The 34th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2017 — official reference solutions

Translator's notes on this file.

- Equation numbers (1), (2), … in each problem follow the circled numbers of the Chinese original. Where the original prints the same circled number twice (Problem 3), the number is repeated here as printed. In Problem 2 the alternative solution restarts the numbering at (5), exactly as in the original.
- The original writes several subscripts in Chinese. They are rendered here as follows: M_{tot} (Problem 2, the mass of the equivalent gravitational source; the Chinese subscript itself is lost in the source file), v_{rel} (Problem 2, "relative"), v_{res} (Problem 5, "resultant") and s_{tot} (Problem 5, "total").
- Symbols and formulas lost in the source file are reconstructed and set in square brackets $[\,\cdot\,]$.

Problem 1 (two balls joined by a spring; the charged balls)

Solution

(1) As shown in figure I, the extension of the spring at the instant t is

$$u = l - l_0,$$

and we have

$$\mu\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = -k_0u, \tag{1}$$

where

$$\mu = \frac{m_\mathrm{a}m_\mathrm{b}}{m_\mathrm{a}+m_\mathrm{b}} \tag{2}$$

is the reduced mass of the two balls.

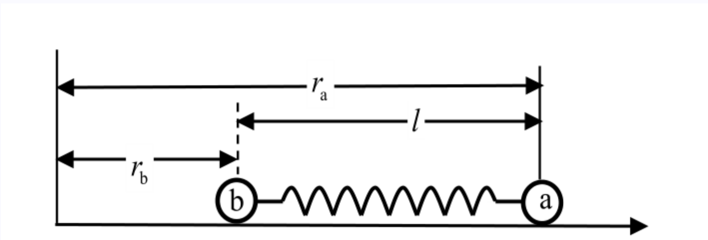


Figure I

By (1) and (2) the extension u of the spring obeys the dynamical equation of simple harmonic motion, and the frequency of the vibration is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_0}{\mu}} = \frac{1}{2\pi}\sqrt{\frac{m_\mathrm{a}+m_\mathrm{b}}{m_\mathrm{a}m_\mathrm{b}}\,k_0}, \tag{3}$$

where the last step used (2). The expression for the extension u of the spring at the instant t is

$$u = A\sin\omega t + B\cos\omega t, \tag{4}$$

in which A and B are constants to be determined. At $t = 0$ the spring is at its natural length, i.e.

$$u(0) = B = 0.$$

Substituting $B = 0$ into (4) gives

$$u = A \sin \omega t. \tag{5}$$

The velocity of a relative to b is

$$v'_{\mathrm{a}} = \frac{\mathrm{d}r_{\mathrm{a}}}{\mathrm{d}t} - \frac{\mathrm{d}r_{\mathrm{b}}}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} = A\omega \cos \omega t. \tag{6}$$

At $t = 0$ we have

$$v'_{\mathrm{a}}(0) = v_0 - 0 = A\omega. \tag{7}$$

From (6) and (7),

$$v'_{\mathrm{a}} = v_0 \cos \omega t. \tag{8}$$

The momentum of the system is conserved during the motion,

$$m_{\mathrm{a}} v_0 = m_{\mathrm{a}} v_{\mathrm{a}} + m_{\mathrm{b}} v_{\mathrm{b}}. \tag{9}$$

The velocity of ball a relative to the ground is

$$v_{\mathrm{a}} = v'_{\mathrm{a}} + v_{\mathrm{b}}. \tag{10}$$

From (3), (8), (9) and (10), the velocities of ball a and of ball b at the instant t are, respectively,

$$v_{\mathrm{a}} = \left[1 + \frac{m_{\mathrm{b}}}{m_{\mathrm{a}}} \cos\left(\sqrt{\frac{(m_{\mathrm{a}} + m_{\mathrm{b}})k_0}{m_{\mathrm{a}} m_{\mathrm{b}}}}\, t\right)\right] \frac{m_{\mathrm{a}}}{m_{\mathrm{a}} + m_{\mathrm{b}}} v_0, \tag{11}$$

$$v_{\mathrm{b}} = \left[1 - \cos\left(\sqrt{\frac{(m_{\mathrm{a}} + m_{\mathrm{b}})k_0}{m_{\mathrm{a}} m_{\mathrm{b}}}}\, t\right)\right] \frac{m_{\mathrm{a}}}{m_{\mathrm{a}} + m_{\mathrm{b}}} v_0. \tag{12}$$

(2) If the two balls carry equal charges of the same sign, of magnitude q, then at equilibrium of the system

$$K \frac{q^2}{L_0^2} = k_0 \left(L_0 - l_0\right). \tag{13}$$

From (13),

$$q = L_0 \sqrt{\frac{k_0}{K} \left(L_0 - l_0\right)}. \tag{14}$$

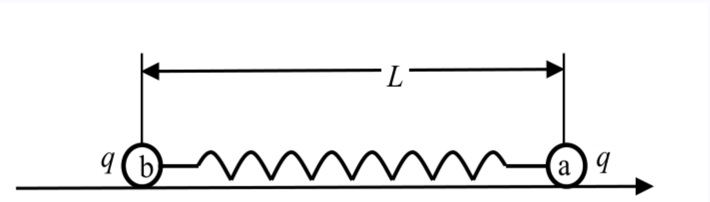


Figure II

Let the length of the spring at the instant t be L (see figure II); then

$$\mu \frac{\mathrm{d}^2 L}{\mathrm{d}t^2} = -k_0 \left(L - l_0\right) + K \frac{q^2}{L^2}. \tag{15}$$

Put $x = L - L_0$, the extension of the spring at the instant t measured from the equilibrium length L_0 of the spring; then (15) can be rewritten as

$$\mu \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k_0 x - k_0 (L_0 - l_0) + K \frac{q^2}{L_0^2} \left(1 + \frac{x}{L_0}\right)^{-2}. \tag{16}$$

When the system performs small oscillations,

$$x \ll L_0,$$

and therefore

$$\left(1 + \frac{x}{L_0}\right)^{-2} = 1 - 2\frac{x}{L_0} + O\left[\left(\frac{x}{L_0}\right)^2\right]. \tag{17}$$

With the help of the above expression, (16) can be written as

$$\mu \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \left[-k_0 (L_0 - l_0) + K \frac{q^2}{L_0^2}\right] - \left(k_0 + 2K \frac{q^2}{L_0^3}\right) x + O\left[\left(\frac{x}{L_0}\right)^2\right]. \tag{18}$$

Dropping $O[(x/L_0)^2]$ and using (13) or (14), (18) can be written as

$$\mu \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\left(k_0 + 2K \frac{q^2}{L_0^3}\right) x = -\frac{3L_0 - 2l_0}{L_0} k_0 x. \tag{19}$$

By (13) we know that $L_0 > l_0$, hence $3L_0 - 2l_0 > 0$. The small oscillations of the system obey the dynamical equation of simple harmonic motion, and the frequency of the vibration is

$$f = \frac{1}{2\pi} \sqrt{\frac{\left(\frac{3L_0 - 2l_0}{L_0}\right) k_0}{\mu}} = \frac{1}{2\pi} \sqrt{\left(\frac{3L_0 - 2l_0}{L_0}\right)\left(\frac{m_\mathrm{a} + m_\mathrm{b}}{m_\mathrm{a} m_\mathrm{b}}\right) k_0}, \tag{20}$$

where the last step used (2).

Problem 2 (binary star system after an explosion)

Solution

(1) The relative motion of a two-body system is equivalent to the motion, in a fixed force field, of a point mass $\mu = \dfrac{Mm}{M+m}$; its equation of motion is

$$\frac{Mm}{M+m} \ddot{\boldsymbol{r}} = -\frac{GMm}{r^3} \boldsymbol{r}, \tag{1}$$

in which $\boldsymbol{r}$ is the relative position vector of the two stars. Equation (1) reduces to

$$\ddot{\boldsymbol{r}} = -\frac{G(M+m)}{r^3} \boldsymbol{r}. \tag{2}$$

By (2) the relative motion of the binary system may be regarded as the motion of a point mass in the gravitational field of a fixed equivalent gravitational source of mass $M + m$. Before the explosion the motion is circular, and its equation of motion is

$$\frac{G(M+m)}{r_0^2} = \left(\frac{2\pi}{T_0}\right)^2 r_0. \tag{3}$$

Solving (3),

$$r_0 = \left(\frac{G(M+m)T_0^2}{4\pi^2}\right)^{1/3}. \tag{4}$$

(2) Before the explosion the magnitude of the velocity of the star m relative to the star M is

$$v_0 = \frac{2\pi r_0}{T_0} = \frac{2\pi}{T_0}\left(\frac{G(M+m)T_0^2}{4\pi^2}\right)^{1/3} = \left(\frac{2\pi G(M+m)}{T_0}\right)^{1/3}, \tag{5}$$

and its direction is perpendicular to the line joining the two stars.
After the explosion the mass of the equivalent gravitational source becomes

$$M_{\mathrm{tot}} = M' + m = M + m - \Delta M. \tag{6}$$

The orbit of the relative motion changes from a circle into an ellipse, a parabola or a hyperbola. From the positions and the states of motion of the two stars at the instant when the explosion has just been completed (taken as the initial instant) we know that the initial distance of the two stars is r_0 and that the magnitude of the initial relative velocity is v_0, its direction being perpendicular to the line joining the two stars; hence the initial position must be a vertex of the ellipse, of the parabola or of the hyperbola. For an elliptical orbit it is one end of the major axis of the ellipse.
Let the distance from the other end of the major axis of the elliptical orbit to the equivalent gravitational source be r_1, and let the speed at r_1 (the minimum speed) be v_{min} (for the reason see (11)). Conservation of angular momentum and conservation of mechanical energy give

$$r_1 v_1 = r_0 v_0, \tag{7}$$

$$\frac{v_1^2}{2} - \frac{GM_{\mathrm{tot}}}{r_1} = \frac{v_0^2}{2} - \frac{GM_{\mathrm{tot}}}{r_0}. \tag{8}$$

From (7) and (8), r_1 satisfies the equation

$$\left(\frac{2GM_{\mathrm{tot}}}{r_0} - v_0^2\right) r_1^2 - 2GM_{\mathrm{tot}}\, r_1 + r_0^2 v_0^2 = 0. \tag{9}$$

Solving (9),

$$\begin{aligned} r_1 &= \left(\frac{2GM_{\mathrm{tot}}}{r_0} - v_0^2\right)^{-1}\left(GM_{\mathrm{tot}} + \sqrt{G^2M_{\mathrm{tot}}^2 - \left(\frac{2GM_{\mathrm{tot}}}{r_0} - v_0^2\right) r_0^2 v_0^2}\right) \\ &= \frac{r_0}{2GM_{\mathrm{tot}} - r_0 v_0^2}\left(GM_{\mathrm{tot}} + r_0 v_0^2 - GM_{\mathrm{tot}}\right) = \frac{r_0 v_0^2}{2GM_{\mathrm{tot}} - r_0 v_0^2}\, r_0. \end{aligned} \tag{10}$$

The other solution, r_0, is obtained by taking the minus sign in front of the square root on the right side of (10). From (10) we see that

$$r_1 > r_0. \tag{11}$$

Using equation (9) together with Viète's theorem (or using (10)), the semi-major axis of the ellipse is

$$a = \frac{r_0 + r_1}{2} = \frac{GM_{\mathrm{tot}}}{2GM_{\mathrm{tot}} - r_0 v_0^2}\, r_0 = \left(\frac{G(M+m)T_0^2}{4\pi^2}\right)^{1/3} \frac{M+m-\Delta M}{M+m-2\Delta M}. \tag{12}$$

For the orbit of the motion to be an ellipse we must have

$$0 < a < \infty. \tag{13}$$

From (12) and (13),

$$M + m - 2\Delta M > 0. \tag{14}$$

By Kepler's third law,

$$T_1 = 2\pi\sqrt{\frac{a^3}{GM_{\text{tot}}}}. \tag{15}$$

Substituting (6) and (12) into (15),

$$T_1 = \sqrt{\frac{(M+m)(M+m-\Delta M)^2}{(M+m-2\Delta M)^3}}\, T_0. \tag{16}$$

[**Method 2.** *(Translator's note: the original encloses this alternative solution in square brackets and restarts its equation numbering at* (5)*; both features are reproduced here.)*
Before the explosion, let the speed of the relative motion between the star M and the star m be v_{rel0}; then

$$v_{\text{rel0}} = \sqrt{\frac{G(M+m)}{r_0}}. \tag{5}$$

At the instant just after the explosion the velocity of the star m has not changed, and the velocity of the star $M - \Delta M$ is equal to its velocity before the explosion. Let the speed of the relative motion between the star $M - \Delta M$ and the star m be v_{rel}; then

$$v_{\text{rel}} = v_{\text{rel0}}. \tag{6}$$

The total kinetic energy in the centre-of-mass frame after the explosion is

$$E'_{\text{k}} = \frac{1}{2}\,\frac{(M-\Delta M)m}{M+m-\Delta M}\, v_{\text{rel}}^2. \tag{7}$$

The total energy in the centre-of-mass frame is

$$E' = E'_{\text{k}} - \frac{G(M-\Delta M)m}{r_0}. \tag{8}$$

For motion in an elliptical orbit,

$$E' = -\frac{G(M-\Delta M)m}{2A}, \tag{9}$$

where

$$A = \frac{M+m-\Delta M}{M+m-2\Delta M}\, r_0. \tag{10}$$

By Kepler's third law,

$$T_0 = 2\pi\sqrt{\frac{r_0^3}{G(M+m)}}. \tag{11}$$

From (10) and (11),

$$T_1 = 2\pi\sqrt{\frac{A_0^3}{G(M+m-\Delta M)}}, \tag{12}$$

(Translator's note: the source prints A_0^3 here; it should read A^3, with A given by (10).) so that

$$T_1 = \left[\frac{(M+m)(M+m-\Delta M)^2}{(M+m-2\Delta M)^3}\right]^{1/2} T_0. \tag{13}$$

]

(3) By (12), when $\Delta M \geq (M+m)/2$ equation (13) no longer holds and the orbit is no longer an ellipse. Hence, if the star M' and the star m are finally to separate for ever, it is necessary that

$$\Delta M \geq (M+m)/2. \tag{17}$$

By the statement of the problem,

$$M > \Delta M. \tag{18}$$

Combining (17) and (18), it is also necessary that

$$M > m. \tag{19}$$

Equations (16) and (19) are the required conditions. *(Translator's note: as printed. The required conditions are those of equations (17) and (19); the reference to (16) is a misprint for (17).)*

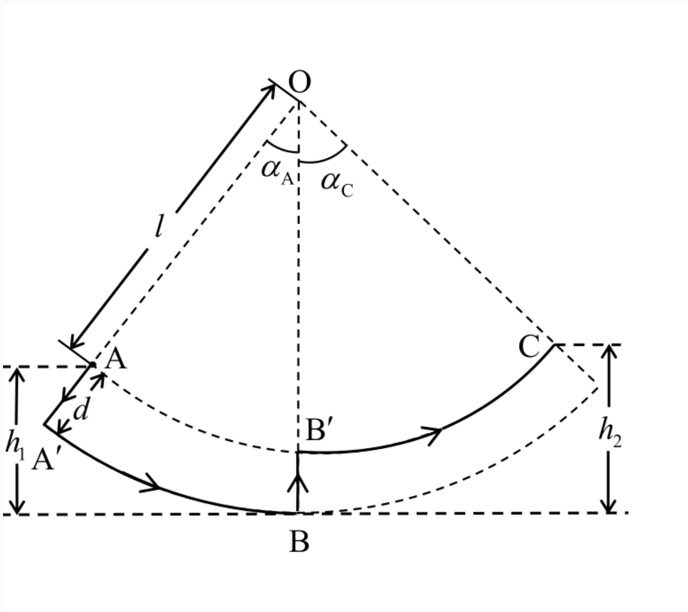
Problem 3 (pumping a swing by standing up and squatting down)

Solution

(1) Assume that there is no loss of mechanical energy during the stages in which the person stays fully squatting and during the stages in which he stays fully standing. Take the person and the swing as one system. At A the person is in the fully standing state; the mechanical energy of the system at that instant is

$$U_\mathrm{A} = K_\mathrm{A} + V_\mathrm{A} = 0 + mg\big[l\,(1-\cos\alpha_\mathrm{A}) + d\big] = mgh_1, \tag{1}$$

in which $K_\mathrm{A} = 0$ and $V_\mathrm{A} = mgh_1$ are, respectively, the kinetic energy and the gravitational potential energy of the system when the person is at A, and α_A is the angle between the swing and the vertical direction (similar notation is used below), as shown in the figure.



The person squats fully at A′. During the process A → A′ the gravitational potential energy of the system decreases while the kinetic energy is still zero (because of the restriction imposed by the board of the swing). The mechanical energy of the system when the person is at A′ is

$$\begin{aligned} U_{\mathrm{A}'} &= K_{\mathrm{A}'} + V_{\mathrm{A}'} = 0 + mg(d+l)\,(1-\cos\alpha_{\mathrm{A}}) \\ &= mg\,\frac{l+d}{l}\,(h_1 - d)\,. \end{aligned} \tag{2}$$

The person at B is still in the fully squatting state, and during the process A′ → B the mechanical energy of the system is conserved. The kinetic energy of the person at B is

$$K_{\mathrm{B1}} = \frac{1}{2} m v_{\mathrm{B1}}^2 = U_{\mathrm{A}'}, \tag{3}$$

where the subscript 1 means that the swing reaches B for the 1st time.
At B′ the person suddenly stands up; the person does work and the mechanical energy increases. Let the angular momenta of the system before and after the person stands up be L_{B1} and $L_{\mathrm{B}'1}$. During the process B → B′ the angular momentum of the system is conserved,

$$\begin{aligned} L_{\mathrm{B1}} &= m v_{\mathrm{B1}}(l+d) = m(l+d)\sqrt{2K_{\mathrm{B1}}/m} \\ &= m(l+d)\sqrt{2g\,\frac{l+d}{l}\,(h_1-d)} \\ &= L_{\mathrm{B}'1} = m l v_{\mathrm{B}'1}. \end{aligned} \tag{4}$$

Hence

$$v_{\mathrm{B}'1} = \frac{l+d}{l}\sqrt{2g\,\frac{l+d}{l}\,(h_1-d)}. \tag{5}$$

The mechanical energy of the person at B′ is

$$\begin{aligned} U_{\mathrm{B}'1} &= \frac{1}{2} m v_{\mathrm{B}'1}^2 + mgd \\ &= \frac{1}{2} m \cdot 2\left(\frac{l+d}{l}\right)^3 g\,(h_1-d) + mgd. \end{aligned} \tag{6}$$

The person at C is still in the fully standing state, and during the process B′ → C the mechanical energy of the system is conserved. The mechanical energy of the system when the person is at C is

$$\begin{aligned} U_{\mathrm{C}} &= mgh_2 \\ &= mg\,\frac{(l+d)^3}{l^3}\,(h_1-d) + mgd = U_{\mathrm{B}'1}. \end{aligned} \tag{7}$$

From (6) and (7),

$$h_2 - d = \left(\frac{l+d}{l}\right)^3 (h_1-d)\,,$$

and therefore

$$U_{\mathrm{C}} - U_{\mathrm{A}} = mg\left[\left(\frac{l+d}{l}\right)^3 - 1\right](h_1-d)\,. \tag{8}$$

(2)(i) Now take into account the loss of mechanical energy during the stages in which the person stays fully squatting and during the stages in which he stays fully standing.

According to the model given in the problem, the kinetic energy of the system when the person is at B for the n-th time is

$$\begin{aligned}\frac{1}{2}mv_{\mathrm{B}n}^2 &= mgh'_n - k_1\left(mgh_0 + mgh'_n\right)\\ &= \left(1-k_1\right)mgh'_n - k_1mgh_0\\ &= \left(1-k_1\right)mg\,\frac{l+d}{l}\left(h_n-d\right) - k_1mgh_0,\end{aligned}\tag{9}$$

that is,

$$v_{\mathrm{B}n}^2 = 2\left(1-k_1\right)g\,\frac{l+d}{l}\left(h_n-d\right) - 2k_1gh_0.\tag{9}$$

(Translator's note: here $h'_n = \dfrac{l+d}{l}(h_n-d)$ is the height above B of the fully squatting turning point that follows the n-th highest point, i.e. the drop Δh of that fully squatting stage; compare (2). The source does not state this definition explicitly.) According to the model given in the problem, the kinetic energy of the system when the person is at B′ for the n-th time is

$$\begin{aligned}\frac{1}{2}mv_{\mathrm{B}'n}^2 &= mg\left(h_{n+1}-d\right) + k_2\big[mgh'_0 + mg\left(h_{n+1}-d\right)\big]\\ &= \left(1+k_2\right)mg\left(h_{n+1}-d\right) + k_2mgh'_0,\end{aligned}\tag{10}$$

that is,

$$v_{\mathrm{B}'n}^2 = 2\left(1+k_2\right)g\left(h_{n+1}-d\right) + 2k_2gh'_0.\tag{10}$$

During the n-th process $\mathrm{B}\to\mathrm{B}'$ the angular momentum of the system is conserved,

$$(l+d)mv_{\mathrm{B}n} = mlv_{\mathrm{B}'n}.\tag{11}$$

From (9), (10) and (11),

$$\begin{aligned}&2\left(1+k_2\right)l^2g\left(h_{n+1}-d\right) + 2k_2l^2gh'_0\\ &\quad= 2\left(1-k_1\right)(l+d)^2g\,\frac{l+d}{l}\left(h_n-d\right) - 2k_1(l+d)^2gh_0.\end{aligned}\tag{12}$$

From (12),

$$\begin{aligned}h_{n+1}-d &= \frac{1-k_1}{1+k_2}\left(\frac{l+d}{l}\right)^3\left(h_n-d\right) - \frac{k_1}{1+k_2}\left(\frac{l+d}{l}\right)^2h_0 - \frac{k_2}{1+k_2}h'_0\\ &= \frac{1-k_1}{1+k_2}\left(\frac{l+d}{l}\right)^3\left(h_n-d\right) - \left[\frac{k_1}{1+k_2}\left(\frac{l+d}{l}\right)^2h_0 + \frac{k_2}{1+k_2}h'_0\right]\\ &= \lambda\left(h_n-d\right) - \mu,\end{aligned}\tag{13}$$

in which

$$\lambda = \frac{1-k_1}{1+k_2}\left(\frac{l+d}{l}\right)^3,\tag{14}$$

$$\mu = \frac{k_1}{1+k_2}\left(\frac{l+d}{l}\right)^2h_0 + \frac{k_2}{1+k_2}h'_0.\tag{15}$$

(ii) From (13), (14) and (15),

$$\begin{aligned} h_{n+1}-d &= \lambda\,(h_n-d)-\mu \\ &= \lambda\big[\lambda\,(h_{n-1}-d)-\mu\big]-\mu \\ &= \lambda^2\,(h_{n-1}-d)-\mu\,(1+\lambda) \\ &= \cdots \\ &= \lambda^n\,(h_1-d)-\mu\left(1+\lambda+\cdots+\lambda^{n-1}\right) \\ &= \lambda^n\,(h_1-d)-\mu\,\frac{1-\lambda^n}{1-\lambda}, \end{aligned} \tag{16}$$

that is,

$$h_{n+1}=\lambda^n\,(h_1-d)+d-\mu\,\frac{1-\lambda^n}{1-\lambda}. \tag{17}$$

Therefore

$$h_{n+1}-h_n=\lambda^{n-1}\big[\,(\lambda-1)\,(h_1-d)-\mu\big]. \tag{18}$$

Problem 4 (current-carrying square loop turning in a magnetic field)

Solution

(1) The moments of inertia J_{ab} and J_{cd} of the side ab and of the side cd of the loop about the axis OO′ are both

$$J_{\mathrm{ab}}=J_{\mathrm{cd}}=\frac{m}{4}\left(\frac{l}{2}\right)^2=\frac{ml^2}{16}.$$

The moments of inertia J_{ad} and J_{bc} of the side ad and of the side bc about the axis OO′ are both

$$J_{\mathrm{ad}}=J_{\mathrm{bc}}=2\int_0^{l/2}\frac{m}{4l}\,r^2\,\mathrm{d}r=\frac{ml^2}{48}.$$

The moment of inertia J of the loop about the axis OO′ is

$$J=J_{\mathrm{ab}}+J_{\mathrm{cd}}+J_{\mathrm{ad}}+J_{\mathrm{bc}}=\frac{ml^2}{6}. \tag{1}$$

(2) When the current I flows in the loop, the two sides ab and cd experience Ampère forces of equal magnitude, directed respectively along the positive and the negative direction of the X axis. Let the angular velocity of the loop be $\dot{\theta}$ and its angular acceleration $\ddot{\theta}$ when the side ab has turned so that it makes the angle θ with the X axis; the linear speed of the side ab and of the side cd is then $v=\dfrac{l}{2}\dot{\theta}$, and the magnitude of the Ampère force is $F=BlI$. The couple acting on the loop is

$$M=-2\cdot\frac{l}{2}BlI\sin\theta\approx -Bl^2I\theta, \tag{2}$$

in which the small-angle approximation $\sin\theta\approx\theta$ has been used, and the minus sign shows that the couple is opposite in direction to the angular displacement θ of the loop. By the theorem of rotation of a rigid body,

$$J\ddot{\theta}=-Bl^2I\theta. \tag{3}$$

This equation has exactly the same form as the dynamical equation of a simple pendulum, so the loop will perform simple harmonic motion, and we may put

$$\theta = \theta_0 \cos \omega t. \tag{4}$$

Substituting it into (3),

$$\omega = l\sqrt{\frac{BI}{J}} = \sqrt{\frac{6BI}{m}}. \tag{5}$$

Hence the angular displacement θ as a function of the time t is

$$\theta = \theta_0 \cos \sqrt{\frac{6BI}{m}}\, t, \tag{6}$$

the angular velocity $\dot{\theta}$ as a function of the time t is

$$\dot{\theta} = -\theta_0 \sqrt{\frac{6BI}{m}} \sin \sqrt{\frac{6BI}{m}}\, t, \tag{7}$$

and the angular acceleration $\ddot{\theta}$ as a function of the time t is

$$\ddot{\theta} = -\theta_0 \frac{6BI}{m} \cos \sqrt{\frac{6BI}{m}}\, t. \tag{8}$$

From (6), (7) and (8) we see that at $t = \bar{t} = \frac{\pi}{2}\sqrt{\frac{m}{6BI}}$ we have $\theta(\bar{t}) = 0$, while $\dot{\theta}(\bar{t}) = -\theta_0\sqrt{\frac{6BI}{m}}$ reaches its greatest value in the reverse direction, and $\ddot{\theta}(\bar{t}) = 0$. We see that the motion of the loop is a simple harmonic motion about $\theta = 0$ as equilibrium position, with angular frequency $\omega = \sqrt{\frac{6BI}{m}}$, period $T = 2\pi\sqrt{\frac{m}{6BI}}$ and amplitude θ_0.

(3) While the loop performs the above simple harmonic motion, when at $t = t_0 > 0$ it has turned counterclockwise so that the side da coincides with the positive direction of the X axis, we have

$$\theta(t_0) = 0, \qquad \text{and} \qquad \dot{\theta}_0 = \dot{\theta}(t_0) = \theta_0\sqrt{\frac{6BI}{m}}, \tag{9}$$

At this instant P and Q form an open circuit; there is no current in the loop, it is no longer acted on by the moment of the Ampère force, and the loop will keep turning with the constant angular velocity $\dot{\theta}_0$,

$$\dot{\theta}(t) = \dot{\theta}_0, \qquad t \geq t_0. \tag{10}$$

Because the sides ab and cd cut the magnetic field lines, the electromotive force induced in the loop varies with the time as

$$\begin{aligned}\varepsilon &= 2Blv \sin\left[\theta_0\sqrt{\frac{6BI}{m}}\,(t-t_0)\right] = 2Bl\,\frac{l}{2}\dot{\theta}\sin\left[\theta_0\sqrt{\frac{6BI}{m}}\,(t-t_0)\right] \\ &= Bl^2\theta_0\sqrt{\frac{6BI}{m}}\sin\left[\theta_0\sqrt{\frac{6BI}{m}}\,(t-t_0)\right],\end{aligned} \tag{11}$$

directed along P → a → b → c → d → Q. The expression giving the time dependence of the voltage V_{PQ} between P and Q is

$$V_{\text{PQ}} = -Bl^2\theta_0\sqrt{\frac{6BI}{m}}\sin\left[\theta_0\sqrt{\frac{6BI}{m}}\,(t-t_0)\right]. \tag{12}$$

(4) After P and Q are short-circuited, let the angular velocity of the loop be $\dot{\theta}$ when it has turned to some angular position θ. Because the sides ab and cd cut the magnetic field lines, the induced electromotive force is $Bl^2\dot{\theta}\sin\theta$, the current produced in the loop is $\dfrac{Bl^2\dot{\theta}}{R}\sin\theta$, and the magnitude of the Ampère force on the side ab and on the side cd is $B\left(\dfrac{Bl^2\dot{\theta}}{R}\right)l\sin\theta = \dfrac{B^2l^3}{R}\dot{\theta}\sin\theta$; these forces are directed respectively along the positive and the negative direction of the X axis, and form the couple

$$M = -\frac{B^2l^4}{R}\dot{\theta}\sin^2\theta. \tag{13}$$

Let t_1 be the instant at which P and Q have just been short-circuited; at that instant the loop is still turning counterclockwise with the angular velocity of (9), and its angular momentum is

$$J\dot{\theta}_0 = \frac{ml^2}{6}\theta_0\sqrt{\frac{6BI}{m}} = l^2\theta_0\sqrt{\frac{mBI}{6}}. \tag{14}$$

Because of the action of the above couple the rotation of the loop slows down until it stops. Let the instant at which it stops turning be t_x and its angular position $\theta_1+\alpha$; at that instant the angular momentum of the loop is 0. By the theorem of angular momentum we must have

$$0 - J\dot{\theta}_0 = \int_{t_1}^{t_x} M\,\mathrm{d}t, \tag{15}$$

from which

$$J\dot{\theta}_0 = \int_{t_1}^{t_x}\frac{B^2l^4}{R}\dot{\theta}\sin^2\theta\,\mathrm{d}t = \int_{\theta_1}^{\theta_1+\alpha}\frac{B^2l^4}{R}\sin^2\theta\,\mathrm{d}\theta \approx \frac{B^2l^4}{R}\alpha\sin^2\theta_1, \tag{16}$$

where in the integral on the right side the condition given in the problem, that α is a small angle, has been applied. From (1) and (16),

$$\alpha = \frac{RJ\dot{\theta}_0}{B^2l^4\sin^2\theta_1} = \frac{R\theta_0}{Bl^2\sin^2\theta_1}\sqrt{\frac{mI}{6B}}. \tag{17}$$

When

$$\theta_1 = \frac{\pi}{2} \tag{18}$$

α reaches its minimum value $\alpha_{\min}$,

$$\alpha_{\min} = \frac{R}{Bl^2}\sqrt{\frac{mI}{6B}}\,\theta_0. \tag{19}$$

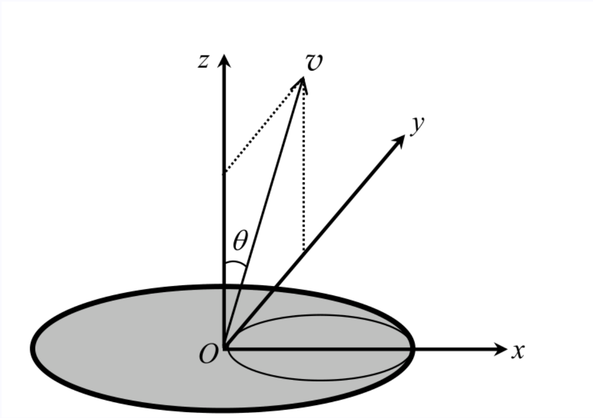
Problem 5 (charged particles trapped above a circular sheet)

Solution

(1) Because in the xOy plane this problem has circular symmetry about the origin, we may take the velocity of some positively charged particle to be

$$(0,\ V\sin\theta,\ V\cos\theta)\,.$$

The particles confined by the electric field and by the magnetic field will move on helices along the z direction inside the cylindrical region whose base is the surface of the material (that is, uniform circular motion in the xOy plane and uniformly accelerated straight-line motion in the z direction).



The figure cropped from the source labels the emission velocity v; in the text it is written v.

The radius of the uniform circular motion of the projection of the particle in the xOy plane is

$$r(\theta) = \frac{mV\sin\theta}{qB}. \tag{1}$$

If the particle can be confined by this electric field and this magnetic field inside the cylindrical region whose base is the circular material, then

$$r(\theta) \le \frac{R_0}{2}. \tag{2}$$

(Translator's note: R_0 in (2) is the radius R of the circular sheet of material; the source writes it R_0 here and R everywhere else.)

Suppose that the emission directions of the particles confined by the electric field and by the magnetic field are uniformly distributed inside the solid angle $\Delta\Omega$: $0 \le \theta \le \Theta$, $0 \le \phi \le 2\pi$. The area, on a sphere whose radius is one unit of length, that corresponds to this solid angle (that is, the value $\Delta\Omega$ of the solid angle) is

$$\Delta\Omega = -2\pi \int_0^{\Theta} \mathrm{d}\cos\theta = 2\pi\left(1-\cos\Theta\right). \tag{3}$$

The positively charged particles emitted uniformly from the origin O in all directions into the half-space above the xOy plane occupy the solid angle 2π. By the statement of the problem the fraction η of the total number of emitted particles that are confined by this electric field and this magnetic field is

$$\eta = \frac{\Delta\Omega}{2\pi} = \frac{\pi}{2\pi} = 50\%. \tag{4}$$

From (3) and (4),

$$\theta \le \Theta = \frac{\pi}{3}. \tag{5}$$

We see that a particle will certainly be confined by the electric field and by the magnetic field as long as the angle θ between the direction of its velocity and the z axis satisfies the condition (5). From (1), (2), (3), (4) and (5),

$$R = \frac{\sqrt{3}\,mV}{qB}. \tag{6}$$

(2)(i) The positively charged particle moves on a helix in the region of the magnetic field and of the electric field. From (1), the time in which the projection of the particle in the xOy plane completes one revolution of the circular motion is

$$T = \frac{2\pi r}{V\sin\theta} = \frac{2\pi m}{qB}. \tag{7}$$

The particle performs uniformly accelerated straight-line motion in the z direction. Taking the instant of emission of the particle as the zero of time, let the instant just before the particle collides with the surface of the material for the first time be $t_1(\theta)$; then the kinematic formula gives

$$V\cos\theta - a\,\frac{t_1(\theta)}{2} = 0, \tag{8}$$

in which

$$a = \frac{qE}{m}. \tag{9}$$

From (8) and (9),

$$t_1(\theta) = \frac{2mV\cos\theta}{qE}. \tag{10}$$

The path length travelled by the projection of the particle in the xOy plane in one revolution of the circular motion is

$$d(\theta) = 2\pi r(\theta) = \frac{2\pi mV\sin\theta}{qB}. \tag{11}$$

The total path length $s_{z=0}(\theta)$ travelled by the projection of the particle in the xOy plane in the time t_1 is

$$s_{z=0}(\theta) = \frac{t_1(\theta)}{T}\,d(\theta) = \frac{mV^2}{qE}\sin 2\theta. \tag{12}$$

When

$$\theta = \frac{\pi}{4} \tag{13}$$

$s_{z=0}$ is largest,

$$s_{z=0}^{\max} = s_{z=0}\Big(\theta = \frac{\pi}{4}\Big) = \frac{mV^2}{qE}. \tag{14}$$

(ii) If the total path length travelled by the projection of the particle in the xOy plane is exactly the greatest, then the value of θ is the one given by (13).
The particle performs uniformly decelerated straight-line motion in the z direction and uniform circular motion in the xOy plane. Taking the instant of emission of the particle as the zero of time, the magnitude of the resultant velocity of the particle at the instant t after the emission is

$$v_{\mathrm{res}} = \sqrt{\Big(V\sin\frac{\pi}{4}\Big)^2 + \Big(V\cos\frac{\pi}{4} - at\Big)^2}. \tag{15}$$

The path length travelled by this particle from its emission to the instant just before its first collision with the surface of the material is

$$s_1 = 2\int_0^{t_1(\theta=\pi/4)/2} \mathrm{d}t\, v_{\mathrm{res}} = \frac{mV^2}{qE}\int_0^1 \mathrm{d}u\,\sqrt{1+u^2}, \tag{16}$$

where (10) and (15) have been used. Using the integral formula given in the problem, (16) gives

$$s_1 = \frac{\sqrt{2} + \ln\big(1+\sqrt{2}\big)}{qE}\,\frac{1}{2}mV^2, \tag{17}$$

which is proportional to the kinetic energy of the particle at its emission.
Before and after each collision of the particle with the surface of the material the velocity in the z direction is reversed while the direction of the velocity along the xOy plane is unchanged, and the velocity along the z direction and the velocity along the xOy plane decrease in the same proportion; since the kinetic energy decreases by 10%, the magnitude of the angle at which the particle leaves the surface again is unchanged. The particle goes on moving according to the same rule; the path length s_n travelled by it from its n-th emission to the instant just before it collides with the surface of the material is

$$s_n = k^{n-1} s_1, \qquad k = 1 - 10\%. \tag{18}$$

Finally its kinetic energy is exhausted (it settles on the surface of the material). Using (18), the total path length s_{tot} travelled by the particle in the process from its emission until its kinetic energy is finally exhausted and it settles on the surface of the material is

$$\begin{aligned} s_{\text{tot}} &= s_1 + s_2 + \cdots + s_n + \cdots \\ &= \left(1 + k + \cdots + k^{n-1} + \cdots\right) s_1 \\ &= \frac{s_1}{1-k} = 10 s_1. \end{aligned} \tag{19}$$

From (17) and (19),

$$s_{\text{tot}} = \left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right] \frac{5mV^2}{qE}. \tag{20}$$

Problem 6 (glass capillary tube: hanging drop, and partial immersion)

Solution

(1) Let the atmospheric pressure be P_0, the inner radius of the capillary tube r, its outer radius R, the density of water ρ_0, the coefficient of surface tension σ and the gravitational acceleration g. *(Translator's note: the source writes "inner diameter r, outer diameter R", but uses r and R as radii throughout — see (6) and (9) — so they are translated as radii here.)* At the upper end of the water column the water wets the wall of the tube completely and the contact angle is zero, $\theta = 0$; the pressure P_1 just inside the upper surface of the water column is

$$P_1 = P_0 - \frac{2\sigma}{r}. \tag{1}$$

For the lower end of the water column, in the same way, the pressure P_2 at the surface is

$$P_2 = P_0 + \frac{2\sigma}{a}. \tag{2}$$

When the water column is in equilibrium,

$$P_2 = P_1 + \rho_0 g h. \tag{3}$$

From (1), (2) and (3),

$$a = \frac{2\sigma r}{\rho_0 g h r - 2\sigma}. \tag{4}$$

Here h $[= 3.50\ \text{cm}]$ is the height of the centre of the surface of the water column above the bottom of the water drop. From (4) and the data given in the problem,

$$a = 2.79 \times 10^{-3}\ \text{m}. \tag{5}$$

(2) After one third of the length of this capillary tube has been immersed vertically in water, let the height difference between the top of the water column inside the tube and the horizontal water surface outside the tube be h_1; the condition of equilibrium gives

$$\rho_0 \cdot \pi r^2 h_1 g = 2\pi r \cdot \sigma, \tag{6}$$

that is

$$h_1 = \frac{2\sigma}{\rho_0 g r}.$$

From the above expression and the data given in the problem,

$$h_1 = 2.97 \text{ cm} < 4.00 \text{ cm}. \tag{7}$$

The magnitude of the adhesion force F_1 acting on the inner wall of the capillary tube is equal to the magnitude of the weight G_1 of that part of the water column inside the tube which lies above the horizontal water surface outside the tube, that is

$$F_1 = G_1 = \rho_0 g \pi r^2 h_1, \tag{8}$$

directed downward. The magnitude of the adhesion force F_2 acting on the outer wall of the glass tube is

$$F_2 = \sigma 2\pi R, \tag{9}$$

directed downward. Let the length of the glass tube be h; the magnitude of the weight of the glass tube is *(Translator's note: from here on the symbol h denotes the length of the glass tube,* 6.00 *cm, not the* 3.50 *cm height of part (1); the source re-uses the same letter.)*

$$G = 2\rho_0 g \pi \left(R^2 - r^2\right) h, \tag{10}$$

directed downward. The magnitude F_0 of the buoyant force acting on the glass tube is

$$F_0 = \frac{1}{3} h \rho_0 g \pi \left(R^2 - r^2\right), \tag{11}$$

directed upward. Let the magnitude of the upward pulling force be F; the condition of equilibrium of the forces gives

$$F = G + G_1 + F_2 - F_0. \tag{12}$$

From (7), (8), (9), (10), (11) and (12),

$$\begin{aligned} F &= \rho g \pi \left(R^2 - r^2\right) h + \rho_0 g \pi r^2 h_1 + \sigma 2\pi R - \frac{1}{3}\rho_0 g \pi \left(R^2 - r^2\right) h \\ &= \frac{5}{3}\rho_0 g \pi h \left(R^2 - r^2\right) + 2\pi\sigma(R + r). \end{aligned} \tag{13}$$

(Translator's note: in the first term of (13) the source writes ρ for the density of the glass, $\rho = 2\rho_0$; this is what makes the coefficient $2 - \frac{1}{3} = \frac{5}{3}$ of the final expression.)
From (13) and the data given in the problem,

$$F = 7.87 \times 10^{-2} \text{ N}. \tag{14}$$

Problem 7 (gravitational redshift and the equivalence principle)

Solution

(1)(i) For a photon of energy $\frac{hc}{\lambda_0}$, the mass–energy relation gives

$$\frac{hc}{\lambda_0} = mc^2, \tag{1}$$

in which m is the equivalent mass of that photon. In the gravitational field of the star, let the gravitational acceleration at the surface of the star be g'; the law of universal gravitation gives

$$g' = \frac{GM}{R^2}. \tag{2}$$

By the conservation of energy the energies of the photon at the two points A and B are equal, i.e.

$$\frac{hc}{\lambda_0} = \frac{hc}{\lambda'} + mg'L. \tag{3}$$

From (1) and (3),

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0}\left(1 - \frac{g'L}{c^2}\right). \tag{4}$$

Solving (4),

$$\lambda' = \lambda_0 \frac{1}{1 - \frac{g'L}{c^2}} \approx \lambda_0\left(1 + \frac{g'L}{c^2}\right). \tag{5}$$

Substituting (2) into (5),

$$\lambda' \approx \lambda_0\left(1 + \frac{GM}{R^2}\frac{L}{c^2}\right). \tag{6}$$

(ii) The magnitude c of the speed of light in vacuum does not depend on the state of motion of the emitter. Let t be the time needed for the light emitted at A, travelling with the speed of light c towards the receiver B, to reach B; then

$$ct = L + \frac{1}{2}at^2. \tag{7}$$

Let the velocity of the receiver B relative to the ground at the moment when it receives the laser light be

$$v = at. \tag{8}$$

From (7) and (8),

$$v \approx \frac{aL}{c}, \tag{9}$$

where small quantities of higher order in $\frac{aL}{c^2}$ have been dropped. By the Doppler shift formula the wavelength λ'' of the light received at the point B satisfies

$$\frac{c}{\lambda''} = \sqrt{\frac{1-\beta}{1+\beta}}\frac{c}{\lambda_0}, \tag{10}$$

in which

$$\beta = \frac{v}{c} = \frac{aL}{c^2}. \tag{11}$$

Since $\beta \ll 1$,

$$\sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{1 - \frac{2\beta}{1+\beta}} \approx 1 - \frac{\beta}{1+\beta} = \frac{1}{1+\beta}. \tag{12}$$

Substituting (11) and (12) into (10), the wavelength of the laser light received by the receiver at the point B is

$$\lambda'' = \lambda_0 \left(1 + a\,\frac{L}{c^2}\right). \tag{13}$$

(iii) By the equivalence principle, if the wavelengths λ' and λ'' of the laser light received by the receiver at B, computed in (i) and in (ii), are regarded as one and the same event, then (6) and (13) must express the same physical law; comparing the results of (i) and of (ii),

$$a = \frac{GM}{R^2}. \tag{14}$$

(2) At the surface of the Earth the magnitude of the gravitational acceleration is g; replacing g' in (4) by g, the frequency ν' of the photon received at the point B is

$$\nu' = \nu_0 \left(1 - \frac{gL}{c^2}\right). \tag{15}$$

From (15) the frequency shift of the photon caused by the gravity of the Earth is

$$\Delta\nu = \nu' - \nu_0 = -\nu_0\,\frac{gL}{c^2}.$$

We see that the action of gravity decreases the frequency by $\nu_0\dfrac{gL}{c^2}$ (a redshift), and that this must be compensated by the motion of the Mössbauer detector relative to the ground, so that the Mössbauer detector can detect that photon. Let the speed at which the Mössbauer detector moves relative to the ground along the direction BA be v; then the frequency of the photon that the Mössbauer detector can detect [is ν_0]. *(Translator's note: this sentence is left incomplete in the Chinese original; the missing predicate has been supplied in square brackets.)* By the Doppler shift formula,

$$\nu_0 = \sqrt{\frac{1+\beta}{1-\beta}}\,\nu'. \tag{16}$$

From (12) and (16),

$$\nu' = \frac{\nu_0}{1+\beta} \approx \nu_0 \left(1 - \frac{v}{c}\right). \tag{17}$$

From (15) and (17),

$$\frac{gL}{c^2} = \frac{v}{c}, \tag{18}$$

that is

$$v = \frac{gL}{c}. \tag{19}$$

From (19) and the data given in the problem,

$$v = \frac{9.81 \times 22.6}{3.00 \times 10^8}\ \mathrm{m\cdot s^{-1}} = 7.39 \times 10^{-7}\ \mathrm{m\cdot s^{-1}}. \tag{20}$$

Problem 8 (silicon-based gallium nitride film: interference, and radiation pressure)

Solution

(1) The wavelength range of the incident light is given as $450-1200$ nm; from the dispersion relation the range of the refractive index is obtained,

$$2.278 \leq n \leq 2.436.$$

We see that the refractive index of gallium nitride is smaller than that of silicon, so that a half-wave loss must be taken into account both at the air–gallium nitride interface and at the gallium nitride–silicon interface; but for the interference of the light reflected at the two interfaces the half-wave loss need not be taken into account.
Let the refractive index of the film for the incident light of wavelength 600 nm be n_1, and let its refractive index for the incident light of wavelength λ be n. For the incident light of wavelength 600 nm, after the light has been reflected at the front and at the back face of the gallium nitride film of thickness d the two beams meet at the surface, and the condition of coherent enhancement is

$$2n_1 d = k_1 \lambda_1 \qquad (k_1 = 1, 2, \cdots) \tag{1}$$

In the same way, for the incident light of wavelength λ the condition of coherent enhancement is

$$2nd = k\lambda \qquad (k = 1, 2, \cdots) \tag{2}$$

From (1) and (2),

$$\frac{k_1}{k}\frac{n}{n_1} = \frac{\lambda}{\lambda_1} \qquad \text{or} \qquad \frac{k_1}{k} = \frac{n_1}{n}\frac{\lambda}{\lambda_1}. \tag{3}$$

From (3), and according to the statement of the problem, let λ' and λ'' correspond respectively to the incident wavelengths 450 nm and 1200 nm, and let the corresponding refractive indices of the film be n' and n''; the dispersion relation given in the problem gives

$$n_1(600 \text{ nm}) \approx 2.342, \qquad n'(450 \text{ nm}) \approx 2.436, \qquad n''(1200 \text{ nm}) \approx 2.278. \tag{4}$$

From (4),

$$\frac{k_1}{k'} = \frac{n_1(600)}{n'(450)}\frac{450}{600} \approx 0.721, \qquad \frac{k_1}{k''} = \frac{n_1(600)}{n''(1200)}\frac{1200}{600} \approx 2.056. \tag{5}$$

From (5) and the statement of the problem,

$$\frac{k_1}{k'} < \frac{k_1}{k} < \frac{k_1}{k''}. \tag{6}$$

From (6) and the conditions given in the problem, only

$$k_1 = 2, \qquad k = 1 \tag{7}$$

satisfies the conditions. Substituting (4) and (7) into (1),

$$d \approx 256 \text{ nm}. \tag{8}$$

Substituting (7) and (8) into (2),

$$\frac{\lambda}{n} \approx 512.38. \tag{9}$$

From the dispersion relation given in the problem and (9),

$$2.41n^4 - 12.97n^2 + 2.32 = 0,$$

or

$$2.41\left(\frac{\lambda}{512.38}\right)^4 - 12.97\left(\frac{\lambda}{512.38}\right)^2 + 2.32 = 0.$$

Solving this equation,

$$\lambda = 1168 \text{ nm}. \tag{10}$$

The other solution, $\lambda = 220$ nm, does not lie in the wavelength range given in the problem and is discarded.

(2) Let the number of photons of the monochromatic light of frequency ν that fall perpendicularly, per second, on each square metre of the surface of the body be N. The momentum that each photon transfers to the body is $\frac{h\nu}{c}$ (h is Planck's constant and c is the speed of light in vacuum). If the incident photons are all absorbed by the body, then the momentum acquired per second by each square metre of that surface, that is the light pressure P_0, is

$$P_0 = N\,\frac{h\nu}{c}.$$

The energy that each photon transfers to the body is $h\nu$, so the energy that falls perpendicularly, per second, on each square metre of the surface of a completely absorbing body is

$$Nh\nu = \bar{w}c,$$

in which $\bar{w}$ is the mean energy density of the electromagnetic wave,

$$\bar{w} = \varepsilon_0 E^2.$$

From the three expressions above, the light pressure P_0 exerted on a completely absorbing surface by monochromatic light of frequency ν incident normally on it is

$$P_0 = \bar{w} = \varepsilon_0 E^2. \tag{11}$$

The resultant moment of the radiation pressure on the sample is

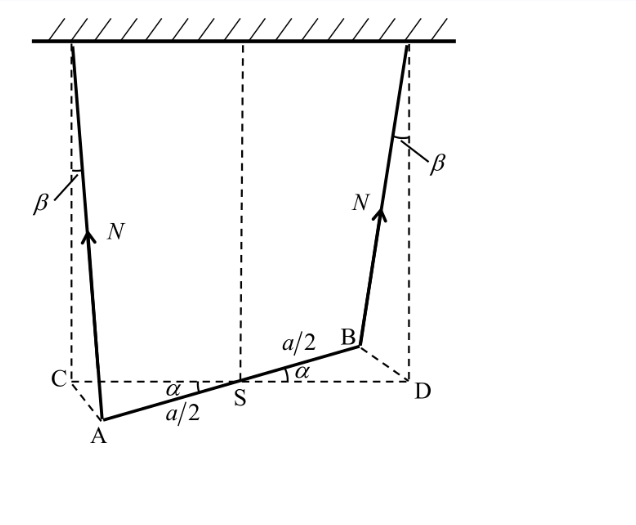
$$\begin{aligned} L_p &= \left(2P_0 \cdot \frac{ab}{2} \cdot \cos\alpha\right) \cdot \frac{a}{4} \cdot \cos\alpha - \left(P_0 \cdot \frac{ab}{2} \cdot \cos\alpha\right) \cdot \frac{a}{4} \cdot \cos\alpha \\ &= P_0 \cdot \frac{a^2 b}{8} \cdot \cos^2\alpha. \end{aligned} \tag{12}$$

As shown in the figure, the geometry gives

$$\mathrm{AC} = a\sin\frac{\alpha}{2},$$

from which

$$\beta = \frac{\alpha}{2}. \tag{13}$$



The sample is in equilibrium of forces in the vertical direction,

$$2N\cos\beta = Mg, \tag{14}$$

in which N is the tension in the thin thread *(translator's note: the source uses the same letter N for the number of photons above and for the tension in the threads from (14) on)* and M is the sum of the mass m' of the wafer and the mass m of the gallium nitride film,

$$M = m' + m = \left(\rho' d' + \rho d\right) ab. \tag{15}$$

The moment supplied to the sample by the tension N of the light thin threads is

$$L_N = 2\left(N\sin\beta\right)\cdot\left(\frac{a}{2}\cos\frac{\alpha}{2}\right). \tag{16}$$

The condition that the moments acting on the sample balance is

$$L_N = L_p.$$

Substituting (13), (14), (15) and (16) into the above expression,

$$\left(\rho' d' + \rho d\right) ab\cdot g\cdot\frac{a}{2}\sin\frac{\alpha}{2} = P_0\cdot\frac{a^2 b}{8}\cdot\cos^2\alpha. \tag{17}$$

From (11) and (17),

$$\frac{\sin\dfrac{\alpha}{2}}{\cos^2\alpha} = \frac{\varepsilon_0 E^2}{4\left(\rho' d' + \rho d\right) g},$$

that is

$$E = 2\left[\frac{\left(\rho' d' + \rho d\right) g}{\varepsilon_0}\,\frac{\sin\dfrac{\alpha}{2}}{\cos^2\alpha}\right]^{1/2}. \tag{18}$$

(3) Substituting the data given in the problem into (18),

$$\frac{\sin\dfrac{\alpha}{2}}{\cos^2\alpha} = 8.12\times10^{-4}. \tag{19}$$

Since the right side of (19) is very small, α is also very small. From (19) and the small-angle approximations $\sin\alpha\approx\alpha$ and $\cos\alpha\approx1$,

$$\alpha\approx1.62\times10^{-3}\ \text{rad}. \tag{20}$$