The 32nd Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2015 — official reference solutions

Equation numbers (1), (2), ... in each problem follow the circled numbers of the Chinese original.

Problem 1 (springs and rod)

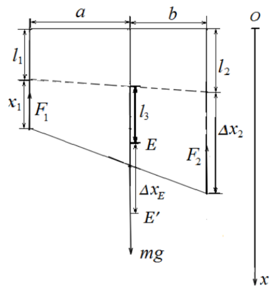
Solution

(1) Set up the coordinate axis Ox vertically, as in the figure (the origin O at the lower edge of the ceiling, direction vertically down). Write l_1, l_2 and l_3 for the natural lengths of the springs A, B and C; without loss of generality take $l_2 > l_1$. Let the extensions of the springs A, B and C relative to their natural lengths, in the equilibrium position of the system, be Δx_1, Δx_2 and Δx_3. The block D is in equilibrium, so

$$mg = k_3 \Delta x_3, \tag{1}$$

that is,

$$\Delta x_3 = \frac{mg}{k_3}. \tag{2}$$



Let F_1 and F_2 be the pulling forces of the ends of the springs A and B on the light rod. With the junction of the spring C and the rod as the pivot, the torque balance gives

$$F_1 a = F_2 b. \tag{3}$$

The resultant vertical force on the light rod is zero:

$$F_1 + F_2 - mg = 0. \tag{4}$$

By Hooke's law,

$$F_1 = k_1 \Delta x_1, \tag{5}$$

$$F_2 = k_2 \Delta x_2. \tag{6}$$

Equations (3), (4), (5), (6) give

$$\Delta x_1 = \frac{mgb}{(a+b)k_1}, \tag{7}$$

$$\Delta x_2 = \frac{mga}{(a+b)k_2}. \tag{8}$$

(2) Before the block D is hung on, the coordinate x_E of the end point E of the spring C in the equilibrium of the system is

$$x_E = l_1 + l_3 + \frac{a(l_2 - l_1)}{a+b}. \tag{9}$$

After the block D is hung on and the system is in its new equilibrium position, the end of the spring C has moved from E to E'. The coordinate $x_{E'}$ of E' is

$$x_{E'} = l_1 + \Delta x_1 + l_3 + \Delta x_3 + \frac{a\,(l_2 - l_1 + \Delta x_2 - \Delta x_1)}{a+b}. \tag{10}$$

Treat the system as one equivalent spring; its extension in the static equilibrium is

$$\Delta x_E = x_{E'} - x_E = \Delta x_1 + \Delta x_3 + \frac{a(\Delta x_2 - \Delta x_1)}{a+b}. \tag{11}$$

Substitution of (2), (7), (8) into (11) gives

$$\Delta x_E = \frac{mg\,(a^2 k_1 + b^2 k_2)}{(a+b)^2 k_1 k_2} + \frac{mg}{k_3}. \tag{12}$$

The equivalent stiffness of the equivalent spring is therefore

$$k_e = \frac{mg}{\Delta x_E} = \frac{(a+b)^2 k_1 k_2 k_3}{a^2 k_1 k_3 + b^2 k_2 k_3 + (a+b)^2 k_1 k_2}. \tag{13}$$

The natural (angular) frequency of the small vertical oscillation of the block D is

$$\omega_e = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{(a+b)^2 k_1 k_2 k_3}{m\big[a^2 k_1 k_3 + b^2 k_2 k_3 + (a+b)^2 k_1 k_2\big]}}. \tag{14}$$

(3) The largest natural frequency of the block D satisfies

$$\frac{\mathrm{d}\omega_e}{\mathrm{d}a} = 0. \tag{15}$$

Substitution of (14) into (15) gives

$$a = \frac{k_2}{k_1}\, b. \tag{16}$$

At this point

$$\frac{\mathrm{d}^2\omega_e}{\mathrm{d}a^2} = -\frac{k_1^3}{k_2 b^2 \sqrt{m}} \left(\frac{k_3}{(k_1+k_2)(k_1+k_2+k_3)}\right)^{3/2} < 0, \tag{17}$$

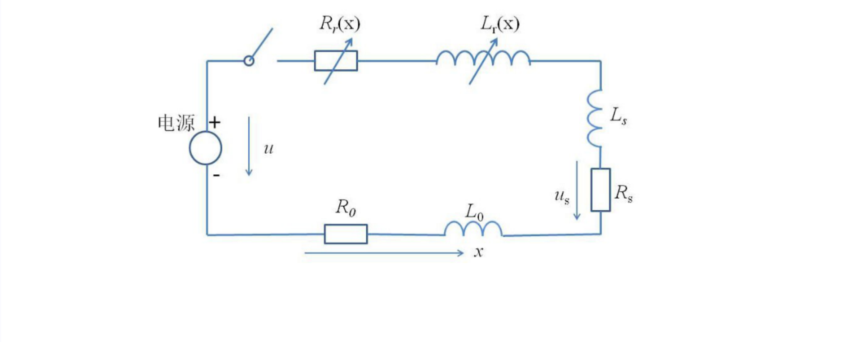
so (16) is the condition for the natural frequency of the block D to take its largest value; the largest natural frequency is

$$\omega_e = \sqrt{\frac{(k_1+k_2)\,k_3}{(k_1+k_2+k_3)\,m}}. \tag{18}$$

Problem 2 (rail-type electromagnetic launcher)

Solution

(1) The equivalent circuit of the rail-type electromagnetic launcher is shown in the figure.



Label: 电源 = *source.*

Take the line of the rails as the x axis, the direction of the motion of the armature as the positive x direction, and the source end of the rails as the origin. Let the total inductance of the loop be $L(x)$ and the total resistance be $R(x)$; then

$$L(x) = L_0 + L'_r x + L_s, \tag{1}$$

$$R(x) = R_0 + R'_r x + R_s. \tag{2}$$

With the equivalent circuit, the loop equation (Kirchhoff's law) is

$$u = \frac{\mathrm{d}}{\mathrm{d}t}\big[L(x)\, i\big] + R(x)\, i = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + iL'_r v_s + iR(x), \tag{3}$$

where u is the voltage at the source end, i is the loop current, and v_s is the speed of the projectile.

(2) The output power of the source is

$$p = iu = L(x)\, i\,\frac{\mathrm{d}i}{\mathrm{d}t} + i^2 L'_r v_s + i^2 R(x). \tag{4}$$

Let $\mathrm{d}E$ be the electric energy that goes into the launcher system in the time $\mathrm{d}t$. Energy conservation gives

$$\mathrm{d}E = \mathrm{d}E_m + \mathrm{d}E_k + \mathrm{d}E_R = \mathrm{d}\Big[\frac{1}{2}L(x)i^2\Big] + \mathrm{d}\Big[\frac{1}{2}(m_s + m_a)v_s^2\Big] + i^2 R(x)\,\mathrm{d}t, \tag{5}$$

where $\mathrm{d}E_m$ and $\mathrm{d}E_k$ are the changes of the magnetic energy of the loop and of the total kinetic energy of the armature and the projectile, and $\mathrm{d}E_R$ is the heat loss of the loop. From (5),

$$\frac{\mathrm{d}E}{\mathrm{d}t} = L(x)i\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{2}L'_r i^2 v_s + (m_s + m_a)v_s\frac{\mathrm{d}v_s}{\mathrm{d}t} + i^2 R(x). \tag{6}$$

With $p = \mathrm{d}E/\mathrm{d}t$ and (4), (6),

$$L(x)i\frac{\mathrm{d}i}{\mathrm{d}t} + i^2 L'_r v_s + i^2 R(x) = L(x)i\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{2}L'_r i^2 v_s + (m_s + m_a)v_s\frac{\mathrm{d}v_s}{\mathrm{d}t} + i^2 R(x). \tag{7}$$

From (7), the relation between the acceleration of the projectile and the loop current is

$$a_s = \frac{\mathrm{d}v_s}{\mathrm{d}t} = \frac{1}{2(m_s + m_a)}\, L'_r\, i^2. \tag{8}$$

(3) For a constant current I, equation (8) gives the acceleration of the armature and the projectile at the time t:

$$a_s = \frac{\mathrm{d}v_s}{\mathrm{d}t} = \frac{1}{2(m_a + m_s)}\, L_r' I^2. \tag{9}$$

The armature and the projectile therefore move along the rails with constant acceleration, from rest. So that the exit speed v_{sm} reaches 3.0×10^3 m/s, the given data lead to

$$a_{sm} = \frac{v_{sm}^2}{2x_{sm}} = \frac{9.0 \times 10^9}{2 \times 5.0}\ \mathrm{m/s^2} = 9.0 \times 10^8\ \mathrm{m/s^2}. \tag{10}$$

From (9) and (10),

$$I = \sqrt{\frac{2(m_s + m_a)\, a_{sm}}{L_r'}} = \sqrt{\frac{2 \times 0.50 \times 9.0 \times 10^6}{1.0 \times 10^{-6}}}\ \mathrm{A} = 3.0 \times 10^6\ \mathrm{A}. \tag{11}$$

The acceleration time is

$$\tau = \frac{v_{sm}}{a_{sm}} = \frac{3.0 \times 10^3}{9.0 \times 10^6}\ \mathrm{s} = 3.3 \times 10^{-4}\ \mathrm{s}. \tag{12}$$

[a]

[a] *Translator's note:* The numbers in (10)–(12) are transcribed exactly from the source document, which is not self-consistent here. For the given data ($v_{sm} = 3.0 \times 10^3$ m/s, $x_{sm} = 500$ m) the consistent values are $a_{sm} = v_{sm}^2/(2x_{sm}) = 9.0 \times 10^3$ m/s^2, $I = \sqrt{2 \times 0.50 \times 9.0 \times 10^3/10^{-6}} \approx 9.5 \times 10^4$ A and $\tau = v_{sm}/a_{sm} \approx 0.33$ s. The values printed in the source correspond to a rail length $x_{sm} = 0.50$ m.

Problem 3 (ideal rocket equation)

Solution

(1) *The velocity–mass-ratio relation from Newtonian mechanics.* By momentum conservation, the change of the momentum of the rocket equals the momentum change caused by the expelled gas of mass dm:

$$\mathrm{d}(MV_1) = V_2\, \mathrm{d}m. \tag{1}$$

By the conservation of mass,

$$\mathrm{d}M + \mathrm{d}m = 0. \tag{2}$$

By the addition of velocities,

$$V_2 = V_e - V_1. \tag{3}$$

Substitution of (3) into (1) gives

$$\mathrm{d}(MV_1) = (V_e - V_1)\, \mathrm{d}m. \tag{4}$$

Substitution of (2) into (4) gives

$$\frac{\mathrm{d}M}{M} = -\frac{\mathrm{d}V_1}{V_e}. \tag{5}$$

Integration of both sides gives

$$\int_{M_i}^{M} \frac{\mathrm{d}M}{M} = -\frac{1}{V_e} \int_0^{V_1} \mathrm{d}V_1. \tag{6}$$

With the given integral formula and the initial conditions,

$$\ln \frac{M}{M_i} = -\frac{V_1}{V_e}, \tag{7}$$

or

$$V_1 = -V_e \ln \frac{M}{M_i}. \tag{8}$$

(2) *The velocity–mass-ratio relation from special relativity.* By the conservation of relativistic momentum, the change of the momentum of the rocket equals the momentum change caused by the expelled gas of mass $\mathrm{d}m$:

$$\mathrm{d}\Big(\frac{MV_1}{\sqrt{1-V_1^2/c^2}}\Big) = V_2\, \frac{\mathrm{d}m}{\sqrt{1-V_2^2/c^2}}. \tag{9}$$

By the conservation of relativistic energy, the decrease of the energy of the rocket equals the energy of the expelled gas of mass $\mathrm{d}m$:

$$-\mathrm{d}\Big(\frac{Mc^2}{\sqrt{1-V_1^2/c^2}}\Big) = \frac{\mathrm{d}m\, c^2}{\sqrt{1-V_2^2/c^2}}. \tag{10}$$

By the relativistic addition of velocities,

$$V_2 = \frac{V_e - V_1}{1 - V_1 V_e/c^2}. \tag{11}$$

Substitution of (11) into (9), with (10), gives

$$\mathrm{d}\Big(\frac{MV_1}{\sqrt{1-V_1^2/c^2}}\Big) = \frac{V_1 - V_e}{1 - V_1 V_e/c^2}\, \mathrm{d}\Big(\frac{M}{\sqrt{1-V_1^2/c^2}}\Big), \tag{12}$$

which reduces to

$$\frac{\mathrm{d}M}{M} = -\frac{\mathrm{d}V_1}{V_e\,(1 - V_1^2/c^2)}. \tag{13}$$

Integrate both sides of (13) with the identity

$$\frac{1}{1-x^2} = \frac{1}{2}\Big(\frac{1}{1-x} + \frac{1}{1+x}\Big):$$

$$\int_{M_i}^{M} \frac{\mathrm{d}M}{M} = -\frac{1}{2V_e}\left(\int_0^{V_1} \frac{\mathrm{d}V_1}{1-V_1/c} + \int_0^{V_1} \frac{\mathrm{d}V_1}{1+V_1/c}\right). \tag{14}$$

With the given integral formula and the initial conditions,

$$\ln \frac{M}{M_i} = -\frac{c}{2V_e}\, \ln\left(\frac{1+V_1/c}{1-V_1/c}\right), \tag{15}$$

or

$$\frac{M}{M_i} = \left(\frac{1-V_1/c}{1+V_1/c}\right)^{\frac{c}{2V_e}}, \tag{16}$$

or

$$V_1 = \frac{1 - \big(M/M_i\big)^{\frac{2V_e}{c}}}{1 + \big(M/M_i\big)^{\frac{2V_e}{c}}}\, c. \tag{17}$$

(3) When the rocket speed V_1 is very small, the formula

$$\ln(1+x) \approx x - \frac{x^2}{2}, \qquad |x| \ll 1$$

gives

$$\ln\Big(1+\frac{V_1}{c}\Big) \approx \frac{V_1}{c} - \frac{1}{2}\Big(\frac{V_1}{c}\Big)^2 \quad \text{and} \quad \ln\Big(1-\frac{V_1}{c}\Big) \approx -\frac{V_1}{c} - \frac{1}{2}\Big(\frac{V_1}{c}\Big)^2. \tag{18}$$

Substitution of (18) into the relativistic result (15), with (8), gives

$$\Big(\ln\frac{M}{M_i}\Big)_{\text{relativistic}}(V_1) - \Big(\ln\frac{M}{M_i}\Big)_{\text{Newtonian}}(V_1) = -\frac{c}{2V_e}\ln\Big(\frac{1+V_1/c}{1-V_1/c}\Big) - \Big(-\frac{V_1}{V_e}\Big) = O\Big(\Big(\frac{V_1}{c}\Big)^3\Big). \tag{19}$$

So, for a very small rocket speed V_1, the difference between the result of relativistic mechanics and the result of Newtonian mechanics is zero up to the order $(V_1/c)^2$; the difference is in fact of the order $(V_1/c)^3$.

Problem 4 (radiation pressure on a prism)

Solution

(1)(a) First calculate the effect of one photon on the body. Let the frequency of the light be ν; the energy and the magnitude of the momentum of one photon are

$$E_0 = h\nu, \tag{1}$$

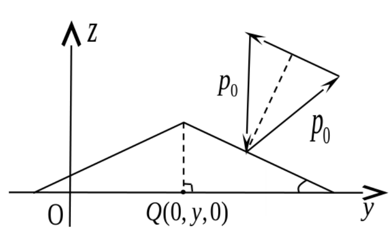
$$p_0 = \frac{E_0}{c} = \frac{h\nu}{c}, \tag{2}$$

where h is the Planck constant.

Now analyse the change of the momentum of a photon in the reflection. As shown in the yz section (figure), a photon falls along the negative z direction onto the face B of the body; the angles of incidence and of reflection both equal the base angle θ of the isosceles triangle, and the direction of the outgoing photon makes the angle 2θ with the z axis. By assumption, the magnitude of the photon momentum is essentially unchanged in the reflection. The x component of the impulse that a photon gives to the body is zero; the z component cancels against the impulse of the supporting force of the xy plane; only the y component is not zero. The xy plane is smooth, so the body moves only along the horizontal y axis. The motion of the whole body can be described by the time dependence of the y coordinate of $Q(0, y, 0)$, the foot of the vertical from the apex of the isosceles triangle on the y axis. The magnitude of the y component of the impulse that a single photon gives to the body at the face B is

$$|I_{0y}| = \frac{h\nu}{c}\sin 2\theta, \tag{3}$$

directed toward the coordinate origin O.



Next calculate the number of photons that fall on the inclined face B in unit time when the point $Q(0,y,0)$ is on the positive half of the y axis ($y \geq 0$). The laser beam that falls on the face B of the body has the cross-section area

$$s = yd, \tag{4}$$

where d is the edge length of the prism. The incident laser power that the body receives is then

$$w = Is = Iyd. \tag{5}$$

The number of photons that fall on the face B in unit time is

$$n = \frac{w}{E_0} = \frac{Iyd}{h\nu}. \tag{6}$$

In the same way, when the point $Q(0,y,0)$ is on the negative half of the y axis ($y \leq 0$), the number of photons that fall on the face A of the body in unit time is

$$n = \frac{w}{E_0} = -\frac{Iyd}{h\nu}. \tag{7}$$

The z component of the force of the light on the body, and the weight of the body, are always balanced by the support of the xy plane. With the result (3), and because the y component of the impulse that a photon gives to the body always points toward the origin, the resultant force F on the body is related to the y coordinate of the point $Q(0,y,0)$ by

$$F = nI_{0y} = -\frac{Id\sin 2\theta}{c}\,y = -ky, \tag{8}$$

where $k = \dfrac{Id\sin 2\theta}{c}$. This is a linear restoring force.
By (8), after the laser field begins to shine on it, the body performs simple harmonic motion along the y axis about the coordinate origin, with the angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{Id\sin 2\theta/c}{\rho L^2 d\tan\theta}} = \sqrt{\frac{2I\cos^2\theta}{c\rho L^2}}, \tag{9}$$

where $m = \rho L^2 d\tan\theta$ is the mass of the body. Only the y coordinate of the point $Q(0,y,0)$ changes with the time t; the dynamical equation of the body is

$$\ddot{y}(t) + \frac{2I\cos^2\theta}{c\rho L^2}\,y(t) = 0. \tag{10}$$

(b) With the initial position $(0,L,0)$ of the point $Q(0,y,0)$, the solution $y(t)$ of the simple-harmonic equation (10) is

$$y(t) = L\cos\left(t\sqrt{\frac{2I\cos^2\theta}{c\rho L^2}}\right). \tag{11}$$

When the body has moved the distance $3L/2$ from its initial position in the negative y direction, $y = -\frac{1}{2}L$; equation (11) gives

$$-\frac{1}{2}L = L\cos\left(t_1\sqrt{\frac{2I\cos^2\theta}{c\rho L^2}}\right),$$

where t_1 is the time the body needs to move the distance $3L/2$ from the initial position in the negative y direction. This gives

$$t_1 = \frac{2\pi L}{3\cos\theta}\sqrt{\frac{c\rho}{2I}}. \tag{12}$$

(2) In the process in which the body moves from the initial position of the figure to the distance L in the negative y direction, the y component of the velocity of the reflected photons is opposite to the direction of the motion of the body; by the Doppler effect, the frequency of the reflected photons is smaller than the frequency of the incident photons, and as the time grows the frequency of the reflected photons becomes smaller and smaller (the difference to the frequency of the incident photons becomes larger and larger).
While the body moves on in the negative y direction, from the distance L to the distance $3L/2$ from the initial position, the y component of the velocity of the reflected photons has the same direction as the motion of the body; by the Doppler effect, the frequency of the reflected photons is larger than the frequency of the incident photons, and as the time grows the frequency of the reflected photons becomes smaller and smaller (the difference to the frequency of the incident photons becomes smaller and smaller).

Problem 5 (capillary rise between two cylinders)

Solution

(1) The projection along the wall direction of the surface tension that acts on the contact lines between the water surface and the two glass walls has the magnitude

$$F_s = 2\pi R_1\alpha\cos\theta + 2\pi R_2\alpha\cos\theta. \tag{1}$$

This force does not change in magnitude, and its direction is vertical (downward on the liquid at the contact line). When the water between the two cylinders has risen, by capillary action, to the height h relative to the water surface of the large vessel, the work done by the force F_s is

$$W_s = -F_s h. \tag{2}$$

The change of the potential energy of the system caused by the capillary action is

$$E_s(h) - E_s(0) = W_s, \tag{3}$$

where $E_s(0)$ is the value of the capillary potential energy of the system at $h = 0$. By assumption,

$$E_s(0) = 0. \tag{4}$$

Equations (1)–(4) give

$$E_s(h) = -2\pi\alpha\cos\theta\,(R_1 + R_2)\,h. \tag{5}$$

(2) The change of the gravitational potential energy caused by the rise h of the water surface between the two cylinders is

$$E_G = \frac{1}{2}\rho g\big(\pi R_1^2 - \pi R_2^2\big)h^2. \tag{6}$$

With (5) and (6), the total potential energy of the system connected with the capillary action is

$$E(h) = E_s(h) + E_G(h) = \frac{1}{2}\pi\rho g\big(R_1^2 - R_2^2\big)h^2 - 2\pi\alpha\cos\theta\,(R_1 + R_2)\,h. \tag{7}$$

By the principle of minimum potential energy at equilibrium,

$$\left.\frac{\mathrm{d}E(h)}{\mathrm{d}h}\right|_{h=h_e} = 0,$$

so

$$\rho g h_e\big(R_1^2 - R_2^2\big) = 2R_1\alpha\cos\theta + 2R_2\alpha\cos\theta. \tag{8}$$

At equilibrium, the height of the rise of the water surface between the two cylinders is

$$h_e = \frac{2\alpha\cos\theta}{\rho g\,(R_1 - R_2)}. \tag{9}$$

At equilibrium, equations (5), (6), (9) give

$$E_s(h_e) = -\frac{4\pi\alpha^2\cos^2\theta\,(R_1 + R_2)}{\rho g\,(R_1 - R_2)}, \tag{10}$$

$$E_G(h_e) = \frac{2\pi\alpha^2\cos^2\theta\,(R_1 + R_2)}{\rho g\,(R_1 - R_2)}. \tag{11}$$

Of the decrease of the capillary potential energy, one part changes into the gravitational potential energy of the water; the other part is used as work against the friction of the system, and this work changes into heat. The heat released by the system up to the equilibrium is therefore

$$Q = \big[-E_s(h_e)\big] - E_G(h_e), \tag{12}$$

and (10), (11), (12) give

$$Q = \frac{2\pi\alpha^2\cos^2\theta\,(R_1 + R_2)}{\rho g\,(R_1 - R_2)}. \tag{13}$$

(3) Let the airplane be at the height z above the ground. From

$$mg_z = \frac{GMm}{(R_\oplus + z)^2}, \qquad g = \frac{GM}{R_\oplus^2}, \tag{14}$$

where $R_\oplus$ is the radius of the Earth, G the gravitational constant and M the mass of the Earth, the gravitational acceleration changes with the height above the ground as

$$g_z = \frac{g}{(1 + z/R_\oplus)^2}. \tag{15}$$

Substitution of (15) into (13) gives the heat released at different heights:

$$Q = \frac{2\pi\alpha^2\cos^2\theta\,(R_1 + R_2)}{\rho g\,(R_1 - R_2)}\left(1 + \frac{z}{R_\oplus}\right)^2. \tag{16}$$

The height z of the airplane above the ground is

$$z = R_\oplus\left(\sqrt{\frac{Q\rho g\,(R_1 - R_2)}{2\pi\alpha^2\cos^2\theta\,(R_1 + R_2)}} - 1\right). \tag{17}$$

Problem 6 (apparent motion of Mars)

Solution

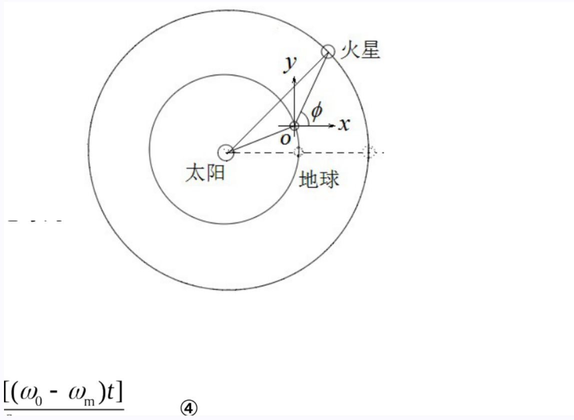
Solution 1. Choose the moment at which the Earth, Mars and the Sun are on one line (an opposition of Mars) as the zero of the time, as in the figure; the counterclockwise direction is the positive direction of the angle ϕ. After the time t, the position of Mars in the translating coordinate system o-xy with the origin at the center of the Earth is

$$x = R_\mathrm{m}\cos\omega_\mathrm{m}t - R_0\cos\omega_0 t, \tag{1}$$

$$y = R_\mathrm{m}\sin\omega_\mathrm{m}t - R_0\sin\omega_0 t, \tag{2}$$

where ω_m and ω_0 are the orbital angular velocities of Mars and of the Earth. The angle ϕ between the Earth–Mars line and the x axis of the Earth system o-xy satisfies

$$\tan\phi = \frac{R_\mathrm{m}\sin\omega_\mathrm{m}t - R_0\sin\omega_0 t}{R_\mathrm{m}\cos\omega_\mathrm{m}t - R_0\cos\omega_0 t}. \tag{3}$$



Labels: 太阳 = Sun; 地球 = Earth; 火星 = Mars.

Differentiation of both sides of (3) with respect to the time gives

$$\frac{\dot\phi}{\cos^2\phi} = \frac{\omega_\mathrm{m}R_\mathrm{m}^2 + \omega_0 R_0^2 - (\omega_\mathrm{m}+\omega_0)R_\mathrm{m}R_0\cos[(\omega_0-\omega_\mathrm{m})t]}{\left(R_\mathrm{m}\cos\omega_\mathrm{m}t - R_0\cos\omega_0 t\right)^2}. \tag{4}$$

The prograde and the retrograde apparent motion of Mars correspond to a positive and a negative $\dot\phi$. With $\dot\phi = 0$,

$$\omega_\mathrm{m}R_\mathrm{m}^2 + \omega_0 R_0^2 - (\omega_\mathrm{m}+\omega_0)R_\mathrm{m}R_0\cos[(\omega_0-\omega_\mathrm{m})t] = 0, \tag{5}$$

which gives

$$t = \pm\frac{1}{\omega_0-\omega_\mathrm{m}}\arccos\frac{\omega_\mathrm{m}R_\mathrm{m}^2+\omega_0R_0^2}{(\omega_\mathrm{m}+\omega_0)R_\mathrm{m}R_0}. \tag{6}$$

Within the time between two neighbouring oppositions, $\dot\phi$ is negative exactly for

$$t \in \left(-\frac{1}{\omega_0-\omega_\mathrm{m}}\arccos\frac{\omega_\mathrm{m}R_\mathrm{m}^2+\omega_0R_0^2}{(\omega_\mathrm{m}+\omega_0)R_\mathrm{m}R_0},\ \frac{1}{\omega_0-\omega_\mathrm{m}}\arccos\frac{\omega_\mathrm{m}R_\mathrm{m}^2+\omega_0R_0^2}{(\omega_\mathrm{m}+\omega_0)R_\mathrm{m}R_0}\right)$$

(at $t = 0$ the numerator of the right side of (4) is smaller than zero). In the course between two neighbouring oppositions, the total time of the retrograde apparent motion of Mars is therefore

$$t_\mathrm{retro} = \frac{2}{\omega_0-\omega_\mathrm{m}}\arccos\frac{\omega_\mathrm{m}R_\mathrm{m}^2+\omega_0R_0^2}{(\omega_\mathrm{m}+\omega_0)R_\mathrm{m}R_0}. \tag{7}$$

By Kepler's third law,

$$\omega_{\mathrm{m}}^2 R_{\mathrm{m}}^3 = \omega_0^2 R_0^3. \tag{8}$$

Substitution of (8) into (7) gives

$$t_{\mathrm{retro}} = \frac{2}{\omega_0 - \omega_{\mathrm{m}}} \arccos \frac{\omega_{\mathrm{m}} R_{\mathrm{m}}^2 + \omega_0 R_0^2}{(\omega_{\mathrm{m}} + \omega_0) R_{\mathrm{m}} R_0} = \frac{T_0}{\pi\,(1 - n^{-3/2})} \arccos \frac{n + n^{1/2}}{n^{3/2} + 1}, \tag{9}$$

where $n = \dfrac{R_{\mathrm{m}}}{R_0}$ and $T_0 = \dfrac{2\pi}{\omega_0}$. The data $n = 1.524$, $T_0 = 365$ d give

$$t_{\mathrm{retro}} = 0.1991\, T_0 = 72.7 \text{ d} \approx 73 \text{ d}. \tag{10}$$

The time T between two neighbouring oppositions of Mars satisfies

$$T = \frac{2\pi}{\omega_0 - \omega_{\mathrm{m}}} = \frac{T_0 T_{\mathrm{m}}}{T_{\mathrm{m}} - T_0} = \frac{T_0}{1 - n^{-3/2}} \approx 779 \text{ d}. \tag{11}$$

The time of the prograde apparent motion of Mars is

$$t_{\mathrm{pro}} = (779 - 73) \text{ d} = 706 \text{ d}. \tag{12}$$

Solution 2. When the direction of the velocity of Mars relative to the Earth,

$$\boldsymbol{v}'_{\mathrm{m}} = \boldsymbol{v}_{\mathrm{m}} - \boldsymbol{v}_E,$$

is parallel to the Earth–Mars line $\overline{\mathrm{EM}}$,

$$\boldsymbol{v}'_{\mathrm{m}} \parallel \overline{\mathrm{EM}}, \tag{1}$$

(that is, when the tangential velocity of Mars relative to the Earth is zero), Mars is at a station. The position of the station is shown in the figure. The position vectors of the Earth and of Mars relative to the Sun satisfy

$$\frac{R_{\mathrm{m}}}{\sin(\theta + \varphi)} = \frac{R_0}{\sin\varphi}. \tag{2}$$

The speeds v_{m} and v_E satisfy

$$\frac{v_{\mathrm{m}}}{\sin\left(\frac{\pi}{2} - \theta - \varphi\right)} = \frac{v_0}{\sin\left(\frac{\pi}{2} + \varphi\right)}. \tag{3}$$

By Kepler's third law,

$$\omega_{\mathrm{m}}^2 R_{\mathrm{m}}^3 = \omega_0^2 R_0^3. \tag{4}$$

Equations (1)–(4) give

$$\cos\theta = \frac{n + n^{1/2}}{n^{3/2} + 1}, \tag{5}$$

where $n = \dfrac{R_{\mathrm{m}}}{R_0}$. The data $n = 1.524$, $T_0 = 365$ d give

$$\theta = \pm 16.79^\circ. \tag{6}$$

Within the time between two neighbouring oppositions, the apparent motion of Mars is retrograde exactly while $\theta \in (-16.79^\circ, 16.79^\circ)$. The time T between two neighbouring oppositions of Mars satisfies

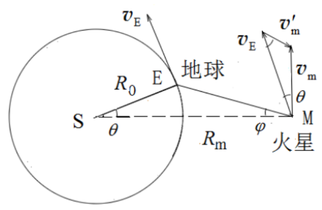
$$T = \frac{2\pi}{\omega_0 - \omega_{\mathrm{m}}} = \frac{T_0 T_{\mathrm{m}}}{T_{\mathrm{m}} - T_0} = \frac{T_0}{1 - n^{-3/2}} \approx 779 \text{ d}, \tag{7}$$

where $T_0 = 365$ d was used. The total time of the retrograde apparent motion of Mars between two neighbouring oppositions is

$$t_{\rm retro} = \frac{2\times 16.79^\circ}{360^\circ} T = 0.09328\, T = 73 \text{ d}. \tag{8}$$

The time of the prograde apparent motion is

$$t_{\rm pro} = T - t_{\rm retro} = (779-73) \text{ d} = 706 \text{ d}. \tag{9}$$



Labels: 地球 = *Earth;* 火星 = *Mars.*

Problem 7 (time-of-flight mass spectrometer)

Solution

(1) In the ion-source region the ion moves with constant acceleration from rest; the magnitude of the acceleration is

$$a_0 = \frac{qE_s}{m}.$$

The flight time of the ion in the ion-source region is

$$t_s = \sqrt{\frac{2S}{a_0}} = \sqrt{\frac{2mS}{qE_s}}. \tag{1}$$

In the drift region the ion moves with constant velocity; the final speed of the acceleration is

$$v_0 = a_0 t_s = \sqrt{\frac{2qE_sS}{m}},$$

so the flight time of the ion in the drift region is

$$t_L = \frac{L}{v_0} = L\sqrt{\frac{m}{2qE_sS}}. \tag{2}$$

The total flight time of the ion is

$$t = t_s + t_L = \sqrt{\frac{m}{q}}\left(\sqrt{\frac{2S}{E_s}} + \frac{L}{\sqrt{2E_sS}}\right). \tag{3}$$

Equation (3) can be written as

$$\frac{m}{q} = \frac{t^2}{k_0^2}, \qquad k_0 = \sqrt{\frac{2S}{E_s}} + \frac{L}{\sqrt{2E_sS}}, \tag{4}$$

so ions with different mass-to-charge ratios m/q have different total flight times t. Differentiation of both sides of (4) gives

$$\frac{1}{q}\,\mathrm{d}m = \frac{2t}{k_0^2}\,\mathrm{d}t \quad \text{or} \quad \mathrm{d}m = \frac{2qt}{k_0^2}\,\mathrm{d}t, \tag{5}$$

so

$$\frac{\mathrm{d}m}{m} = \frac{2qt}{k_0^2}\,\mathrm{d}t \cdot \frac{k_0^2}{qt^2} = \frac{2\,\mathrm{d}t}{t}. \tag{6}$$

The resolution of the detector, in its relation to t and Δt, is therefore

$$R = \frac{m}{\Delta m} = \frac{t}{2\Delta t}. \tag{7}$$

(2) The flight time t_s of an ion in the ion-source region is still given by (1). In the second accelerating region the ion moves with constant acceleration and initial speed v_0; the magnitude of the acceleration is

$$a_1 = \frac{qE_1}{m}.$$

Let t_{d1} be the flight time of the ion in the second accelerating region. The kinematic formula gives

$$\left(v_0 + a_1 t_{d1}\right)^2 - v_0^2 = 2a_1 D_1, \tag{8}$$

which gives

$$t_{d1} = \sqrt{\frac{2m}{q}}\,\frac{1}{E_1}\left(\sqrt{E_sS + E_1D_1} - \sqrt{E_sS}\right). \tag{9}$$

In the same way, the flight time of the ion in the drift region is

$$t_L = \sqrt{\frac{2m}{q}}\,\frac{L}{2}\,\frac{1}{\sqrt{E_sS + E_1D_1}}. \tag{10}$$

The total time up to the arrival at the detector is

$$t = t_s + t_{d1} + t_L.$$

Differentiation of both sides with respect to S gives

$$\frac{\mathrm{d}t}{\mathrm{d}S} = \sqrt{\frac{2m}{q}}\,\frac{1}{2}\left(\frac{1}{\sqrt{E_sS}} + \frac{E_s}{E_1\sqrt{E_sS + E_1D_1}} - \frac{E_s}{E_1\sqrt{E_sS}} - \frac{L}{2}\,\frac{E_s}{\sqrt{(E_sS + E_1D_1)^3}}\right). \tag{11}$$

The effect of $\mathrm{d}S$ on $\mathrm{d}t$ is smallest when $\dfrac{\mathrm{d}t}{\mathrm{d}S} = 0$, that is,

$$\frac{1}{\sqrt{E_sS}} + \frac{E_s}{E_1\sqrt{E_sS + E_1D_1}} - \frac{E_s}{E_1\sqrt{E_sS}} - \frac{L}{2}\,\frac{E_s}{\sqrt{(E_sS + E_1D_1)^3}} = 0. \tag{12}$$

From (12),

$$\begin{aligned} L &= 2\sqrt{(E_sS + E_1D_1)^3}\left(\frac{1}{E_s\sqrt{E_sS}} + \frac{1}{E_1\sqrt{E_sS + E_1D_1}} - \frac{1}{E_1\sqrt{E_sS}}\right) \\ &= 2\left(\sqrt{k^3} - k\left(\sqrt{k} - 1\right)\frac{E_s}{E_1}\right)S, \end{aligned} \tag{13}$$

where $k = \dfrac{E_s S + E_1 D_1}{E_s S}$.

(3) The total time up to the arrival at the detector is

$$t = t_s + t_{d1} + 2t_L + 2t_{d2} + 2t_{d3}, \tag{14}$$

where t_s, t_{d1} and t_L are the one-way flight times of the ion in the ion-source region, in the second accelerating region and in the drift region, given by (1), (9) and (10), and t_{d2} and t_{d3} are the one-way flight times of the ion in the two field regions of the reflector; the factor 2 comes from the symmetry of the back-and-forth flight. In the same way as before,

$$t_{d2} = \sqrt{\frac{2m}{q}}\,\frac{1}{E_2}\left(\sqrt{E_s S + E_1 D_1} - \sqrt{E_s S + E_1 D_1 - E_2 D_2}\right), \tag{15}$$

$$t_{d3} = \sqrt{\frac{2m}{q}}\,\frac{1}{E_3}\sqrt{E_s S + E_1 D_1 - E_2 D_2}. \tag{16}$$

Differentiation of both sides of (14) with respect to S, with (1), (9), (10), (15), (16), gives

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}S} = \sqrt{\frac{2m}{q}}\,\frac{1}{2}\Bigg[&\frac{1}{\sqrt{E_s S}} + \frac{1}{E_1}\left(\frac{E_s}{\sqrt{E_s S + E_1 D_1}} - \frac{E_s}{\sqrt{E_s S}}\right) - L\,\frac{E_s}{\sqrt{(E_s S + E_1 D_1)^3}}\\ &+ \frac{2}{E_2}\left(\frac{E_s}{\sqrt{E_s S + E_1 D_1}} - \frac{E_s}{\sqrt{E_s S + E_1 D_1 - E_2 D_2}}\right)\\ &+ \frac{2}{E_3}\,\frac{E_s}{\sqrt{E_s S + E_1 D_1 - E_2 D_2}}\Bigg].\end{aligned} \tag{17}$$

The requirement $\mathrm{d}t/\mathrm{d}S = 0$ gives, in the same way,

$$\begin{aligned}L = \sqrt{(E_s S + E_1 D_1)^3}\Bigg[&\frac{1}{E_s\sqrt{E_s S}} + \frac{1}{E_1}\left(\frac{1}{\sqrt{E_s S + E_1 D_1}} - \frac{1}{\sqrt{E_s S}}\right)\\ &+ \frac{2}{E_2}\left(\frac{1}{\sqrt{E_s S + E_1 D_1}} - \frac{1}{\sqrt{E_s S + E_1 D_1 - E_2 D_2}}\right)\\ &+ \frac{2}{E_3}\,\frac{1}{\sqrt{E_s S + E_1 D_1 - E_2 D_2}}\Bigg]\\ = (E_s S + E_1 D_1)&\left(\frac{\sqrt{k}}{E_s} + \frac{1-\sqrt{k}}{E_1} + \frac{2\,(1-\sqrt{k'})}{E_2} + \frac{2\sqrt{k'}}{E_3}\right),\end{aligned} \tag{18}$$

where $k = \dfrac{E_s S + E_1 D_1}{E_s S}$ and $k' = \dfrac{E_s S + E_1 D_1}{E_s S + E_1 D_1 - E_2 D_2}$.

Problem 8 (laser aiming in a stratified atmosphere)

Solution

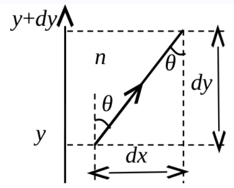
(1) Divide the atmosphere into many thin horizontal layers; let the heights of the bottom and the top of the i-th layer be y_i and $y_{i+1} = y_i + \mathrm{d}y$, and let the refractive index of the air of this layer be $n(y_i)$, as in the figure. The angles of refraction and of incidence at the lower and the upper interface of a thin layer are both $\theta(y_i)$. The law of refraction and the

geometry give

$$n(y_{i-1})\sin\theta(y_{i-1}) = n(y_i)\sin\theta(y_i) = n(y_{i+1})\sin\theta(y_{i+1}) = \text{constant},$$

that is,

$$n(y)\sin\theta(y) = \text{constant}. \tag{1}$$



The geometry gives

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{\dfrac{1}{\sin^2\theta}-1}}. \tag{2}$$

The two equations above give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{n^2}{n_0^2\sin^2\theta_0}-1}. \tag{3}$$

Substitution of the given law $n^2 = n_0^2 + \alpha^2 y$ into (3) gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{n_0^2+\alpha^2 y}{n_0^2\sin^2\theta_0}-1} = \frac{\alpha}{n_0\sin\theta_0}\sqrt{\frac{n_0^2}{\alpha^2}\cos^2\theta_0 + y}, \tag{4}$$

that is,

$$\frac{\mathrm{d}y}{\sqrt{\dfrac{n_0^2}{\alpha^2}\cos^2\theta_0 + y}} = \frac{\alpha}{n_0\sin\theta_0}\,\mathrm{d}x. \tag{5}$$

Integration gives

$$2\sqrt{\frac{n_0^2}{\alpha^2}\cos^2\theta_0 + y} + C = \frac{\alpha}{n_0\sin\theta_0}\,x. \tag{6}$$

The laser leaves from the coordinate origin O, so $y = 0$ at $x = 0$:

$$2\sqrt{\frac{n_0^2}{\alpha^2}\cos^2\theta_0 + 0} + C = 0. \tag{7}$$

Because n_0 and α are positive and $0 \le \theta_0 \le \pi/2$,

$$C = -2\sqrt{\frac{n_0^2}{\alpha^2}\cos^2\theta_0} = -2\,\frac{n_0}{\alpha}\cos\theta_0. \tag{8}$$

Substitution of (8) into (6) and rearrangement give the path equation of the laser light:

$$y = \left(\frac{\alpha}{2n_0\sin\theta_0}\,x + \frac{n_0}{\alpha}\cos\theta_0\right)^2 - \frac{n_0^2}{\alpha^2}\cos^2\theta_0 = \frac{\alpha^2}{4n_0^2\sin^2\theta_0}\,x^2 + \frac{x}{\tan\theta_0}. \tag{9}$$

This is a piece of a parabola through the point O, opening upward, with the vertex $\Big(-\dfrac{n_0^2}{\alpha^2}\sin 2\theta_0,\ -\dfrac{n_0^2}{\alpha^2}\cos^2\theta_0\Big)$ in the third quadrant.

(2) Equation (9) shows: for a given y, if θ_0 grows, then x also grows monotonically; at $\theta_0 = \pi/2$ the vertex of the parabola arrives at the point O, and the curve reaches the farthest distance in the x direction. Substitution of $\theta_0 = \pi/2$ and the coordinates (x_a, y_a) of the point A into the path equation (9) gives

$$x_{a\ \max} = \frac{2n_0}{\alpha}\sqrt{y_a}. \tag{10}$$

This is the largest horizontal distance at which a target aircraft A that flies at the height y_a can be illuminated or attacked by the laser.

(3) Substitution of the coordinates (x_a, y_a) of the target A into the path equation (9) gives

$$y_a = \frac{\alpha^2}{4n_0^2\sin^2\theta_0}\,x_a^2 + \frac{x_a}{\tan\theta_0}. \tag{11}$$

Substitution of the trigonometric relation $\sin^2\theta_0 = \dfrac{\tan^2\theta_0}{1+\tan^2\theta_0}$ into this equation gives

$$y_a = \frac{\alpha^2 x_a^2}{4n_0^2\tan^2\theta_0}\big(1+\tan^2\theta_0\big) + \frac{x_a}{\tan\theta_0}, \tag{12}$$

which rearranges into a quadratic equation for $\tan\theta_0$:

$$\big(4n_0^2 y_a - \alpha^2 x_a^2\big)\tan^2\theta_0 - 4n_0^2 x_a\tan\theta_0 - \alpha^2 x_a^2 = 0. \tag{13}$$

The solution is

$$\tan\theta_0 = \frac{2n_0^2 x_a}{4n_0^2 y_a - \alpha^2 x_a^2} \pm \sqrt{\left(\frac{2n_0^2 x_a}{4n_0^2 y_a - \alpha^2 x_a^2}\right)^2 + \frac{\alpha^2 x_a^2}{4n_0^2 y_a - \alpha^2 x_a^2}}.$$

The target A is in the first quadrant and inside the attack range, so by the result of part (2)

$$0 \le x_a \le \frac{2n_0}{\alpha}\sqrt{y_a},$$

and at the same time $0 \le \theta_0 \le \dfrac{\pi}{2}$, so $\tan\theta_0 \ge 0$; the root with the positive sign must be taken:

$$\tan\theta_0 = \frac{2n_0^2 x_a}{4n_0^2 y_a - \alpha^2 x_a^2} + \sqrt{\left(\frac{2n_0^2 x_a}{4n_0^2 y_a - \alpha^2 x_a^2}\right)^2 + \frac{\alpha^2 x_a^2}{4n_0^2 y_a - \alpha^2 x_a^2}}. \tag{14}$$

By the problem, the trajectory of the target A is

$$x_a = x_0 - vt, \qquad y_a = y_0. \tag{15}$$

Substitution of (15) into (14) gives the law by which the emission angle θ_0 of the laser must change with the time t:

$$\theta_0 = \arctan\left(\frac{2n_0^2(x_0-vt)}{4n_0^2 y_0 - \alpha^2(x_0-vt)^2} + \sqrt{\left(\frac{2n_0^2(x_0-vt)}{4n_0^2 y_0 - \alpha^2(x_0-vt)^2}\right)^2 + \frac{\alpha^2(x_0-vt)^2}{4n_0^2 y_0 - \alpha^2(x_0-vt)^2}}\right). \tag{16}$$

(4) Assume the flying target aircraft just succeeds with a bomb attack on the laser device. Its flight time then has two parts: the first is the time t_a in which it is illuminated by the laser until it is destroyed — the aircraft releases the bomb exactly at the moment of its destruction; the second is the time t_g from the release of the bomb to its landing on the horizontal ground. After the release the bomb moves like a horizontal projectile, so

$$t_g = \sqrt{\frac{2y_a}{g}}. \tag{17}$$

The horizontal distance that the target aircraft A covers in the whole process, from the beginning of the laser illumination to the landing of the bomb, is

$$s = v\,(t_a + t_g) = v\left(t_a + \sqrt{\frac{2y_a}{g}}\right). \tag{18}$$

The laser device at the coordinate origin destroys the target aircraft A safely if

$$s < x_{a\ \max}, \tag{19}$$

where $x_{a\ \max}$ is the largest horizontal distance at which an aircraft A that flies at the height y_a can be illuminated by the laser (see (10)). Equations (10), (18), (19) give the requirement on the horizontal speed v of the target aircraft A:

$$v < \frac{\dfrac{2n_0}{\alpha}\sqrt{y_a}}{t_a + \sqrt{\dfrac{2y_a}{g}}} = \frac{2n_0\sqrt{g}\,\left(t_a\sqrt{gy_a} - \sqrt{2}\,y_a\right)}{\alpha\left(gt_a^2 - 2y_a\right)}.$$

In other words, the speed range of the target aircraft A is

$$\left(0,\ \frac{2n_0\sqrt{g}\,\left(t_a\sqrt{gy_a} - \sqrt{2}\,y_a\right)}{\alpha\left(gt_a^2 - 2y_a\right)}\right). \tag{20}$$

[a]

[a] *Translator's note:* The rationalized form printed in the source omits the factor $\sqrt{g}$; the first (unrationalized) expression is the authoritative one.