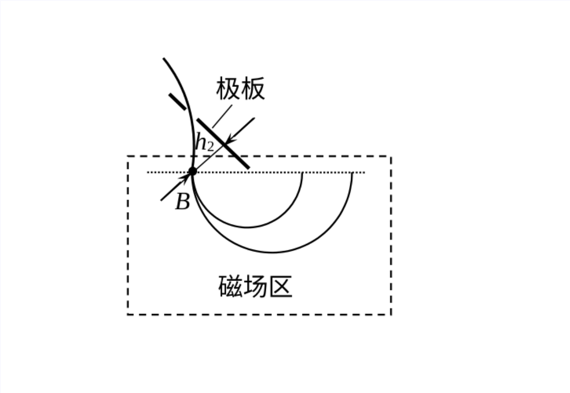
$$d = \frac{\sqrt{3}}{18}\,l_{\max}. \tag{12}$$

(4) One can add, as shown in the figure, a semicircular region of a uniform magnetic field of magnitude B, perpendicular to the plane of the motion; the ions then move in circles, with the equation of motion

$$\frac{mv^2}{R} = qvB, \tag{13}$$

that is, $mv = qBR$. A measurement of the cyclotron radius R of the ion gives

$$m = \frac{(mv)^2}{2E} = \frac{qB^2R^2\,d\sin(2\theta)\big[1+\cos(2\theta)\big]}{Vl} = \frac{qB^2R^2\,l_{\max}\sin^2\theta}{2Vl}. \tag{14}$$



Labels: 极板 *= plates;* 磁场区 *= magnetic field region.*

Problem 6 (flux vortices in a type-II superconductor)

Solution

(1) The magnetic flux is

$$\Phi = B_a \cdot A. \tag{1}$$

By the geometry, two equilateral triangles of the lattice contain one flux quantum, so the area per flux quantum is

$$A = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} a^2 = \frac{\sqrt{3}}{2} a^2. \tag{2}$$

Here $\Phi = \Phi_0$, so

$$\Phi_0 = B_a \cdot \frac{\sqrt{3}}{2} a^2, \tag{3}$$

$$a = \sqrt{\frac{2}{\sqrt{3}} \cdot \frac{\Phi_0}{B_a}} = 2.19 \times 10^{-7}\ \text{m} = 0.219\ \mu\text{m}. \tag{4}$$

(2) As the external field grows, the area of each flux quantum becomes smaller, that is, the flux density grows. When the normal cores of two neighbouring flux vortex lines begin to overlap, that is, when

$$a \le 2\xi, \tag{5}$$

the whole superconductor becomes normal. The corresponding field is the upper critical field; by (3) with $a = 2\xi$,

$$B_{c2} = \frac{2\Phi_0}{\sqrt{3}\,(2\xi)^2} = \frac{\Phi_0}{2\sqrt{3}\,\xi^2} \approx 24\ \text{T}. \tag{6}$$

(3) As shown in figure 3, when the current I flows in the superconducting tape, the flux vortex lines move, under the drive of the Ampère force, with the speed v to the right (perpendicular to both $\boldsymbol{B}_a$ and the current). This is equivalent to a superconductor that moves with speed v to the left and cuts the field lines, so a potential difference appears between the two ends of the tape:

$$U = B_a v d. \tag{7}$$

The driving force per unit volume of vortex line comes from the current density $I/(bc)$, so with the given law,

$$v = \frac{f_A \Phi_0}{\eta B_a} = \frac{1}{\eta} \cdot \frac{I d\, \Phi_0}{bcd}. \tag{8}$$

The resistance produced by the flux flow is

$$R = \frac{U}{I} = \frac{B_a v d}{I} = \frac{\Phi_0 B_a I d^2}{\eta I(bcd)} = \frac{\Phi_0 B_a d^2}{\eta(bcd)}. \tag{9}$$

By the relation between the resistance and the resistivity ρ,

$$R = \rho \frac{d}{bc}, \tag{10}$$

we get the flux-flow resistivity

$$\rho = \frac{\Phi_0 B_a bcd}{\eta(bcd)} = \frac{\Phi_0 B_a}{\eta}. \tag{11}$$

(4) Defects — vacancies, dislocations, interstitial atoms, and so on — are introduced into the superconductor. When a flux vortex line falls into such a defect, a flux-pinning effect appears, and the vortex lines cannot flow easily; this is the non-ideal type-II superconductor.

Problem 7 (rod with two balls thrown in the wind)

Solution

(1) Let $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ be the velocities of A and B. The components of the air resistance on A and on B are

$$\begin{cases} f_{Ax} = -k(v_{Ax} - u) \\ f_{Ay} = -k v_{Ay} \end{cases} \qquad \begin{cases} f_{Bx} = -k(v_{Bx} - u) \\ f_{By} = -k v_{By} \end{cases}$$

The forces between the rod and the balls are internal forces and have no effect on the total external force on the system. With gravity, the components of the total external force on the system are

$$\begin{cases} F_x = -k(v_{Ax} - u) - k(v_{Bx} - u) = -2k v_x' \\ F_y = -2mg - k v_{Ay} - k v_{By} = -2mg - 2k v_y \end{cases} \tag{1}$$

where $v_x' = v_x - u$ is the velocity of the center of mass C along x relative to the wind, and

$$v_x = \frac{v_{Ax} + v_{Bx}}{2}, \qquad v_y = \frac{v_{Ay} + v_{By}}{2}$$

are the velocity components of the center of mass C along x and along y.

(i) *Ascent: motion of C along y.* By the center-of-mass theorem,

$$F_y = -2mg - 2k v_y = 2m \frac{\Delta v_y}{\Delta t}. \tag{2}$$

Simplification and summation over the ascent give

$$-mg \sum \Delta t - k \sum v_y \Delta t = m \sum \Delta v_y.$$

Let y_1 be the height of the rise of C up to the highest point and T_1 the time needed. Then

$$-mgT_1 - k y_1 = m(0 - v_0). \tag{3}$$

(ii) *Collision.* Let v_{y1} be the velocity of the center of mass C along y just after the collision and u_2 the velocity of the pebble just after the collision. By the momentum conservation of the particle system along y,

$$m_1 u_1 = 2m v_{y1} + m_1 u_2. \tag{4}$$

Let v_{Ay_1} and v_{By_1} be the y velocity components of A and B just after the collision; then

$$v_{y1} = \frac{v_{Ay_1} + v_{By_1}}{2}.$$

Let v_{c0} be the y component of the rotation velocity of A about the center of mass C just after the collision; the y component for B is then $-v_{c0}$. By the relative-motion relations,

$$v_{Ay_1} = v_{y1} + v_{c0}, \qquad v_{By_1} = v_{y1} - v_{c0}.$$

The kinetic energy that the collision gives to the rotation and the y motion of the system is

$$E_{ky} = \frac{1}{2}m(v_{y1} + v_{c0})^2 + \frac{1}{2}m(v_{y1} - v_{c0})^2 = 2 \times \frac{1}{2}mv_{y1}^2 + 2 \times \frac{1}{2}mv_{c0}^2.$$

The x velocities do not change in the collision; call them v_x. The collision is elastic, so

$$\frac{1}{2}m_1 u_1^2 + 2 \times \frac{1}{2}mv_x^2 = \frac{1}{2}(2m)v_{y1}^2 + 2 \times \frac{1}{2}mv_{c0}^2 + 2 \times \frac{1}{2}mv_x^2 + \frac{1}{2}m_1 u_2^2. \tag{5}$$

About the point of an inertial frame that coincides with the center of mass C at the instant of the collision, the angular momentum is conserved:

$$m_1 l u_1 = 2ml v_{c0} + m_1 l u_2. \tag{6}$$

By the relation between the angular velocity and the linear velocity,

$$v_{c0} = l\omega_0.$$

[Or: by the conservation of mechanical energy,

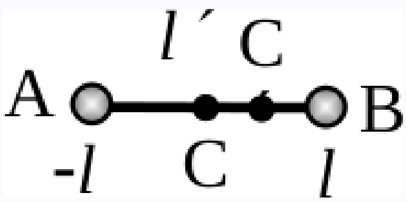
$$\frac{1}{2}m_1 u_1^2 + 2 \times \frac{1}{2}mv_x^2 = \frac{1}{2}m\left(v_{Ay_1}^2 + v_{By_1}^2\right) + 2 \times \frac{1}{2}mv_x^2 + \frac{1}{2}m_1 u_2^2. \tag{5$'$}$$

As in the figure, let C' be the point of the rod that is at rest at the instant just after the collision (the instantaneous center of rotation), at distance l' from A. The angular momentum about C' is conserved:

$$m_1 l' u_1 = m l' v_{Ay_1} + m(2l - l')v_{By_1} + m_1 l' u_2. \tag{6$'$}$$

After the collision the particle system rotates about C':

$$v_{Ay_1} = l'\omega_0, \qquad v_{By_1} = (2l - l')\omega_0, \qquad v_{y1} = (l' - l)\omega_0. \,]$$



Combination of the equations above gives

$$v_{Ay_1} = \frac{2m_1}{m+m_1}u_1, \qquad v_{By_1} = 0, \tag{7}$$

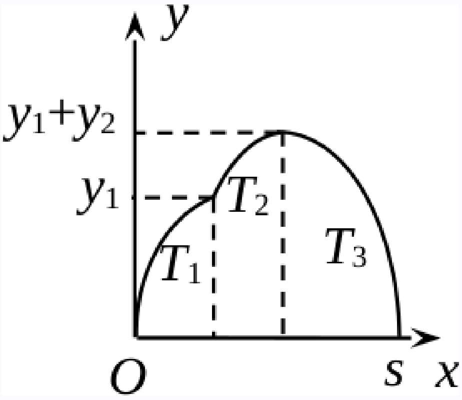
$$v_{y1} = v_{c0} = \frac{m_1}{m+m_1}u_1, \tag{8}$$

$$u_2 = \frac{m_1 - m}{m+m_1}u_1, \tag{9}$$

$$\omega_0 = \frac{m_1}{m+m_1} \cdot \frac{u_1}{l}. \tag{10}$$

(iii) *Second ascent: motion of C along y.* The trajectory of the center of mass C is shown in the figure. After the collision, C makes a second upward flight along y with initial velocity v_{y1}. Let y_2 be the additional height of the rise of C up to the new highest point and T_2 the time needed. In analogy with (3),

$$-mgT_2 - ky_2 = m(0 - v_{y1}). \tag{11}$$



(iv) *Fall: motion of C along y.* After the second highest point, the motion of C is like a horizontal throw. Let T_3 be the duration of this fall. Along y the force on C is

$$F_y = -2mg - 2kv_y. \tag{12}$$

By Newton's second law,

$$-2mg - 2kv_y = 2m\,\frac{\Delta v_y}{\Delta t}. \tag{13}$$

Simplification and summation give

$$-mg\sum \Delta t - k\sum v_y\Delta t = m\sum \Delta v_y.$$

In the fall $\sum v_y \Delta t = -(y_1 + y_2)$, so

$$-mgT_3 + k(y_1 + y_2) = m(-v_{ys} - 0), \tag{14}$$

where v_{ys} is the magnitude of the y velocity of C at the landing (the direction is downward). From (3), (8), (11), (14) and

$$T = T_1 + T_2 + T_3$$

we get

$$v_{ys} = gT - v_0 - \frac{m_1}{m+m_1}u_1. \tag{15}$$

(v) *The whole process: motion of C along x.* The velocity of C along x relative to the wind is $v'_x = v_x - u$. Along x the displacement of C is s (given). Let v_{xs} be the final x velocity. By the center-of-mass theorem,

$$F_x = -2kv'_x = -2k(v_x - u) = 2m\,\frac{\Delta v_x}{\Delta t}. \tag{16}$$

Simplification and summation give

$$-\frac{k}{m}\Big(\sum v_x \Delta t - u\sum \Delta t\Big) = \sum \Delta v_x,$$

and with $\sum v_x\Delta t = s$, $\sum \Delta t = T$, $\sum \Delta v_x = v_{xs} - 0$,

$$v_{xs} = \frac{k}{m}\,(uT - s). \tag{17}$$

[Or: set up the inertial reference frame $O'x'y'$ that moves with the wind; at the start the $O'x'y'$ frame coincides with the Oxy frame. In the whole process, along x' the center of mass C feels only the air resistance; its initial velocity along x' is u, in the negative x' direction. By the center-of-mass theorem and Newton's second law,

$$2kv'_x = 2m\,\frac{\Delta v'_x}{\Delta t}. \tag{16$'$}$$

Simplification and summation give

$$k\sum v'_x\Delta t = m\sum \Delta v'_x, \qquad \text{that is} \qquad ks' = m\big[-v'_{xs} - (-u)\big],$$

where s' and v'_{xs} are the magnitudes of the displacement and of the final velocity of C along x' in the $O'x'y'$ frame at the end of the whole process. With

$$s = uT - s', \qquad v_{xs} = u - v'_{xs},$$

the final velocity of C along x in the Oxy frame is

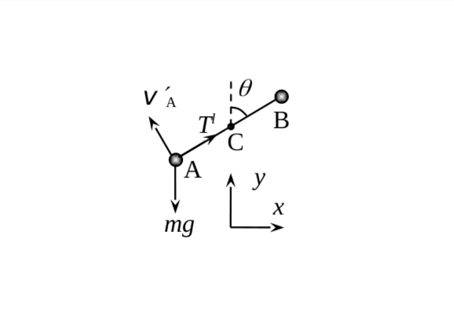
$$v_{xs} = \frac{k}{m}(uT - s).\,] \tag{17$'$}$$

(vi) *Rotation.* As in the figure, the total force and the total torque on the light rod are zero, so the forces from the rod on A and on B are along the rod, equal in magnitude and opposite in direction; write them T_l. Then

$$\begin{cases} F_{Ax} = -k(v_{Ax} - u) + T_{lx} = m\,\dfrac{\Delta v_{Ax}}{\Delta t} \\ F_{Ay} = -mg - kv_{Ay} + T_{ly} = m\,\dfrac{\Delta v_{Ay}}{\Delta t} \end{cases} \qquad \begin{cases} F_{Bx} = -k(v_{Bx} - u) - T_{lx} = m\,\dfrac{\Delta v_{Bx}}{\Delta t} \\ F_{By} = -mg - kv_{By} - T_{ly} = m\,\dfrac{\Delta v_{By}}{\Delta t} \end{cases} \tag{18}$$

By the center-of-mass theorem,

$$\begin{cases} F_x = -2kv_x + 2ku = 2m\,\dfrac{\Delta v_x}{\Delta t} \\ F_y = -2mg - 2kv_y = 2m\,\dfrac{\Delta v_y}{\Delta t} \end{cases} \tag{19}$$



The velocity of A relative to the center of mass is

$$\begin{cases} v'_{Ax} = v_{Ax} - v_x \\ v'_{Ay} = v_{Ay} - v_y \end{cases}$$

From (18) and (19),

$$\begin{cases} \dfrac{\Delta v'_{Ax}}{\Delta t} = \dfrac{\Delta v_{Ax}}{\Delta t} - \dfrac{\Delta v_x}{\Delta t} = -\dfrac{k}{m}\big(v_{Ax} - v_x\big) + \dfrac{T_{lx}}{m} = -\dfrac{k}{m} v'_{Ax} + \dfrac{T_{lx}}{m} \\ \dfrac{\Delta v'_{Ay}}{\Delta t} = \dfrac{\Delta v_{Ay}}{\Delta t} - \dfrac{\Delta v_y}{\Delta t} = -\dfrac{k}{m}\big(v_{Ay} - v_y\big) + \dfrac{T_{ly}}{m} = -\dfrac{k}{m} v'_{Ay} + \dfrac{T_{ly}}{m} \end{cases}$$

A moves in a circle about the center of mass C, so the direction of $\boldsymbol{v}'_A$ is perpendicular to the rod, as in the figure. With the angle θ of the figure, the equations above can be rewritten as

$$\begin{cases} \dfrac{\Delta v'_{Ax}}{\Delta t} = -\dfrac{k}{m}\, v'_A \cos\theta + \dfrac{T_l \sin\theta}{m} \\ \dfrac{\Delta v'_{Ay}}{\Delta t} = -\dfrac{k}{m}\, v'_A \sin\theta + \dfrac{T_l \cos\theta}{m} \end{cases}$$

From these, the tangential acceleration of the circular motion of A about the center of mass C (the projection on the direction of $\boldsymbol{v}'_A$, in which the rod force drops out) is

$$\frac{\Delta v'_A}{\Delta t} = -\frac{k}{m}\, v'_A. \tag{20}$$

With $v'_A = \omega l$ and the tangential acceleration $\dfrac{\Delta v'_A}{\Delta t} = \dfrac{\Delta(\omega l)}{\Delta t} = l\,\dfrac{\Delta \omega}{\Delta t}$, equation (20) becomes

$$\frac{\Delta \omega}{\Delta t} = -\frac{k}{m}\,\omega.$$

Simplification and summation over the whole process give

$$\sum \Delta\omega = -\frac{k}{m} \sum \omega\,\Delta t.$$

By the problem, $\sum \omega\,\Delta t = 2n\pi$, so

$$\omega_s - \omega_0 = -\frac{k}{m}\, 2n\pi.$$

Substitution of ω_0 from (10) gives

$$\omega_s = \omega_0 - \frac{2kn\pi}{m} = \frac{m_1}{m + m_1}\,\frac{u_1}{l} - \frac{2kn\pi}{m}.$$

(2) When the point C falls back to the position s, the square of the speed of the center of mass is

$$v_s^2 = v_{xs}^2 + v_{ys}^2.$$

In the center-of-mass frame, the rotation velocities $\boldsymbol{v}_{Ac}$ and $\boldsymbol{v}_{Bc}$ of A and of B about C are always equal in magnitude and opposite in direction, $\boldsymbol{v}_{Ac} = -\boldsymbol{v}_{Bc}$. When C falls back to the position s, the kinetic energy of the system is

$$E_{ks} = \frac{1}{2}m(\boldsymbol{v}_s + \boldsymbol{v}_{Ac})^2 + \frac{1}{2}m(\boldsymbol{v}_s + \boldsymbol{v}_{Bc})^2 = 2\times\frac{1}{2}mv_s^2 + \frac{1}{2}mv_{Ac}^2 + \frac{1}{2}mv_{Bc}^2,$$

where

$$v_{Ac} = \omega_s l, \qquad v_{Bc} = \omega_s l$$

are the speeds of A and of B in the center-of-mass frame at that moment. Therefore

$$E_{ks} = \frac{1}{2}(2m)\left(v_{xs}^2 + v_{ys}^2\right) + \frac{1}{2}(2m)(\omega_s l)^2.$$

The work done by the pebble on the particle system in the collision equals the kinetic energy the collision contributes; by (7),

$$E_{km} = \frac{1}{2}mv_{Ay_1}^2 + \frac{1}{2}mv_{By_1}^2 = 2m\left(\frac{m_1}{m+m_1}u_1\right)^2.$$

The initial kinetic energy of the throw is $E_{k0} = \frac{1}{2}(2m)v_0^2 = mv_0^2$. By the work–energy theorem, with (15) and (17), the work done by the air resistance in the whole process is

$$\begin{aligned} W_f &= E_{ks} - E_{km} - E_{k0} \\ &= m\left(gT - v_0 - \frac{m_1}{m+m_1}u_1\right)^2 + \frac{k^2}{m}\left(uT - s\right)^2 + m\left(\frac{m_1}{m+m_1}u_1 - \frac{2kn\pi l}{m}\right)^2 \\ &\quad - mv_0^2 - 2m\left(\frac{m_1}{m+m_1}u_1\right)^2. \end{aligned}$$

Problem 8 (nuclear fusion in the Sun)

Solution

(1) In their thermal motion the protons, the deuterons, the helium-3 and helium-4 nuclei and the electrons behave like a monatomic ideal gas; rotational degrees of freedom need not be considered. The mean kinetic energy of the thermal motion of each kind of particle is therefore

$$E_k = \frac{3}{2}kT = \frac{1.5\times1.381\times10^{-23}\times1.5\times10^7}{1.602\times10^{-19}}\ \text{keV} \approx 2\ \text{keV}. \tag{1}$$

The system consists mainly of protons and electrons; the molar mass of the protons is 1 g. From the given density, the molar volume of the protons is

$$V_{\text{mol}} = \frac{1}{1.6\times10^8}\ \text{m}^3, \tag{2}$$

so the pressure in the core region is

$$P = 2\times\frac{RT}{V_{\text{mol}}} = 2\times1.6\times10^8\times1.5\times10^7\times8.31\ \text{N/m}^2 = 2\times2\times10^{16}\ \text{N/m}^2 = 4\times10^{11}\ \text{atm}. \tag{3}$$

The factor 2 appears because the contribution of the electrons of the plasma is included.
(2) By the conservation of charge in the reaction, the charge of the particle x is

$$Q_x = 0, \tag{4}$$

so the particle is neutral. By part (1), the mean kinetic energies of the thermal motion of the deuteron and the proton are only about 0.002 MeV; within the error range their kinetic energies before the reaction are negligible, so the deuteron and the proton can be treated as at rest. Let m_x be the mass of the particle x and p_x its momentum. Energy and momentum conservation give

$$m_{\mathrm{D}}c^2 + m_{\mathrm{H}}c^2 = \sqrt{m_{\mathrm{He3}}^2 c^4 + p_{\mathrm{He3}}^2 c^2} + \sqrt{m_x^2 c^4 + p_x^2 c^2}, \tag{5}$$

$$p_{\mathrm{He3}} = p_x. \tag{6}$$

From (5),

$$p_x c < m_{\mathrm{D}}c^2 + m_{\mathrm{H}}c^2 - m_{\mathrm{He3}}c^2 = 5.5 \text{ MeV}, \tag{7}$$

so $p_x c \ll m_{\mathrm{He3}}c^2$. With (6), equation (5) can be rewritten as

$$m_{\mathrm{D}}c^2 + m_{\mathrm{H}}c^2 - m_{\mathrm{He3}}c^2 = \frac{p_x^2}{2m_{\mathrm{He3}}} + \sqrt{m_x^2 c^4 + p_x^2 c^2} \approx \sqrt{m_x^2 c^4 + p_x^2 c^2}, \tag{8}$$

that is, the kinetic energy of the helium-3 nucleus is $E_{k\,\mathrm{He3}} \approx 0$ and is negligible. The total energy of the particle x (rest energy included) is 5.5 MeV, so its rest mass satisfies $m_x < 5.5 \text{ MeV}/c^2$. Among the choices, such a light neutral particle can only be the γ photon, with $m_x = 0$; its kinetic energy and momentum are

$$E_{k\,x} = 5.5 \text{ MeV}, \qquad p_x = 5.5 \text{ MeV}/c. \tag{9}$$

A photon moves with the speed of light in every reference frame, so no reference frame exists in which the kinetic energy of the particle x is zero.
(3) By the energy conservation in reaction (I), the total kinetic energy of the reaction products is

$$E_k = 2m_{\mathrm{H}}c^2 - m_{\mathrm{D}}c^2 - m_e c^2 - m_\nu c^2 = 0.42 \text{ MeV}, \tag{10}$$

which is much smaller than the rest energy of the deuteron. By the method of part (2), the kinetic energy of the deuteron is negligible:

$$E_{k\,\mathrm{D}} < 0.01 \text{ MeV} \approx 0. \tag{11}$$

The momentum of the deuteron need not be zero, but $\boldsymbol{p}_{\mathrm{D}} + \boldsymbol{p}_e + \boldsymbol{p}_\nu = 0$ must hold, so the kinetic energies of the positron and of the neutrino cannot be fixed separately; only the relations

$$E_{k\,e} + E_{k\,\nu} = 0.42 \text{ MeV}, \qquad 0 < E_{k\,e} < 0.42 \text{ MeV}, \qquad 0 < E_{k\,\nu} < 0.42 \text{ MeV} \tag{12}$$

hold.
(4) An electron–positron pair annihilates into two photons of equal energies and of equal, opposite momenta:

$$h\nu_0 = h\frac{c}{\lambda_0} = mc^2 = 0.51 \text{ MeV},$$

$$\lambda_0 = \frac{hc}{mc^2} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{0.51 \times 1.6 \times 10^{-13}} \text{ m} = 2.43 \times 10^{-12} \text{ m}. \tag{13}$$

By the analysis above, a collision between a γ photon and an ion is almost perfectly elastic — the photon energy is much smaller than the rest energy of an ion, there is no energy loss, and the photon wavelength does not change. For the change of the photon wavelength, only the scattering of the photon on the electrons matters. When a photon of energy $E_0 = h\nu_0$ makes a Compton scattering on an approximately resting electron, an electron with momentum P_e and kinetic energy T and a photon with energy $E = h\nu$ are produced. Energy and momentum conservation give

$$h\nu_0 = h\nu + T, \tag{14}$$

$$\frac{h\nu_0}{c} = \frac{h\nu}{c}\cos\phi + P_e\cos\theta, \tag{15}$$

$$0 = -\frac{h\nu}{c}\sin\phi + P_e\sin\theta. \tag{16}$$

Elimination of θ from (15) and (16) gives

$$P_e^2 = \frac{h^2}{c^2}\big(\nu_0^2 + \nu^2 - 2\nu_0\nu\cos\phi\big). \tag{17}$$

By the relativistic energy–momentum relation

$$P_e^2c^2 = (mc^2+T)^2 - m^2c^4 = T^2 + 2Tmc^2$$

and (14),

$$(E_0-E)^2 + 2(E_0-E)mc^2 = E_0^2 + E^2 - 2E_0E\cos\phi.$$

Rearrangement gives

$$E = \frac{E_0}{1+\dfrac{E_0}{mc^2}(1-\cos\phi)}, \tag{18}$$

which can be simplified to

$$\lambda - \lambda_0 = \frac{h}{mc}(1-\cos\phi). \tag{19}$$

The wavelength after the last Compton scattering is $\lambda_f = 5.4\times10^{-7}$ m. Let ϕ be the (equal) scattering angle of one Compton scattering; then

$$\lambda_f - \lambda_0 = \frac{h\times10^{26}}{mc}\big(1-\cos\phi\big). \tag{20}$$

Assume ϕ is very small; then

$$1-\cos\phi \approx \frac{\phi^2}{2} = \frac{\lambda_f\, mc}{h\times10^{26}},$$

so

$$\phi = \sqrt{\frac{2\lambda_f\, mc}{h\times10^{26}}} = 6.7\times10^{-11}\ \text{rad}.$$

The assumption of a small ϕ is therefore consistent.