The 33rd Chinese Physics Olympiad

Final — Theoretical Examination (Problems)

2016 — 8 problems

Translator's note: the source file of this paper (a Word document) lost a small number of embedded symbols and formulas. All reconstructed symbols and formulas are set in square brackets [·]; they follow unambiguously from the official solutions.

Physical constants and formulas that may be needed: the speed of light in vacuum $c = 2.998 \times 10^8$ m/s; the magnitude of the gravitational acceleration at the Earth's surface is g; the Planck constant is h, and $\hbar = \dfrac{h}{2\pi}$;

$$\int \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\frac{1+x}{1-x} + C, \qquad |x| < 1.$$

Problem 1

A coal mine at Datong (Shanxi) lies at the height h_c relative to Qinhuangdao. A train of mass m_t, loaded with coal of mass m_c, travels the distance l from Datong along the Daqin railway line to Qinhuangdao, unloads, and returns empty. On the way from Datong to Qinhuangdao, one part of the total potential-energy change of the train and the coal does work against the rail and air resistance; the generator converts the remaining part into electric energy, with the average conversion efficiency η_1; all this electric energy is stored in a battery for the return trip. On the return trip the stored electric energy drives the electric motor of the empty train, which does work against gravity and resistance; the average conversion efficiency from stored electric energy to output work is η_2. Assume that, on the Daqin line, the work needed per unit distance of travel against the resistance is proportional to the total weight in motion (the train, or the train and the coal), with a constant coefficient μ; the train carries zero electric energy when it leaves Datong.

(1) If the empty train has remaining electric energy when it returns to Datong, find this energy E.

(2) At least how much coal must the train load, so that it can just return to Datong without an extra energy supply?

(3) The average speed of the train on the rails from Datong to Qinhuangdao is V. Give the relation between the average output power P of the generator and the other given quantities.

Problem 2

As shown in figure a, AB is a uniform thin rod of mass m and length l_2; the upper end B of the rod hangs from the fixed point O by an inextensible soft light rope of length l_1. At the start the rope and the rod hang at rest, vertically. All subsequent motion stays in one vertical plane.

(1) An instantaneous horizontal impulse I is applied at the point D of the rod. If, at the instant just after the impulse, the angular velocity of B around the suspension point O and the angular velocity of the rod around its center of mass are the same, find the distance x from D to B and the initial angular velocity ω_0 of B around the suspension point O.

(2) Suppose that at some moment the rope and the rod make the angles θ_1 and θ_2 with the vertical direction (as in figure b), and that the angular velocities of the rope around the fixed point O and of the rod around its center of mass are ω_1 and ω_2. Find the angular accelerations α_1 (rope, around O) and α_2 (rod, around its center of mass).

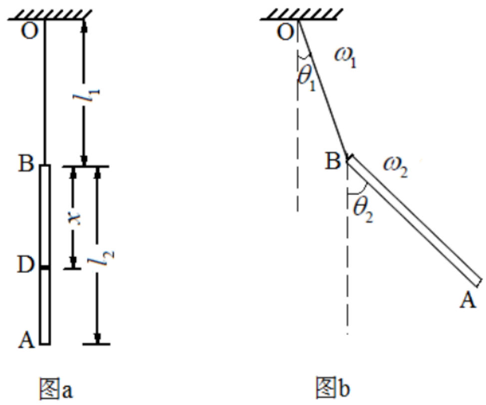


图 *a/b = figure a/b.*

Problem 3 (15 points)

The atmosphere of Mars can be treated as made of very dilute CO_2 only; the molar mass of this atmosphere is [μ], and at any one height the atmosphere can be treated as an ideal gas in an equilibrium state. The mass of Mars is [M_m] (much larger than the total mass of the Martian atmosphere), and its radius is [R_m]. Assume that the mass density of the Martian atmosphere depends on the height h above the surface of Mars as

$$\left[\ \rho(h) = \rho_0\left(1 + \frac{h}{R_\mathrm{m}}\right)^{1-n}\ \right],$$

where ρ_0 is a constant and n ($n > 4$) is a constant.

(1) Find the expression for the temperature of the Martian atmosphere as a function of the height h.

(2) For the exploration of the Martian surface, a lander of mass [m_t] and of large mass density must be brought to the Martian surface. The lander is released from rest at some height much smaller than R_m; after the time t_l it lands on the Martian surface. During the descent the lander does not rotate, and the gravitational acceleration of Mars and the density of the atmosphere can be treated as equal to their values at the Martian surface. When the descent speed of the lander is [v], the atmospheric drag on it is proportional to the product of the atmospheric density and v^2, with a constant proportionality coefficient [k]. Find the speed of the lander at the moment just before the landing.

Problem 4

A photon with energy and momentum also has angular momentum. The magnitude of the angular momentum of a photon of circularly polarized light is $\hbar$. When a body absorbs a photon, the energy, the momentum and the angular momentum of the photon all pass to the body. The angular momentum that the body receives can turn the body. With this principle, scientists have made nanoparticles rotate at high speed under continuous circularly polarized laser light, and have obtained the micro-scale rotor with the highest rotation speed in a liquid environment so far.

As shown in the figure, a gold nanosphere lies between two horizontal smooth glass plates, and the whole system (glass plates included) is immersed in water. A beam of circularly polarized laser light falls from above onto the gold nanoparticle. The wavelength of the incident laser light in vacuum is $\lambda = 830$ nm; after focusing by the microscope objective (the light is still assumed to be a plane wave, and each photon has the angular momentum $\hbar$ along the propagation direction) the light-spot diameter is $d = 1.20 \times 10^{-6}$ m and the power is $P = 20.0$ mW. The radius of the gold nanosphere is $R = 100$ nm, and the density of gold is $\rho = 19.32 \times 10^3$ kg/m^3. Neglect the reflection of the light at the interfaces of the media and the absorption of the light by the glass and the water; analyse the following questions only from the point of view that the gold nanoparticle absorbs photons, receives their angular momentum, and is driven into rotation (give the numerical results to 3 significant figures):

(1) Find the frequency ν of the laser light and the intensity I (the power transported through a unit cross-section in the propagation direction).

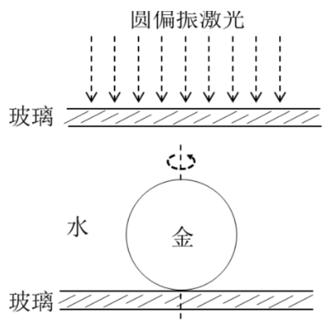
(2) Give the values of the mass m of the gold nanosphere and of its moment of inertia J around its symmetry axis.

(3) Assume the absorption cross-section of the particle for the light (the ratio of the light power absorbed by the particle to the incident intensity) is $\sigma_{\text{abs}} = 0.123\,\pi R^2$. Find the magnitude M of the torque, along the rotation symmetry axis, that the laser beam exerts on the particle.

(4) A spherical particle (radius r) that rotates with angular velocity ω in a liquid of viscosity η feels a viscous friction torque of magnitude $M_{\text{f}} = 8\pi\eta r^3\omega$. The viscosity of water is $\eta = 8.00 \times 10^{-4}$ Pa·s. Find the rotation speed f (revolutions per second) of the gold nanosphere in this light beam when the rotation has become steady.

(5) Take the moment when the light begins to shine on the resting gold nanoparticle as the zero of time, $t_0 = 0$. Find the expression $f(t)$ for the rotation speed of the particle at any time t ($t > 0$).

(6) Replace the incident laser beam by a square-wave pulsed laser beam with pulse width T_1 (the intensity is still I within a pulse) and pause T_2 between pulses. Take the moment when the light of the first pulse begins to shine on the particle as the zero of time, $t_0 = 0$. Find the expression for the rotation speed f_n of the particle at the end of the n-th full pulse period ($T_1 + T_2$), and give the expression for the limit $f_{n\to\infty}$.



Labels: 圆偏振激光 = *circularly polarized laser light;* 玻璃 = *glass;* 水 = *water;* 金 = *gold.*

Problem 5

Many racing cars carry an adjustable aerofoil (wing), which can give the car on a horizontal road an additional vertical press — upward or downward. If the speed of the car is v, the magnitude of this press is $f_B = c_B v^2$, where c_B is a constant. When the front of the aerofoil tilts up, the press points upward; when the rear of the aerofoil tilts up, the press points downward. A moving car also feels the resistance of the oncoming air, of magnitude $f_A = c_A v^2$, where c_A is a constant. The mass of the car is m, and the coefficient of static friction between the tires and the road is μ_S ($\mu_S < 1$).

(1) When the car drives with constant speed on a horizontal straight road, the deformation of the tires during the motion makes the road resist the car; the magnitude of this resistance is proportional to the normal press of the car on the road, with the proportionality coefficient μ_R ($\mu_R < \mu_S$). The aerofoil is adjusted with the front tilted up. At what driving speed is the output power of the engine of the car largest? What is this largest output power?

(2) Without regard to the road resistance caused by the tire deformation, find the largest speed $v_{\max}$ that the car may have while it drives with constant speed along a circular road of radius r on horizontal ground, without slipping on the road surface and without leaving the ground. In this situation, must the aerofoil be adjusted front-up or rear-up so that $v_{\max}$ can be larger? Assume the output power of the engine can be large enough.

Problem 6

Hyperloop is a system for super-high-speed transport that uses a capsule-shaped transport car moving in a horizontal tube (see figure a). It uses the "air cushion" technique and the principle of the "linear electric motor".

In the "air cushion" technique, high-pressure gas from inside the car is blown out quickly through the fine holes in the lower half of the horizontally placed transport car (see figure b), so that the whole car is lifted a very small distance off the tube wall, and friction becomes negligible. The cross-section of the car is a circle of radius [R]; on the wall of the lower half of the car, radial fine holes are distributed uniformly, with n holes per unit area ($n \gg 1$); the area of a single hole is [s]. The length of the car is [l] and its mass is [M]. The gas flow can be assumed to obey the Bernoulli equation, at constant temperature; at the exits of the fine holes the gas pressure is the lower ambient pressure P_{low}.

As in figure c, two parallel horizontal smooth current-supplying rails (thick full lines) are fixed in the horizontal tube; on the car, two conducting wires (thin full lines) perpendicular to the rails are fixed. The rail cross-section is circular, with radius r_{d} and resistivity ρ_{d}; the distance between the axes of the two rails is $2(D + r_{\text{d}})$. The thickness of the two wires is negligible; their separation is D; the resistance of each wire equals 2 times the resistance of a rail piece of length D. The two wires and the rail axes lie in one horizontal plane. The electric contact between the rails and the wires is good, and all contact resistances are negligible.

(1) Find the internal high pressure [P] at which the transport car is just lifted off the tube wall.

(2) As shown in figure d, when the transport car comes into the station with the speed v_0, it slides along the horizontal smooth rails into the region of a strong uniform magnetic field; the field boundary is parallel to the wires, and the magnetic induction, of magnitude B, is perpendicular to the rail–wire plane, downward. Find the sliding speed of the car after both wires have entered the field region.

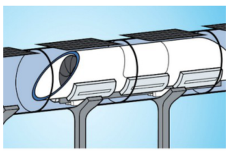
(3) When the transport car is at rest on the horizontal smooth rails, ready to leave the station, a constant-voltage dc source of emf $\mathcal{E}$, fixed on the rails at the distance D behind wire 2, is switched on (as in figure e). The volume of the source and the resistance of its connecting wires are negligible. Find the magnitude of the acceleration of the transport car at the moment just after the source is switched on.

It is known that a straight cylindrical segment of wire in a closed constant-current circuit, with the current distributed uniformly over the cross-section, produces at a point at distance r_0 from the wire axis a magnetic induction perpendicular to the plane through this point and the wire axis, of magnitude approximately

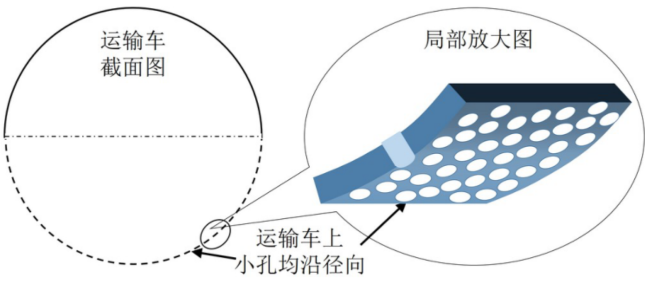
$$B = \frac{\mu_0 I}{4\pi r_0}(\cos\theta_1 + \cos\theta_2),$$

where I is the current and θ_1, θ_2 are the angles between the wire axis and the lines from the point to the two ends of the segment (figure f). An integral formula that may be needed:

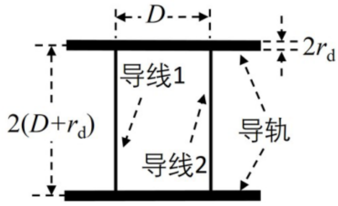
$$\int_a^b \frac{1}{x}\frac{c}{\sqrt{x^2+c^2}}\,\mathrm{d}x = \ln\left(\frac{b}{a}\frac{c+\sqrt{c^2+a^2}}{c+\sqrt{c^2+b^2}}\right), \qquad a, b, c > 0.$$



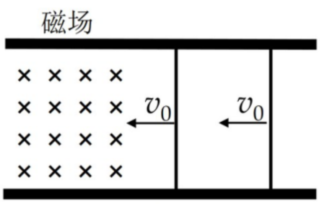
图a



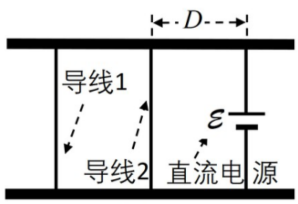
图b



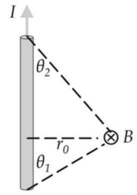
图c



图d



图e



图f

Labels: 运输车截面图 = *cross-section of the car;* 局部放大图 = *enlarged view;* 运输车上小孔均沿径向 = *the holes on the car are all radial;* 导线 *1/2* = *wire 1/2;* 导轨 = *rail;* 磁场 = *magnetic field;* 直流电源 = *dc source;* 图 *a–f* = *figures a–f.*

Problem 7

Einstein's theory of gravitation predicts that changes of the distribution of matter cause waves of the geometric structure of space-time — gravitational waves. For simplicity, consider a plane gravitational wave that travels along the z axis. For any given z, the distance $\mathrm{d}r$ between two infinitely close points (x, y) and $(x + \mathrm{d}x, y + \mathrm{d}y)$ of the two-dimensional x-y space is

$$\mathrm{d}r = \sqrt{(1 + f_1)(\mathrm{d}x)^2 + f_2\,(\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y\,\mathrm{d}x) + (1 - f_1)(\mathrm{d}y)^2}.$$

From the gravitational wave, f_1 and f_2 change (wave).

(1) Assume that when a plane gravitational wave passes, f_1 and f_2 can be written as

$$f_1 = A\sin\left[\omega\left(t - \frac{z}{c}\right)\right], \qquad f_2 = 0; \qquad 0 < A \ll 1,$$

where A and ω are the amplitude and the angular frequency of the wave, and c is the propagation speed of the gravitational wave (its value equals the speed of light).

(i) When no gravitational wave passes, one micro-detector is placed at the origin O of the x-y plane and one at the distance R from O, in the direction at the angle θ to the x axis. Find the approximate expression for the deviation from R of the distance between the two detectors while the gravitational wave passes.

(ii) When no gravitational wave passes, an array of micro-detectors is placed, in the x-y plane, on the circle of radius R around the origin O. While the gravitational wave passes, one can define new space coordinates (X, Y) such that the X-Y two-dimensional space is formally flat: the distance between two close points (X, Y) and $(X + \mathrm{d}X, Y + \mathrm{d}Y)$ is $\sqrt{(\mathrm{d}X)^2 + (\mathrm{d}Y)^2}$. Find the shape of the distribution of the detector array in the new coordinates at a given moment t.

(iii) If one plane gravitational wave

$$f_1 = A\sin\left[\omega\left(t - \frac{z}{c}\right)\right], \quad f_2 = 0$$

and a second plane gravitational wave

$$f_1 = A\sin\left[\omega\left(t - \frac{z}{c}\right) + \phi\right], \quad f_2 = 0, \qquad \phi\ (0 \le \phi < 2\pi) \text{ a constant}$$

travel along the positive z direction at the same time, what conditions must ϕ and θ satisfy, so that the amplitude of the disturbance of the distance between the two micro-detectors — one at the origin O and one at the point with coordinates $(R\cos\theta, R\sin\theta)$ of the x-y space — is largest or smallest?

(2) Assume the source of the gravitational wave is a binary star system; the two stars have the equal mass M. Take the center of mass of the binary system as the coordinate origin O'; the binary system rotates in the two-dimensional x'-y' space. Under specific conditions, f_1 and f_2 can be written approximately as

$$f_1 = \frac{8\pi G}{c^4}\,\frac{2}{l}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[I_1\Big(t-\frac{l}{c}\Big)\Big], \qquad f_2 = \frac{8\pi G}{c^4}\,\frac{2}{l}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[I_2\Big(t-\frac{l}{c}\Big)\Big],$$

where G is the gravitational constant, l is the distance (on the z' axis) from the detection point of the gravitational wave to the point O' (much larger than the distance between the two stars), and

$$I_1(t) = \big[x_1'^2(t) + x_2'^2(t)\big]M, \qquad I_2(t) = x_1'(t)\,y_2'(t)\,M,$$

with $(x_1'(t), y_1'(t))$, $(x_2'(t), y_2'(t))$ the coordinates of the two stars in their center-of-mass system. (Note: with these definitions, the coefficients in the expression for $\mathrm{d}r$ must be replaced as $f_1 \to f_1$, $f_2 \to 2f_2$.) The frequency of this gravitational wave is ω; assume the effect of the gravitational-wave radiation on the orbital motion of the binary system is negligible. Find the amplitudes of f_1 and f_2 at the detection point.

Problem 8

The positronium atom (Ps) is a quantum bound system made of an electron e^- and a positron e^+ (the antiparticle of the electron, with the same mass as the electron and a charge of equal magnitude and opposite sign); its energy levels can be obtained in analogy with the hydrogen atom. By the Bohr theory of the hydrogen atom, the orbital angular momentum of the electron on its circular motion around the proton is quantized: it is an integer multiple of $\hbar$. Because the proton mass is finite, the quantization condition of the hydrogen atom must be corrected: in the electron–proton center-of-mass system, the total orbital angular momentum relative to the center of mass is an integer multiple of $\hbar$. This quantization condition extends directly to other two-body bound systems, such as positronium. Keep 4 significant figures in all numerical results below.

(1) Write down the ground-state energy of positronium.

(2) A ground-state positronium atom is unstable; it can annihilate quickly into two photons:

$$\mathrm{Ps} \;\to\; \gamma_1 + \gamma_2.$$

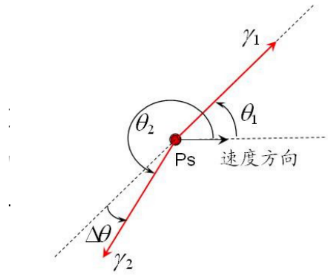
When a ground-state positronium atom that moves, relative to the laboratory frame, with some speed much smaller than the speed of light annihilates, one photon γ_1 is seen in the direction at the angle θ_1 to the velocity, and the other photon γ_2 is seen in the direction at the angle $\Delta\theta$ ($\Delta\theta \ll 1$) to the direction opposite to the velocity of the photon γ_1, as shown in the figure.

(i) Find the magnitude of the speed v_0 of the ground-state positronium atom and the expressions for the energies E_1 and E_2 of the two photons γ_1 and γ_2; give the values of v_0, E_1 and E_2 for $\theta_1 = \dfrac{\pi}{3}$, $\Delta\theta = 3.464 \times 10^{-3}$.

(ii) When the ground-state positronium atom is at rest, find the difference between the energy E_1 (or E_2) of each of the two photons γ_1, γ_2 and the rest energy of one free electron.

(3) When a ground-state positronium atom that moves relative to the laboratory frame with a speed v_0 comparable with the speed of light c annihilates into two photons, find the energies E_1 and E_2 of the two photons γ_1 and γ_2, and the relation between θ_1 and the deflection angle θ_2 of the direction of motion of the photon γ_2 relative to the direction of v_0 (as shown in the figure); give the values of E_1, E_2 and θ_2 for $v_0 = \dfrac{c}{2}$, $\theta_1 = \dfrac{\pi}{3}$.

Given: the ground-state energy of the hydrogen atom $E^{\mathrm{H}}_{n=1} = -13.60$ eV; the electron mass $m_{\mathrm{e}} = 0.5110\ \mathrm{MeV}/c^2$; the proton-to-electron mass ratio is 1836.



Labels: 速度方向 = *direction of the velocity.*