The 33rd Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2016 — official reference solutions

Equation numbers (1), (2), … in each problem follow the circled numbers of the Chinese original; primed numbers mark alternative versions. Symbols and formulas lost in the source file are reconstructed and set in square brackets $[\,\cdot\,]$.

Problem 1 (coal transport with energy recovery)

Solution

(1) One part of the total change of the gravitational potential energy of the train and the coal on the trip from Datong to Qinhuangdao (the outward trip) does the work against the resistance on the outward trip; the other part drives the generator of the train. The input energy of the generator is therefore

$$E_1 = (m_\mathrm{c} + m_\mathrm{t})g(h_\mathrm{c} - 0) - \mu(m_\mathrm{c} + m_\mathrm{t})gl. \tag{1}$$

The electric energy produced by the generator is

$$E_2 = \eta_1 E_1. \tag{2}$$

Let E be the electric energy that remains after the empty train has returned. By the problem, the energy given to the electric motor of the train on the return trip is $(E_2 - E)$, and the output energy of the motor is

$$E_3 = \eta_2\,(E_2 - E). \tag{3}$$

E_3 is used as the work against the resistance and against gravity on the return trip, so

$$E_3 = \mu m_\mathrm{t} gl + m_\mathrm{t} g(h_\mathrm{c} - 0). \tag{4}$$

Equations (1)–(4) give

$$E = \eta_1\big[(m_\mathrm{c}+m_\mathrm{t})gh_\mathrm{c} - \mu(m_\mathrm{c}+m_\mathrm{t})gl\big] - \frac{\mu m_\mathrm{t} gl + m_\mathrm{t} g h_\mathrm{c}}{\eta_2} = \eta_1(m_\mathrm{c}+m_\mathrm{t})g(h_\mathrm{c}-\mu l) - m_\mathrm{t} g\,\frac{h_\mathrm{c}+\mu l}{\eta_2}. \tag{5}$$

(2) From $E_1 \geq 0$, the resistance coefficient μ must satisfy $\mu < h_\mathrm{c}/l$. By the problem, the train can return if $E \geq 0$, which gives

$$m_\mathrm{c} \geq m_{\mathrm{c\,min}} = \left(\frac{1}{\eta_1\eta_2}\,\frac{h_\mathrm{c}+\mu l}{h_\mathrm{c}-\mu l} - 1\right) m_\mathrm{t}. \tag{6}$$

The right side of (6) is the smallest coal load $m_{\mathrm{c\,min}}$.

(3) The travel time is

$$t = \frac{l}{V}. \tag{7}$$

With (1) and (2), the average output power P of the generator is

$$P = \frac{E_2}{t} = \frac{\eta_1\big[(m_\mathrm{c} + m_\mathrm{t})gh_\mathrm{c} - \mu(m_\mathrm{c} + m_\mathrm{t})gl\big]}{t}. \tag{8}$$

Equations (7) and (8) give

$$P = \eta_1(m_\mathrm{c} + m_\mathrm{t})\,gV\left(\frac{h_\mathrm{c}}{l} - \mu\right). \tag{9}$$

Problem 2 (rod on a rope)

Solution

(1) The distance from the point D to the center of mass C is $x - l_2/2$. Let v_C be the speed of the translation of the center of mass at the instant just after the impulse. B and C turn around the point O with one angular velocity, so the speed v_B of the point B satisfies

$$v_B = \omega_0 l_1, \tag{1}$$

$$v_B = v_C - \omega_0 \frac{l_2}{2}. \tag{2}$$

By the impulse theorem,

$$I = mv_C. \tag{3}$$

The point of application of the impulse is not the center of mass, so the moment of the impulse I also causes a rotation of the rod around its center of mass. By the rigid-body rotation theorem,

$$\left(\frac{1}{12} ml_2^2\right)\omega_0 = I\left(x - \frac{l_2}{2}\right). \tag{4}$$

[Or, with the moment of momentum taken about the suspension point O:

$$\left[\frac{1}{12} ml_2^2 + m\left(l_1 + \frac{l_2}{2}\right)^2\right]\omega_0 = I\,(l_1 + x).\,] \tag{4'}$$

Equations (1)–(4) give

$$x = \left[\frac{1}{2} + \frac{l_2}{6\,(2l_1 + l_2)}\right] l_2, \tag{5}$$

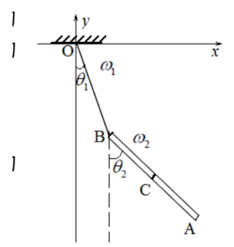
$$\omega_0 = \frac{2I}{m\,(2l_1 + l_2)}. \tag{6}$$

(2) As in the figure, set up plane coordinates in the vertical plane of O, B, A: the origin at O, the x axis horizontal (to the right), the y axis vertical (upward). The acceleration (a_{Cx}, a_{Cy}) of the center of mass C of the rod satisfies the center-of-mass theorem

$$ma_{Cx} = -T\sin\theta_1, \qquad ma_{Cy} = -mg + T\cos\theta_1, \tag{7}$$

where T is the tension of the rope. At the same time the rod turns around its center of mass under the torque of the rope tension T relative to the center of mass; the rotation theorem gives

$$\frac{1}{12} ml_2^2\,\alpha_2 = -T\,\frac{1}{2}\,l_2\sin(\theta_2 - \theta_1). \tag{8}$$



The geometry gives

$$x_{\mathrm{B}}(t)=x_{\mathrm{C}}(t)-\frac{1}{2}\,l_2\sin\theta_2(t),\qquad y_{\mathrm{B}}(t)=y_{\mathrm{C}}(t)+\frac{1}{2}\,l_2\cos\theta_2(t).\tag{9}$$

Differentiation of both sides with respect to the time t shows that the velocity of the point B satisfies

$$v_{\mathrm{B}x}(t)=v_{\mathrm{C}x}(t)-\frac{1}{2}\,\omega_2(t)\,l_2\cos\theta_2(t),\qquad v_{\mathrm{B}y}(t)=v_{\mathrm{C}y}(t)-\frac{1}{2}\,\omega_2(t)\,l_2\sin\theta_2(t).\tag{10}$$

A second differentiation shows that the acceleration of B satisfies

$$a_{\mathrm{B}x}=a_{\mathrm{C}x}-\frac{1}{2}\alpha_2 l_2\cos\theta_2+\frac{1}{2}\omega_2^2 l_2\sin\theta_2,\qquad a_{\mathrm{B}y}=a_{\mathrm{C}y}-\frac{1}{2}\alpha_2 l_2\sin\theta_2-\frac{1}{2}\omega_2^2 l_2\cos\theta_2.\tag{11}$$

At the same time B turns around O on the inextensible rope, so

$$x_{\mathrm{B}}(t)=l_1\sin\theta_1(t),\qquad y_{\mathrm{B}}(t)=-l_1\cos\theta_1(t).\tag{12}$$

Differentiation gives

$$v_{\mathrm{B}x}(t)=\omega_1(t)\,l_1\cos\theta_1(t),\qquad v_{\mathrm{B}y}(t)=\omega_1(t)\,l_1\sin\theta_1(t),\tag{13}$$

and a second differentiation gives

$$a_{\mathrm{B}x}=\alpha_1 l_1\cos\theta_1-\omega_1^2 l_1\sin\theta_1,\qquad a_{\mathrm{B}y}=\alpha_1 l_1\sin\theta_1+\omega_1^2 l_1\cos\theta_1.\tag{14}$$

[Or, without the point B, directly with the position of C:

$$x_{\mathrm{C}}(t)=l_1\sin\theta_1(t)+\frac{1}{2}l_2\sin\theta_2(t),\qquad y_{\mathrm{C}}(t)=-l_1\cos\theta_1(t)-\frac{1}{2}l_2\cos\theta_2(t),$$

$$a_{\mathrm{C}x}=l_1\big(-\alpha_1\cos\theta_1+\omega_1^2\sin\theta_1\big)\cdot(-1)+\frac{l_2}{2}\big(-\alpha_2\cos\theta_2+\omega_2^2\sin\theta_2\big)\cdot(-1)\quad\text{etc. }]$$

Combination of (7), (8), (11), (14) gives the angular accelerations of the rope around the suspension point and of the rod around its center of mass:

$$\alpha_1=-\frac{g\sin\theta_1+\sin(\theta_1-\theta_2)\left[3g\cos\theta_2+3l_1\omega_1^2\cos(\theta_1-\theta_2)+2l_2\omega_2^2\right]}{l_1\left[1+3\sin^2(\theta_1-\theta_2)\right]},$$

$$\alpha_2=\frac{3\sin(\theta_1-\theta_2)\left[2g\cos\theta_1+2l_1\omega_1^2+l_2\omega_2^2\cos(\theta_1-\theta_2)\right]}{l_2\left[1+3\sin^2(\theta_1-\theta_2)\right]}.\tag{15}$$

Problem 3 (atmosphere of Mars)

Solution

(1) Mark out a thin horizontal box-shaped region in the atmosphere at some height of Mars. The volume of this region is [V], the mass of the atmosphere inside it is [m], its pressure is [P], and its temperature is $T(h)$. The box is very thin, so the density, the pressure and the temperature of the gas can be treated as constants. By the equation of

state of an ideal gas,

$$\left[PV = \frac{m}{\mu}\,RT(h)\right]. \tag{1}$$

The volume of the Martian atmosphere inside the box is

$$\left[V = \frac{m}{\rho(h)}\right]; \tag{2}$$

the number of moles of the Martian atmosphere inside the box is

$$\left[\nu = \frac{m}{\mu}\right]. \tag{3}$$

Equations (1)–(3) give

$$\left[T(h) = \frac{\mu P(h)}{R\,\rho(h)}\right]. \tag{4}$$

Now calculate $P(h)$.

Method 1. Consider the thin spherical-cap shell of atmosphere between the heights x and $x + \mathrm{d}x$ that fills the solid angle $\Delta\Omega$; take $\Delta\Omega$ so small that the radial straight lines on the side surface of the thin cap shell can be treated as parallel to each other. The force analysis gives

$$(P(x) + \mathrm{d}P)\cdot\Delta\Omega\,(R_\mathrm{m} + x)^2 + \rho(x)g(x)\cdot\Delta\Omega\,(R_\mathrm{m} + x)^2\,\mathrm{d}x = P(x)\cdot\Delta\Omega\,(R_\mathrm{m} + x)^2. \tag{5}$$

Keeping the terms of first order in $\mathrm{d}x$ in (5) gives

$$\mathrm{d}P = -\rho(x)\,g(x)\,\mathrm{d}x. \tag{6}$$

At the height h of Mars, the gravitational acceleration is

$$g(h) = \frac{GM_\mathrm{m}}{(R_\mathrm{m} + h)^2}. \tag{7}$$

Substitution into (6) and integration give

$$P(h) = GM_\mathrm{m}\int_h^\infty \frac{\rho(x)}{(R_\mathrm{m} + x)^2}\,\mathrm{d}x. \tag{8}$$

Method 2. Consider the spherical-cap shell between the heights x and $x + \mathrm{d}x$ that subtends the angle α at the center of Mars, as in the solution figure a. Analyse the forces along the central axis of the cap. Let the magnitude of the vertical downward component of the atmospheric press on the upper cap be $F_{1.1}$, and the magnitude of the weight of the shell atmosphere be $F_{1.2}$:

$$F_{1.1} = \int_0^\alpha 2\pi(R_\mathrm{m} + x + \mathrm{d}x)^2\sin\theta\,[P(x) + \mathrm{d}P]\cos\theta\,\mathrm{d}\theta,$$

$$F_{1.2} = \int_0^\alpha \frac{GM_\mathrm{m}}{(R_\mathrm{m} + x)^2}\,2\pi(R_\mathrm{m} + x)^2\sin\theta\cos\theta\,\mathrm{d}x\,\rho(x)\,\mathrm{d}\theta.$$

Here $P(x)$ is the pressure at the height x and $P(x) + \mathrm{d}P$ the pressure at the height $x + \mathrm{d}x$. Let the magnitude of the vertical upward component of the atmospheric press on the lower cap be $F_{2.1}$, and the magnitude of the vertical upward component of the atmospheric press on the edge cone band be $F_{2.2}$:

$$F_{2.1} = \int_0^\alpha 2\pi(R_\mathrm{m} + x)^2\sin\theta\,P(x)\cos\theta\,\mathrm{d}\theta,$$

$$F_{2.2} = (R_\mathrm{m} + x)\sin\alpha \cdot 2\pi\,\mathrm{d}x\,P(x)\sin\alpha.$$

The balance of the forces along the vertical direction gives

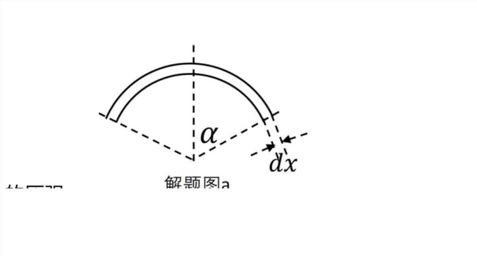
$$F_{1.1} + F_{1.2} = F_{2.1} + F_{2.2}. \tag{5$'$}$$

Keeping the terms up to the order $\mathrm{d}x^1$ gives

$$\mathrm{d}P = -\rho(x)\,g(x)\,\mathrm{d}x, \tag{6$'$}$$

and therefore again

$$P(h) = GM_\mathrm{m}\int_h^\infty \frac{\rho(x)}{(R_\mathrm{m}+x)^2}\,\mathrm{d}x. \tag{7$'$}$$



解题图 a = *solution figure a.*

Substitution of the result into (4), with the given expression for the atmospheric density, and integration give

$$T(h) = \frac{\mu\,GM_\mathrm{m}}{nR\,(R_\mathrm{m}+h)}. \tag{9}$$

(2) Take the release moment as the zero of time; let the height of the lander at the time t be $[h(t)]$, and let the magnitudes of the velocity and the acceleration of the lander be $[v(t)$ and $\dot v(t)]$. The atmospheric drag on the lander points upward, with the magnitude

$$\left[\,f = k\rho\,v^2(t)\,\right]. \tag{10}$$

Neglecting the gravitational pull of the atmosphere on the lander, the gravitational force of Mars on the lander is

$$\left[\,F = \frac{GM_\mathrm{m}m_\mathrm{t}}{(R_\mathrm{m}+h)^2}\,\right]. \tag{11}$$

By the problem, the buoyancy of the atmosphere need not be considered. By Newton's second law,

$$\left[\,m_\mathrm{t}\dot v(t) = k\rho\,v(t)^2 - \frac{GM_\mathrm{m}m_\mathrm{t}}{(R_\mathrm{m}+h)^2}\,\right] \tag{12}$$

(the positive direction of v points downward). Equations (9)–(12) give the equation of motion

$$\left[\,\dot v(t) = \frac{k\rho(h)}{m_\mathrm{t}}\,v(t)^2 - \frac{GM_\mathrm{m}}{(R_\mathrm{m}+h)^2}\,\right]. \tag{13}$$

By the problem, the density and the gravitational acceleration can be approximated by their values at the Martian surface, so (13) becomes

$$\dot v(t) = \frac{k\rho_0}{m_\mathrm{t}}\,v(t)^2 - \frac{GM_\mathrm{m}}{R_\mathrm{m}^2}. \tag{14}$$

Integration gives

$$v(t) = -\sqrt{\frac{GM_{\mathrm{m}}m_{\mathrm{t}}}{k\rho_0 R_{\mathrm{m}}^2}} \tanh\left[\sqrt{\frac{k\rho_0 GM_{\mathrm{m}}}{R_{\mathrm{m}}^2 m_{\mathrm{t}}}}(t+C)\right], \tag{15}$$

where C is a constant. The initial condition $v(t=0)=0$ gives $C=0$. Therefore the velocity of the lander relative to Mars at the moment of the landing is

$$v(t_l) = -\sqrt{\frac{GM_{\mathrm{m}}m_{\mathrm{t}}}{k\rho_0 R_{\mathrm{m}}^2}} \tanh\left[\sqrt{\frac{k\rho_0 GM_{\mathrm{m}}}{R_{\mathrm{m}}^2 m_{\mathrm{t}}}}\, t_l\right]. \tag{16}$$

Problem 4 (rotating gold nanoparticle)

Solution

(1) The frequency of the laser light is

$$\nu = \frac{c}{\lambda} = 3.61 \times 10^{14}\ \mathrm{Hz}. \tag{1}$$

The intensity of the incident laser light is

$$I = \frac{P}{\pi\left(\frac{d}{2}\right)^2} = 1.77 \times 10^{10}\ \mathrm{W/m^2}. \tag{2}$$

(2) The mass of the gold nanosphere is

$$m = \frac{4}{3}\pi R^3 \rho = 8.09 \times 10^{-17}\ \mathrm{kg}. \tag{3}$$

The moment of inertia of the particle around the axis through the center of the sphere is

$$J = \frac{2mR^2}{5} = 3.24 \times 10^{-31}\ \mathrm{kg \cdot m^2}. \tag{4}$$

(3) The light energy that the gold nanosphere absorbs in the time $\mathrm{d}t$ is

$$\mathrm{d}E_{\mathrm{abs}} = I\sigma_{\mathrm{abs}}\,\mathrm{d}t. \tag{5}$$

The energy of one photon is $h\nu$, so the number of photons that the gold particle absorbs in the time $\mathrm{d}t$ is

$$\mathrm{d}N = \frac{\mathrm{d}E_{\mathrm{abs}}}{h\nu} = \frac{I\sigma_{\mathrm{abs}}\,\mathrm{d}t}{h\nu}. \tag{6}$$

By the conservation of angular momentum, each absorbed photon transfers the angular momentum $\hbar$ to the particle. In the time $\mathrm{d}t$, the total angular momentum of the gold particle changes by

$$\mathrm{d}L = \hbar\,\mathrm{d}N = \frac{I\sigma_{\mathrm{abs}}\,\mathrm{d}t}{2\pi\nu}. \tag{7}$$

By the angular-momentum theorem, the magnitude of the torque produced by the light is

$$M = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{I\sigma_{\mathrm{abs}}}{2\pi\nu} = 3.02 \times 10^{-20}\ \mathrm{N \cdot m}. \tag{8}$$

(4) The rotation of the gold nanosphere becomes steady when the total torque on it is zero, that is, when the torque M of the light and the viscous friction torque M_{f} of the water are equal:

$$M - M_{\mathrm{f}} = 0. \tag{9}$$

With the given expression for the friction torque and with (8), (9), the rotation speed of the steadily rotating gold nanosphere is

$$f = \frac{\omega}{2\pi} = \frac{I\sigma_{\mathrm{abs}}}{32\pi^3\eta R^3\nu} = 2.39 \times 10^2\ \mathrm{s}^{-1}. \tag{10}$$

(5) By the rotation theorem of a rigid body around a fixed axis,

$$J\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = M - M_{\mathrm{f}}. \tag{11}$$

With (8),

$$J\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{I\sigma_{\mathrm{abs}}}{2\pi\nu} - 8\pi\eta R^3\omega. \tag{12}$$

Substitution of (4) into (12) shows that, when the rotation rate of the gold particle is ω, the rate of change of the angular velocity satisfies

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = A\omega + B, \tag{13}$$

where

$$A = -\frac{20\pi\eta R}{m}, \qquad B = \frac{5\sigma_{\mathrm{abs}}}{4\pi m R^2\nu}\,I.$$

Equation (13) can be written as

$$\frac{\mathrm{d}\omega}{A\omega + B} = \mathrm{d}t.$$

Integration of both sides from $t = 0$ to t gives

$$\int_{t=0}^{t} \frac{\mathrm{d}\omega}{A\omega + B} = \int_0^t \mathrm{d}t', \tag{14}$$

that is,

$$\ln\frac{A\omega + B}{B} = At,$$

where the initial condition $\omega = 0$ at $t = 0$ was used. This can be written as

$$\omega = \frac{B}{A}\,e^{At} - \frac{B}{A},$$

that is,

$$f(t) = \frac{\omega}{2\pi} = \frac{I\sigma_{\mathrm{abs}}}{32\pi^3\eta R^3\nu}\left(1 - e^{-\frac{20\pi\eta R}{m}\,t}\right). \tag{15}$$

(6) Integration of (13) from t_1 to t_2,

$$\int_{t_1}^{t_2} \frac{\mathrm{d}\omega}{A\omega + B} = \int_{t_1}^{t_2} \mathrm{d}t',$$

has the solution

$$\omega(t_2) = \omega(t_1)\,e^{A(t_2 - t_1)} - \frac{B}{A}\left[1 - e^{A(t_2 - t_1)}\right],$$

where A is a constant and B is proportional to I (see the expressions above). This gives

$$\omega(nT+T_1) = \omega(nT)\, e^{AT_1} - \frac{B}{A}\,(1-e^{AT_1}), \tag{16}$$

$$\omega[(n+1)T] = \omega(nT+T_1)\, e^{AT_2}, \tag{17}$$

where $T = T_1 + T_2$. Under the square-wave pulsed laser light, in each pulse period T the particle first turns with acceleration for the time T_1, then with deceleration for the time T_2. Therefore

$$\omega[(n+1)T] = \omega(nT)\, e^{AT} - \frac{B}{A}\, e^{AT_2}\,(1-e^{AT_1}), \tag{18}$$

with $\omega_n \equiv \omega(nT)$. In the same way,

$$\begin{aligned} \omega_n &= \omega_{n-1}e^{A(T_1+T_2)} - \frac{B}{A}e^{AT_2}\big(1-e^{AT_1}\big) \\ \omega_{n-1}e^{A(T_1+T_2)} &= \omega_{n-2}e^{2A(T_1+T_2)} - \frac{B}{A}e^{AT_2}\big(1-e^{AT_1}\big)e^{A(T_1+T_2)} \\ &\vdots \\ \omega_1 e^{(n-1)A(T_1+T_2)} &= \omega_0 e^{nA(T_1+T_2)} - \frac{B}{A}e^{AT_2}\big(1-e^{AT_1}\big)e^{(n-1)A(T_1+T_2)}, \end{aligned}$$

where ω_0 is the rotation rate of the particle before the first pulse, $\omega_0 = 0$. The sum of all these equations gives

$$f_n = \frac{\omega_n}{2\pi} = -\frac{B}{2\pi A}\,\frac{1-e^{AT_1}}{e^{-AT_2}-e^{AT_1}}\,\big(1-e^{nA(T_1+T_2)}\big). \tag{19}$$

[**Method 2**: from the recursion (18) the general term can also be written out by induction:

$$\begin{aligned} \omega_n &= -\frac{B}{A}e^{AT_2}\big(1-e^{AT_1}\big) + \omega_{n-1}e^{AT} = -\frac{B}{A}e^{AT_2}\big(1-e^{AT_1}\big)\big(1+e^{AT}+\cdots+e^{(n-1)AT}\big) + \omega_0 e^{nAT} \\ &= -\frac{B}{A}e^{AT_2}\big(1-e^{AT_1}\big)\,\frac{1-e^{nAT}}{1-e^{AT}} = -\frac{B}{A}\,\frac{1-e^{AT_1}}{e^{-AT_2}-e^{AT_1}}\,\big(1-e^{nA(T_1+T_2)}\big). \,] \end{aligned}$$

[**Method 3**: write (18) as $\omega_{n+1} = C\omega_n + D$ with $C = e^{AT}$, $D = -\frac{B}{A}e^{AT_2}(1-e^{AT_1})$, that is, $\omega_{n+1} + \frac{D}{C-1} = C\Big(\omega_n + \frac{D}{C-1}\Big)$. The sequence $\Big\{\omega_n + \frac{D}{C-1}\Big\}$ is geometric with the ratio C, so

$$\omega_n = \Big(\omega_0 + \frac{D}{C-1}\Big)C^{n-1} - \frac{D}{C-1} = \frac{D}{C-1}\,(C^{n-1}-1)\cdot C = -\frac{B}{A}\,\frac{1-e^{AT_1}}{e^{-AT_2}-e^{AT_1}}\,\big(1-e^{nAT}\big). \,]$$

From this,

$$f_{n+1} > f_n,$$

and for $n \to \infty$ the limit of f_n is

$$f_{n\to\infty} = -\frac{B}{2\pi A}\,\frac{1-e^{AT_1}}{e^{-AT_2}-e^{AT_1}} = \frac{I\sigma_{\mathrm{abs}}}{32\pi^3\eta R^3\nu}\,\frac{1-e^{-\frac{20\pi\eta R}{m}T_1}}{e^{\frac{20\pi\eta R}{m}T_2}-e^{-\frac{20\pi\eta R}{m}T_1}}. \tag{20}$$

Problem 5 (racing car with aerofoil)

Solution

(1) Let N be the vertical upward support of the ground on the car. The force balance in the vertical direction gives

$$mg = N + c_B v^2. \tag{1}$$

The car drives on a horizontal straight road, so $N \geq 0$, and therefore

$$v \leq v_1 = \sqrt{\frac{mg}{c_B}}. \tag{2}$$

Write F_p for the magnitude of the traction produced by the engine (realized through the static friction between the car and the ground). When the car drives with the constant speed v, the engine power P is

$$P = F_p v. \tag{3}$$

Because the car drives with constant speed, in the horizontal direction the traction balances the rolling resistance and the air drag:

$$F_p = \mu_R N + c_A v^2. \tag{4}$$

The traction of the car is realized through the static friction of the ground, so it must be smaller than the maximum static friction:

$$F_p \leq \mu_S N. \tag{5}$$

Equations (1), (4), (5) give

$$v \leq v_5 = \sqrt{\frac{(\mu_S - \mu_R)\, mg}{(\mu_S - \mu_R)\, c_B + c_A}}. \tag{6}$$

A comparison of (2) and (6) with $\mu_S > \mu_R$, $c_B > 0$, $c_A > 0$ shows that (6) is the stronger restriction; v only needs to satisfy (6). Equations (1), (3), (4) express the power P as the following function of v:

$$\left[\ P = \mu_R m g v + (c_A - \mu_R c_B)\, v^3\ \right]. \tag{7}$$

To find from (7) the largest power $P_{\max}$ under the condition (2)/(6), discuss as follows:
(A) If $c_A \geq \mu_R c_B$, then P grows monotonically with v. The power is largest when v takes the largest value v_5 of (6):

$$P_{\max} = c_A \mu_S \sqrt{(\mu_S - \mu_R)} \left(\frac{mg}{c_B(\mu_S - \mu_R) + c_A}\right)^{3/2}. \tag{8}$$

(B) If $c_A < \mu_R c_B$, then P first grows and then falls with v. Without regard to the condition (6) for the moment, write v_2 for the speed at which P is largest. Differentiation of (7) with respect to v, set to zero,

$$\left.\frac{\mathrm{d}P}{\mathrm{d}v}\right|_{v=v_2} = \mu_R m g - 3(\mu_R c_B - c_A)\, v_2^2 = 0,$$

gives

$$v_2 = \sqrt{\frac{\mu_R m g}{3(\mu_R c_B - c_A)}}. \tag{9}$$

A comparison of (6) and (9) leads to two cases:

(B1) If $\mu_R c_B > c_A \geq \frac{2\mu_R c_B}{3}\left[1-\frac{\mu_R}{3\mu_S-2\mu_R}\right]$, then $v_5 < v_2$; the largest power is still taken at v_5, and $P_{\max} = P(v_5)$ is still given by (8).
(B2) If $c_A < \frac{2\mu_R c_B}{3}\left[1-\frac{\mu_R}{3\mu_S-2\mu_R}\right]$, then $v_5 > v_2$; the largest power is taken at v_2:

$$P_{\max} = P(v_2) = \frac{2\sqrt{3}\,(mg\mu_R)^{3/2}}{9\sqrt{\mu_R c_B - c_A}}. \tag{10}$$

(2) Let v be the magnitude of the tangential velocity of the car on its uniform circular motion, and N' the vertical upward support of the ground. The vertical force balance gives

$$mg + \varepsilon c_B v^2 = N', \tag{11}$$

where $\varepsilon = 1$ for the rear-up aerofoil and $\varepsilon = -1$ for the front-up aerofoil. From $N' \geq 0$: for $\varepsilon = -1$,

$$v \leq \sqrt{\frac{mg}{c_B}}. \tag{12}$$

Let $F'_\parallel$ be the forward pull of the engine on the car along the direction of motion (realized through the static friction between the car and the ground); it balances the oncoming air drag:

$$F'_\parallel = c_A v^2.$$

On the uniform circular motion in the horizontal plane, the radial component $F'_\perp$ of the static friction provides the centripetal force:

$$F'_\perp = \frac{mv^2}{r}. \tag{13}$$

The total friction that the ground can give the car is not larger than the maximum static friction $\mu_S N'$, so

$$F'^2_\parallel + F'^2_\perp \leq \left(\mu_S N'\right)^2.$$

Substitution of (11) and (13) gives

$$a\,v^4 + b\,v^2 + c \geq 0, \tag{14}$$

where

$$a = c_B^2\mu_S^2 - \frac{m^2}{r^2} - c_A^2, \qquad b = 2\varepsilon\, mgc_B\mu_S^2, \qquad c = m^2g^2\mu_S^2 > 0.$$

The left side of the inequality (14) is a linear or quadratic function of v^2. The speed ranges that satisfy (14) are found case by case:
Case (A): $a = 0$.
Case (A.1): if $\varepsilon = 1$, every speed v satisfies (14), so

$$v_{\max} = \infty. \tag{15}$$

Case (A.2): if $\varepsilon = -1$, then

$$v \leq \sqrt{\frac{mg}{2c_B}}. \tag{16}$$

Case (B): $a \neq 0$. Write v_1^2 and v_2^2 for the two roots of (14) with the equality sign:

$$v_1^2 = -\frac{mg\mu_S}{\varepsilon c_B\mu_S + \sqrt{m^2/r^2 + c_A^2}}, \qquad v_2^2 = -\frac{mg\mu_S}{\varepsilon c_B\mu_S - \sqrt{m^2/r^2 + c_A^2}}. \tag{17}$$

Further cases:

Case (B1): if $a > 0$, that is $c_B\mu_S > \sqrt{m^2/r^2 + c_A^2}$, the left side of (14) is an upward-opening quadratic function of v^2, and $v_1^2 v_2^2 = c/a > 0$.

Case (B1.1): if $\varepsilon = 1$, then $v_2^2 < v_1^2 < 0$ — both intersections with the horizontal axis are negative, so every speed v satisfies (14):

$$v_{\max} = \infty. \tag{18}$$

Case (B1.2): if $\varepsilon = -1$, then $0 < v_2^2 < v_1^2$ — both intersections are positive, and (14) requires

$$v \geq \sqrt{v_1^2} \quad \text{or} \quad 0 \leq v \leq v_{\max} = \sqrt{v_2^2}.$$

The former does not satisfy (12); the latter does, so

$$0 \leq v \leq v_{\max} = \sqrt{v_2^2} = \sqrt{\frac{mg\mu_S}{c_B\mu_S + \sqrt{m^2/r^2 + c_A^2}}}. \tag{19}$$

Case (B2): if $a < 0$, then for both $\varepsilon = \pm 1$,

$$v_1^2 < 0 < v_2^2,$$

and the left side of (14) is a downward-opening quadratic function of v^2 with one positive and one negative intersection. (14) requires $0 \leq v^2 \leq v_2^2$, that is,

$$0 \leq v \leq v_{\max} = \sqrt{v_2^2} = \sqrt{\frac{mg\mu_S}{-\varepsilon c_B\mu_S + \sqrt{m^2/r^2 + c_A^2}}}. \tag{20}$$

Summary of the cases: for the rear-up aerofoil, if $a < 0$ the allowed speed range is given by (20) with $\varepsilon = 1$; in the other cases the allowed range is unbounded, by (18). For the front-up aerofoil, the requirements of (16), (19), (20) combine into one expression: the allowed speed range is given by (19). A comparison of all cases shows: under all conditions, the rear-up aerofoil allows a larger maximum speed than the front-up aerofoil.

Problem 6 (Hyperloop)

Solution

(1) For each gas-blowing fine hole, the pressures inside and outside are P and P_{low}; the gas flow speed inside the hole is zero, and the height difference between inside and outside is negligible. Let v be the steady flow speed just outside the fine hole. The Bernoulli equation for the flow tube through the hole gives

$$P = P_{\text{low}} + \frac{1}{2}\rho v^2. \tag{1}$$

For a single fine hole, the magnitude of the effective thrust perpendicular to the hole cross-section (radial) caused by the ejected gas is

$$f = -\frac{\mathrm{d}m}{\mathrm{d}t}\, v = \rho s v^2 = 2\,(P - P_{\text{low}})\, s. \tag{2}$$

For the transport car, the net effective pressure provided by the fine holes is

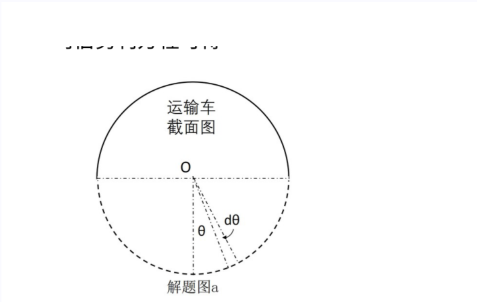
$$P_{\text{eff}} = \frac{f}{s} = 2\,(P - P_{\text{low}}).$$

In the vertical direction the car feels gravity and the net effective press. Set up polar coordinates with the origin at the center of the cross-section and the polar axis vertical, downward; write θ for the polar angle (see the solution figure a). The hole area is very small, so the separate-variable continuum method can be used. By the problem, all car surfaces except the fine holes feel the ambient pressure P_{low}. The vertical upward force provided by the narrow strip of the car between the polar angles [θ and $\theta + \mathrm{d}\theta$] is

$$\left[\,\mathrm{d}N = P_{\text{eff}}\, nslR\, \mathrm{d}\theta\, \cos\theta\,\right],$$

so the total upward force provided by the fine holes between [$-\pi/2$ and $\pi/2$] is

$$N = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{\text{eff}}\, nslR\, \mathrm{d}\theta \cos\theta = 2nslR\, P_{\text{eff}} = 4nslR\,(P - P_{\text{low}}). \tag{3}$$



Labels: 运输车截面图 *= cross-section of the car;* 解题图 *a = solution figure a.*

So that the car is just lifted off the tube wall, the vertical force balance gives

$$\left[\,N = Mg\,\right], \tag{4}$$

and therefore

$$\left[\,P = P_{\text{low}} + \frac{Mg}{4nslR}\,\right]. \tag{5}$$

(2) Let the resistance of a rail piece of length D be [R_1], and the resistance of each wire on the car $R_2 = 2R_1$. Then

$$R_1 = \rho_{\text{d}}\, \frac{D}{\pi r_{\text{d}}^2}. \tag{6}$$

Take the moment when wire 1 (see figure d) enters the field as the zero of time. At a time t before wire 2 has entered the field, let the speed of the wires be $v(t)$; the emf induced by the cutting of the field lines is

$$E = B \cdot 2D \cdot v(t). \tag{7}$$

Let I be the current in the loop of the wires 1, 2 and the rails. By Ohm's law,

$$E = I\,(2R_2 + 2R_1). \tag{8}$$

The force on the current of wire 1 in the field opposes the motion of the loop. By Newton's law,

$$-M\dot{v}(t) = I \cdot 2D \cdot B. \tag{9}$$

Equations (6)–(9), with the initial condition $v(t=0)=v_0$, give

$$v(t) = v_0 \exp\Big(-\frac{2B^2D^2}{3MR_1}\,t\Big). \tag{10}$$

Let t_1 be the moment at which wire 2 reaches the field boundary; then

$$\int_0^{t_1} v(t)\,\mathrm{d}t = \int_0^{t_1} v_0 \exp\Big(-\frac{2B^2D^2}{3MR_1}\,t\Big)\mathrm{d}t = D. \tag{11}$$

Equation (11) gives t_1, and (10) gives the corresponding speed v_1, the sliding speed of the car after wire 2 has entered the field:

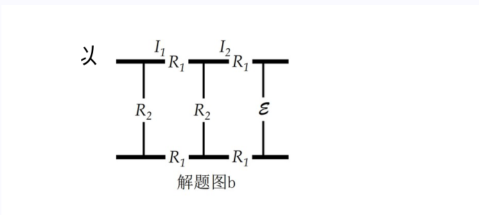
$$t_1 = -\frac{3MR_1}{2B^2D^2}\ln\bigg(1-\frac{2B^2D^3}{3MR_1v_0}\bigg), \qquad v_1 = v(t=t_1) = v_0 - \frac{2\pi B^2D^2r_\mathrm{d}^2}{3M\rho_\mathrm{d}}. \tag{12}$$

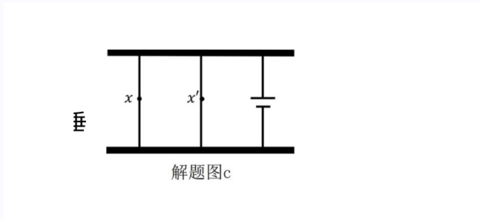
(After both wires are in the uniform field, the total emf of the loop is zero and the car slides on with the constant speed v_1.) [If the current is approximated as flowing exactly along the rail axes, then D in (7)–(10) and (12) must be written as $D + r_\mathrm{d}$.]
(3) Write I_1 and I_2 for the currents on the sliding rails (as in the equivalent circuit, solution figure b); by the left–right symmetry of the wires, the currents on the corresponding rail pieces of the two sides are equal. For the loop of the source, the rails and wire 1, and for the loop of the source, the rails and wire 2, Ohm's law gives

$$\big[\,\mathcal{E} = I_2\cdot 2R_1 + I_1\,(2R_1+R_2), \qquad \mathcal{E} = I_2\cdot 2R_1 + (I_2-I_1)\,R_2\,\big]. \tag{13}$$

Equations (6) and (13) give

$$I_1 = \frac{1}{10}\,\frac{\mathcal{E}}{R_1}, \qquad I_2 = \frac{3}{10}\,\frac{\mathcal{E}}{R_1}. \tag{14}$$





解题图 *b/c = solution figures b/c.*

When the rails and the source line carry currents, they produce at the wire positions a magnetic field perpendicular to the rail–wire plane, and the current-carrying wires feel Ampère forces in this field. First calculate the field produced by the rails at the wire positions. By the given formula, the geometry gives, for a rail piece adjacent to a wire and for the next-adjacent piece, the angles

$$\Big[\text{adjacent: } |\cos\theta_1| = 0,\ \ |\cos\theta_2| = \frac{D}{\sqrt{r_0^2+D^2}}\,\Big]; \qquad \text{next-adjacent: } |\cos\theta_1| = \frac{D}{\sqrt{r_0^2+D^2}},\ \ |\cos\theta_2| = \frac{2D}{\sqrt{r_0^2+4\cdots}} \tag{$*$}$$

Therefore the magnitudes of the magnetic induction produced by one rail piece at the position of the adjacent and of the next-adjacent wire are

$$\Big[\text{adjacent: } |B_n| = \frac{\mu_0}{4\pi}\frac{I}{r_0}\,\frac{D}{\sqrt{r_0^2+D^2}}\,\Big]; \qquad \text{next-adjacent: } |B_{nn}| = \frac{\mu_0}{4\pi}\frac{I}{r_0}\left(\frac{2D}{\sqrt{r_0^2+4D^2}} - \frac{D}{\sqrt{r_0^2+D^2}}\right). \tag{15}$$

By the Ampère formula, the force of the field of the adjacent and of the next-adjacent rail piece on a wire with the current I_c is along the rail direction, with the magnitudes

$$\text{adjacent:}\quad F=\int_{r_d}^{2D+r_d}|B_n|\,I_c\,\mathrm{d}r_0=\int_{r_d}^{2D+r_d}\frac{\mu_0}{4\pi}\frac{I\,I_c}{r_0}\frac{D}{\sqrt{r_0^2+D^2}}\,\mathrm{d}r_0=\frac{\mu_0 I\,I_c}{4\pi}\,\Pi_n,$$

$$\text{next-adjacent:}\quad F=\int_{r_d}^{2D+r_d}|B_{nn}|\,I_c\,\mathrm{d}r_0=\frac{\mu_0 I\,I_c}{4\pi}\,(\Pi_{nn}-\Pi_n), \tag{16}$$

where the given integral formula was used, with

$$\Big[\,\Pi_n=\ln\left(\frac{2D+r_d}{r_d}\,\frac{D+\sqrt{D^2+r_d^2}}{D+\sqrt{D^2+(2D+r_d)^2}}\right)\Big],\qquad \Pi_{nn}=\ln\left(\frac{2D+r_d}{r_d}\,\frac{2D+\sqrt{4D^2+r_d^2}}{2D+\sqrt{4D^2+(2D+r_d)^2}}\right).$$

For wire 1, the current is $[I_1]$; the current on the adjacent rail piece of one side is $[I_1]$, and the current on the next-adjacent piece is I_2. For wire 2, the current is $[I_2-I_1]$, and the currents on the two adjacent rail pieces of one side are $[I_1$ and $I_2]$. By the problem the upper end of the emf $\mathcal{E}$ is positive, and the Ampère forces of the rails of both sides on wires 1 and 2 point to the left. The magnitudes of the total Ampère forces F_{1G} and F_{2G} of the rails on wires 1 and 2 are

$$|F_{1G}|=\frac{\mu_0 I_1}{2\pi}\big[I_1\Pi_n+I_2(\Pi_{nn}-\Pi_n)\big],\qquad |F_{2G}|=\frac{\mu_0(I_2-I_1)}{2\pi}\,(I_2+I_1)\,\Pi_n. \tag{17}$$

Second, the source line produces at the point x of wire 1 and at the point x' of wire 2 (see solution figure c) magnetic inductions perpendicular to the wires, of magnitudes

$$|B_1|=\frac{\mu_0}{4\pi}\frac{I_2}{2D}\left(\frac{2D-x}{\sqrt{(2D-x)^2+(2D)^2}}+\frac{x}{\sqrt{x^2+(2D)^2}}\right),\qquad |B_2|=\frac{\mu_0}{4\pi}\frac{I_2}{D}\left(\frac{2D-x'}{\sqrt{(2D-x')^2+D^2}}+\frac{x'}{\sqrt{x'^2+D^2}}\right) \tag{18}$$

By the left-hand rule, the Ampère forces F_{1E} and F_{2E} of the source line on the wires 1 and 2 fixed on the car point to the left; by the Ampère formula their magnitudes are

$$|F_{1E}|=\int_0^{2D}I_1\,|B_1(x)|\,\mathrm{d}x=\frac{1}{2\pi}\,\mu_0 I_1 I_2\,(\sqrt{2}-1),\qquad |F_{2E}|=\int_0^{2D}(I_2-I_1)\,|B_2(x)|\,\mathrm{d}x=\frac{1}{2\pi}\,\mu_0(I_2-I_1)I_2\,(\sqrt{5}- \tag{19}$$

The forces between the wires themselves are internal forces of the system and cause no acceleration of the whole; they need not be considered. From (17) and (19), the total Ampère force on the wires 1 and 2 is

$$\big[\,F_{\text{total}}=|F_{1G}|+|F_{2G}|+|F_{1E}|+|F_{2E}|\,\big].$$

By Newton's second law, the total Ampère force F_{total} gives the transport car the acceleration

$$a=\frac{F_{\text{total}}}{M}=\frac{3\pi\mu_0 r_{\mathrm{d}}^4\,\mathcal{E}^2}{200\,(\rho_{\mathrm{d}}D)^2M}\left[\sqrt{2}+2\sqrt{5}-3+\ln\left(\frac{(2D+r_d)^3}{r_d^3}\,\frac{\big(D+\sqrt{D^2+r_d^2}\big)^2}{\big(D+\sqrt{D^2+(2D+r_d)^2}\big)^2}\,\frac{2D+\sqrt{4D^2+}}{2D+\sqrt{4D^2+(2D}}\right.\right. \tag{20}$$

Problem 7 (gravitational waves)

Solution

(1)(i) While the gravitational wave passes, near the origin the distance dr between two neighbouring points (x, y) and $(x + \mathrm{d}x, y + \mathrm{d}y)$ of the x-y plane satisfies

$$\mathrm{d}r = \sqrt{(1 + A\sin\omega t)(\mathrm{d}x)^2 + (1 - A\sin\omega t)(\mathrm{d}y)^2}. \tag{1}$$

When there is no gravitational wave, (x, y) can be written in polar coordinates (ρ, θ):

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta. \tag{2}$$

While the gravitational wave passes, for a fixed direction θ,

$$\mathrm{d}r = \sqrt{(1 + A\sin\omega t)\cos^2\theta + (1 - A\sin\omega t)\sin^2\theta}\,\mathrm{d}\rho = \sqrt{1 + A\sin\omega t\,(\cos^2\theta - \sin^2\theta)}\,\mathrm{d}\rho = \sqrt{1 + A\sin\omega t\cos} \tag{3}$$

In the direction at the angle θ to the x axis, the distance between the two micro-detectors is

$$D = R\sqrt{1 + A\sin\omega t\cos 2\theta}. \tag{4}$$

With the amplitude $A \ll 1$,

$$D = R\Big(1 + \frac{1}{2}A\cos 2\theta\sin\omega t\Big). \tag{5}$$

The deviation of the distance between the two micro-detectors from R is

$$D - R = R\Big(\frac{1}{2}A\cos 2\theta\Big)\sin\omega t. \tag{6}$$

This is a simple harmonic oscillation around R with the amplitude $\frac{1}{2}RA\cos 2\theta$ and the angular frequency ω.

(ii) When no gravitational wave passes, the micro-detector array lies on the circle of radius R around the origin in the x-y plane:

$$x^2 + y^2 = R^2. \tag{7}$$

When the gravitational wave passes the origin, the coefficients f_1 and f_2 change; so that space-time becomes formally flat, define

$$X = \sqrt{1 + A\sin\omega t}\,x, \qquad Y = \sqrt{1 - A\sin\omega t}\,y. \tag{8}$$

Equations (7) and (8) give

$$\frac{X^2}{\big[R\sqrt{1 + A\sin\omega t}\,\big]^2} + \frac{Y^2}{\big[R\sqrt{1 - A\sin\omega t}\,\big]^2} = 1. \tag{9}$$

(The equation

$$\frac{X^2}{\big[R(1 + \frac{1}{2}A\sin\omega t)\big]^2} + \frac{Y^2}{\big[R(1 - \frac{1}{2}A\sin\omega t)\big]^2} = 1$$

is also treated as correct.) At a given moment t, the micro-detector array is therefore distributed on the ellipse (9); the lengths of the major and the minor axis of this ellipse change with time.

(iii) At $z = 0$ the two waves interfere; the combined result is

$$A_{\text{tot}}\sin(\omega_{\text{tot}}t + \Phi) = A\sin\omega t + A\sin(\omega t + \phi). \tag{10}$$

This is equivalent to the disturbance, at $z = 0$, of one gravitational wave of amplitude A_{tot} and frequency ω_{tot}, with Φ independent of the time t. By the trigonometric sum-to-product formula, the right side of (10) is

$$2A\cos\frac{\phi}{2}\,\sin\Big(\omega t+\frac{\phi}{2}\Big). \tag{11}$$

By the result of part (1)(i), the disturbance of the distance between the two micro-detectors is

$$\Delta r = D - R = \Big(AR\cos\frac{\phi}{2}\cos 2\theta\Big)\sin\Big(\omega t+\frac{\phi}{2}\Big). \tag{12}$$

The amplitude of the disturbance of the distance is smallest when

$$AR\cos\frac{\phi}{2}\cos 2\theta = 0,$$

with the solutions

$$\theta = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4},\quad \phi \text{ arbitrary;}\quad \text{or}\quad \theta \text{ arbitrary,}\quad \phi=\pi. \tag{13}$$

(Or: $\theta = \dfrac{(2k+1)\pi}{4}$, ϕ arbitrary; or θ arbitrary, $\phi = (2k+1)\pi$.) The amplitude of the disturbance of the distance is largest when

$$\Big|AR\cos\frac{\phi}{2}\cos 2\theta\Big| = AR,$$

with the solutions

$$\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\qquad \text{and}\qquad \phi = 0. \tag{14}$$

(Note: the answer $\theta = \dfrac{k\pi}{2}$ and $\phi = 2k\pi$ is also correct.)

(2) The two stars of the binary system have equal masses, so the center of mass is at the equal distance $\dfrac{L}{2}$ from the two stars. Assume that at the time $t = 0$ the coordinates of the two stars in their center-of-mass system are approximately

$$x_1'(0) = \frac{L}{2},\ y_1'(0) = 0;\qquad x_2'(0) = -\frac{L}{2},\ y_2'(0) = 0. \tag{15}$$

At any time t the two coordinate pairs are

$$x_1'(t) = \frac{L}{2}\cos\Omega t,\quad y_1'(t) = \frac{L}{2}\sin\Omega t;\qquad x_2'(t) = -\frac{L}{2}\cos\Omega t,\quad y_2'(t) = -\frac{L}{2}\sin\Omega t. \tag{16}$$

Substitution into the given expressions for I_1, I_2 gives

$$f_1(t) = \frac{8\pi G}{c^4}\frac{2}{l}\frac{\mathrm{d}^2}{\mathrm{d}t^2}M\big[x_1'^2(t)+x_2'^2(t)\big] = -\frac{16\pi G}{c^4}\,\frac{\Omega^2 ML^2}{l}\cos 2\Omega t,$$

$$f_2(t) = \frac{8\pi G}{c^4}\frac{2}{l}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x_1'(t)\,y_2'(t)\big]M = \frac{8\pi G}{c^4}\,\frac{\Omega^2 ML^2}{l}\sin 2\Omega t.$$

The gravitational wave at the distance l from the center of mass of the binary system is therefore

$$f_1 = -\frac{16\pi G}{c^4}\,\frac{\Omega^2 ML^2}{l}\cos 2\Omega\Big(t-\frac{l}{c}\Big),\qquad f_2 = \frac{8\pi G}{c^4}\,\frac{\Omega^2 ML^2}{l}\sin 2\Omega\Big(t-\frac{l}{c}\Big). \tag{17}$$

By (17), the amplitudes of f_1, f_2 are $\dfrac{16\pi G}{c^4}\dfrac{\Omega^2 ML^2}{l}$ and $\dfrac{8\pi G}{c^4}\dfrac{\Omega^2 ML^2}{l}$; the angular frequency Ω of the binary system is one half of the gravitational-wave frequency ω: $\Omega = \dfrac{\omega}{2}$. Newton's law of gravitation and Newton's second law give

$$G\,\frac{M^2}{L^2} = M\Omega^2\frac{L}{2} = M\Big(\frac{\omega}{2}\Big)^2\frac{L}{2}, \tag{18}$$

so the distance between the two stars is

$$L = 2\left(\frac{GM}{\omega^2}\right)^{1/3}. \tag{19}$$

Substitution into the amplitude expressions gives the amplitudes of the gravitational wave f_1, f_2:

$$\frac{16\pi}{lc^4}\,\omega^{2/3}\,(GM)^{5/3}, \qquad \frac{8\pi}{lc^4}\,\omega^{2/3}\,(GM)^{5/3}. \tag{20}$$

Problem 8 (positronium)

Solution

(1) Consider the ground state of the hydrogen atom. Let r be the distance of the electron from the proton and v the speed of the electron relative to the proton. In the center-of-mass system of the p–e$^-$ system, the equations of motion of the proton and of the electron are

$$k\,\frac{e^2}{r^2} = m_i\,\frac{v_i^2}{r_i} = \mu_{\mathrm{H}}\,\frac{v^2}{r}, \quad i = \mathrm{p}, \mathrm{e}^-; \qquad \mu_{\mathrm{H}} = \frac{m_{\mathrm{p}} m_{\mathrm{e}}}{m_{\mathrm{p}} + m_{\mathrm{e}}}, \tag{1}$$

where m_{p}, m_{e} are the masses of the proton and the electron, and r_{p}, r_{e} and v_{p}, v_{e} are the distances and the speeds of the proton and the electron relative to the center of mass:

$$r_{\mathrm{p}} = \frac{m_{\mathrm{e}}}{m_{\mathrm{p}} + m_{\mathrm{e}}}\,r, \quad r_{\mathrm{e}} = \frac{m_{\mathrm{p}}}{m_{\mathrm{p}} + m_{\mathrm{e}}}\,r, \qquad v_{\mathrm{p}} = \frac{m_{\mathrm{e}}}{m_{\mathrm{p}} + m_{\mathrm{e}}}\,v, \quad v_{\mathrm{e}} = \frac{m_{\mathrm{p}}}{m_{\mathrm{p}} + m_{\mathrm{e}}}\,v;$$

e is the proton charge, and μ_{H} is the reduced mass of the p–e$^-$ system. With the quantization condition of the hydrogen atom that takes the finite proton mass into account, the total orbital angular momentum in the center-of-mass system is

$$L = m_1 v_1 r_1 + m_2 v_2 r_2 = \mu_{\mathrm{H}} v r = n\,\frac{h}{2\pi}, \qquad n = 1, 2, \ldots, \tag{2}$$

where n is the quantum number of the hydrogen energy level. The ground-state ($n = 1$) energy of the hydrogen atom is

$$E^{\mathrm{H}}_{n=1} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{ke^2}{r} = \frac{1}{2}\mu_{\mathrm{H}} v^2 - \frac{ke^2}{r} = -\mu_{\mathrm{H}}\Big(\frac{2\pi}{h}\Big)^2\frac{\left(ke^2\right)^2}{2}. \tag{3}$$

The ground-state energy of hydrogen is proportional to its reduced mass μ_{H}. The reduced mass of the e$^+$–e$^-$ system is $\mu_{\mathrm{Ps}} = m_{\mathrm{e}}/2$. In analogy with (3), with the given data, the ground-state ($n = 1$) energy of positronium (Ps) is

$$E^{\mathrm{Ps}}_{n=1} = -\mu_{\mathrm{Ps}}\Big(\frac{2\pi}{h}\Big)^2\frac{\left(ke^2\right)^2}{2} = \frac{\mu_{\mathrm{Ps}}}{\mu_{\mathrm{H}}}\,E^{\mathrm{H}}_{n=1} = \frac{m_{\mathrm{p}} + m_{\mathrm{e}}}{2m_{\mathrm{p}}}\,E^{\mathrm{H}}_{n=1} = -\frac{1}{2}\times\frac{1837}{1836}\times 13.60\ \mathrm{eV} \approx -6.804\ \mathrm{eV}. \tag{4}$$

(2)(i) The rest mass m_{Ps} of Ps is

$$m_{\mathrm{Ps}} = 2m_{\mathrm{e}} + \frac{E_{n=1}^{\mathrm{Ps}}}{c^2} \approx 2m_{\mathrm{e}}, \tag{5}$$

where $E_{n=1}^{\mathrm{Ps}} \ll m_{\mathrm{e}}c^2$ (see (4)) was used. Let the ground-state positronium atom move relative to the laboratory frame with the speed v_0 ($v_0 \ll c$) and annihilate into the two photons γ_1 and γ_2. The conservation of energy and momentum in the annihilation gives

$$2m_{\mathrm{e}}c^2 = E_1 + E_2, \tag{6}$$

$$2m_{\mathrm{e}}v_0 = p_1\cos\theta_1 + p_2\cos\theta_2, \tag{7}$$

$$0 = p_1\sin\theta_1 + p_2\sin\theta_2, \tag{8}$$

where p_1 and p_2 are the magnitudes of the momenta of the two photons γ_1 and γ_2, with

$$E_1 = p_1c, \qquad E_2 = p_2c. \tag{9}$$

Equations (6)–(9) give

$$E_1 = m_{\mathrm{e}}c^2\left(1 + \frac{v_0}{c}\cos\theta_1\right), \tag{10}$$

$$E_2 = m_{\mathrm{e}}c^2\left(1 - \frac{v_0}{c}\cos\theta_1\right), \tag{11}$$

$$v_0 = \frac{\Delta\theta}{2\sin\theta_1}\,c, \tag{12}$$

where $\Delta\theta = \theta_2 - (\theta_1 + \pi)$, and $\Delta\theta \ll 1$ was used in the derivation (if $v_0 = 0$, then $\Delta\theta = 0$; for $v_0 \ll c$, $\Delta\theta \propto \dfrac{v_0}{c} \ll 1$). With the given data $\theta_1 = \dfrac{\pi}{3}$, $\Delta\theta = 3.462 \times 10^{-3}$, equations (10)–(12) give

$$v_0 = 2.000\times10^{-3}c = 5.996\times10^5\ \mathrm{m/s}, \qquad E_1 = 0.5115\ \mathrm{MeV}, \qquad E_2 = 0.5105\ \mathrm{MeV}. \tag{13}$$

(ii) From (5) and the exact versions of (10), (11) (with the equality, not the approximation),

$$E_{1,2} = \frac{m_{\mathrm{Ps}}c^2}{2} = m_{\mathrm{e}}c^2 + \frac{E_{n=1}^{\mathrm{Ps}}}{2},$$

so with (4)

$$\Delta E = E_{1,2} - m_{\mathrm{e}}c^2 = \frac{E_{n=1}^{\mathrm{Ps}}}{2} = -3.402\ \mathrm{eV}. \tag{14}$$

(3) A ground-state positronium atom (Ps) that moves with the speed $v_0 = c/2$ annihilates. Let p_1 and p_2 be the momenta of the two photons; the direction of the photon γ_1 makes the angle θ_1 with the momentum of the atom, and the direction of the photon γ_2 makes the angle θ_2 with the direction of the momentum of the ground-state Ps. Energy and momentum conservation in the annihilation give

$$\frac{m_{\mathrm{Ps}}c^2}{\sqrt{1 - \left(\frac{v_0}{c}\right)^2}} = E_1 + E_2, \tag{15}$$

$$p_{\mathrm{Ps}} = \frac{m_{\mathrm{Ps}}v_0}{\sqrt{1 - \left(\frac{v_0}{c}\right)^2}} = p_1\cos\theta_1 + p_2\cos\theta_2, \tag{16}$$

and the energies and momenta of the two photons still satisfy (8), (9). Equations (8), (9), (15), (16) give

$$E_1 = \frac{m_{\mathrm{Ps}}c^2}{2\sqrt{1-\left(\frac{v_0}{c}\right)^2}} \frac{1-\left(\frac{v_0}{c}\right)^2}{1-\frac{v_0}{c}\cos\theta_1}, \tag{17}$$

$$E_2 = \frac{m_{\mathrm{Ps}}c^2}{2\sqrt{1-\left(\frac{v_0}{c}\right)^2}} \frac{1-2\frac{v_0}{c}\cos\theta_1+\left(\frac{v_0}{c}\right)^2}{1-\frac{v_0}{c}\cos\theta_1}, \tag{18}$$

$$\sin\theta_2 = -\frac{1-\left(\frac{v_0}{c}\right)^2}{1-2\frac{v_0}{c}\cos\theta_1+\left(\frac{v_0}{c}\right)^2}\sin\theta_1 \quad \text{or} \quad \sin\theta_1 = -\frac{1-2\frac{v_0}{c}\cos\theta_1+\left(\frac{v_0}{c}\right)^2}{1-\left(\frac{v_0}{c}\right)^2}\sin\theta_2. \tag{19}$$

Substitution of the given data $v_0 = \frac{c}{2}$, $\theta_1 = \frac{\pi}{3}$ and of (5) into (17), (18), (19) gives

$$E_1 = E_2 = 0.5900 \text{ MeV}, \qquad \theta_2 = \frac{5\pi}{3}. \tag{20}$$

The other solution of (19), $\theta_2 = \frac{4\pi}{3}$, contradicts (16) and is rejected.