The 35th Chinese Physics Olympiad

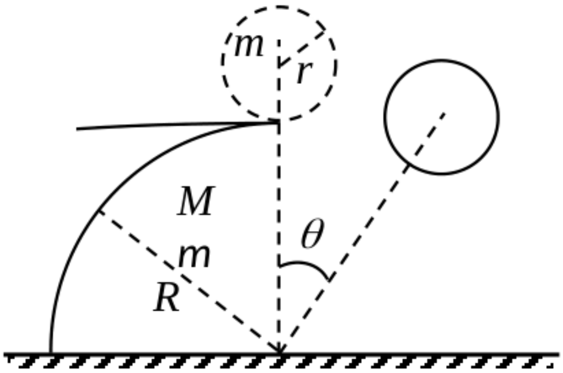
Final — Theoretical Examination (Problems)

2018 — 8 problems

Translator's note: the source of this paper is the official document that prints each problem together with its reference answer. The problem statements below are reproduced from that source; the reference solutions are collected in the companion file. A few symbols lost in the conversion are reconstructed and set in square brackets [·].

Problem 1

As shown in the figure, a hemisphere of radius R and mass M rests on a smooth horizontal table. A uniform small ball of mass m and radius r sits at the top of the hemisphere. At some instant the ball receives a tiny disturbance and starts to move from rest along the surface of the hemisphere. During the motion the position of the ball relative to the hemisphere is described by the angular position θ, the angle between the line joining the two centres and the vertical direction. The moment of inertia of the ball about its symmetry axis is $\frac{2}{5}mr^2$. The coefficient of kinetic friction between the ball and the hemisphere is μ; assume the maximum static friction equals the sliding friction. The magnitude of the gravitational acceleration is g.



(1) After the ball starts to move it rolls without slipping for some time. During this stage, when the angular position of the ball is θ_1, find the magnitude $V_M(\theta_1)$ of the velocity and the magnitude $a_M(\theta_1)$ of the acceleration of the hemisphere.

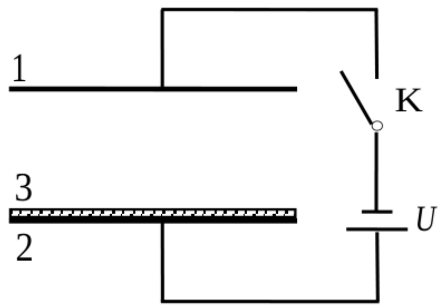
(2) When the rolling ball reaches the angular position θ_2 it begins to slip relative to the hemisphere. Find the equation satisfied by θ_2 (it may be written in terms of the speed $V_M(\theta_2)$ and the acceleration $a_M(\theta_2)$ of the hemisphere together with the given quantities).

(3) The ball leaves the hemisphere exactly when it reaches the angular position θ_3. Find the magnitude $v_m(\theta_3)$ of the velocity of the centre of mass of the ball relative to the hemisphere at that moment.

Problem 2

The plates 1 and 2 of a parallel-plate capacitor both have area S; they are fixed horizontally and the distance between them is d. They are connected in the circuit shown in the figure; the electromotive force of the source is denoted by U. An uncharged thin conducting plate 3, of mass

m, has the same size as the capacitor plates. Plate 3 lies flat directly on top of plate 2 and is in good electrical contact with plate 2. The whole system is placed in a vacuum chamber; the permittivity of vacuum is ε_0.



After the key K is closed, plate 3 collides with plates 1 and 2 in turn and moves up and down repeatedly. Assume that the electric field between any two conducting plates may be regarded as uniform; that the resistance of the wires and the internal resistance of the source are small enough that the charging and discharging times can be neglected; that after plate 3 collides with plate 1 or plate 2 electrostatic equilibrium is reached immediately, in an extremely short time; and that all collisions are perfectly inelastic. The magnitude of the gravitational acceleration is g.

(1) How large must the electromotive force U of the source be at least?

(2) Find the period of the motion of plate 3 (expressed in terms of U and the given quantities).

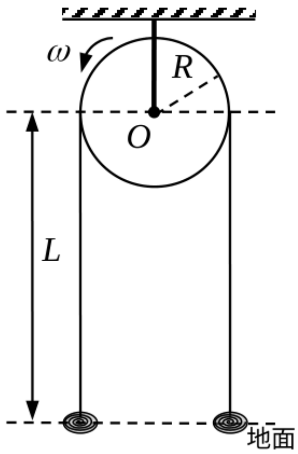
The following integral formula is given:

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx}} = \frac{1}{\sqrt{a}} \ln\left(2ax + b + 2\sqrt{a}\sqrt{ax^2+bx}\right) + C,$$

where $a > 0$ and C is a constant of integration.

Problem 3

As shown in the figure, an inextensible soft thin rope of linear mass density λ passes over a disc-shaped fixed pulley. The radius of the pulley is R and its axle is at the height L above the ground. The system is initially at rest. At $t = 0$ the pulley begins to rotate counterclockwise with the constant angular velocity ω, and the rope, driven by the pulley, starts to move. The coefficient of kinetic friction between the rope and the pulley is μ. During the motion the rope on both sides of the pulley may always be regarded as vertical, neither end of the rope leaves the ground, and the rope lying on the ground may be regarded as concentrated at one point. The magnitude of the gravitational acceleration is g. Denote by T_1 and T_2 the magnitudes of the tension of the rope at the points where it is tangent to the left and to the right side of the pulley, respectively.



Labels: 地面 = *ground.*

(1) Write down, for the stage before the speed of the rope reaches its maximum value, the set of dynamical equations satisfied by the motion of the vertical parts of the rope on the two sides of the pulley and by the motion of an arbitrary small element of the rope on the pulley.

(2) Find the maximum speed that the rope can reach.

Mathematical relations that may be used:

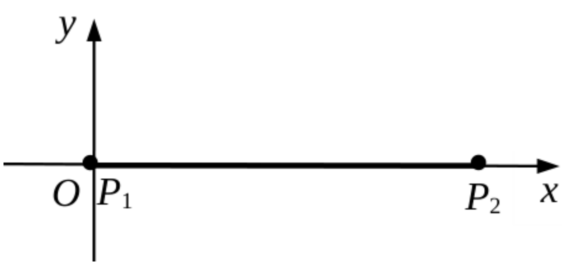
$$\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha y = \mathrm{e}^{-\alpha x}\,\frac{\mathrm{d}(y\mathrm{e}^{\alpha x})}{\mathrm{d}x};$$

$$\int \mathrm{e}^{\alpha x}\cos x\,\mathrm{d}x = \frac{\mathrm{e}^{\alpha x}}{1+\alpha^2}(\alpha\cos x + \sin x) + C_1, \qquad C_1 \text{ a constant of integration;}$$

$$\int \mathrm{e}^{\alpha x}\sin x\,\mathrm{d}x = \frac{\mathrm{e}^{\alpha x}}{1+\alpha^2}(\alpha\sin x - \cos x) + C_2, \qquad C_2 \text{ a constant of integration.}$$

Problem 4

As shown in the figure, a taut string of length L is placed horizontally along the x axis. The left end of the string is at the origin of coordinates. The string can be connected to a vibration source through its left or its right end, so that the string performs transverse forced vibrations along the y direction; the propagation speed of the vibration is u.



(1) Fix the right end P_2 of the string and connect its left end P_1 to the vibration source. In the steady state the vibration of the left end P_1 is $y(x=0,t) = A_0\cos(\omega t)$, where A_0 is the amplitude and ω the angular frequency.

(i) The amplitude of the transverse wave on the string is known to decay along the propagation direction, with decay constant γ ($\gamma > 0$). Find the amplitude of the vibration at every point of the string. (It is given that on an infinite string the transverse wave with gradually decaying amplitude travelling in the positive x direction is

$$y(x,t) = A\mathrm{e}^{-\gamma x}\cos\Big(\omega t - \frac{\omega x}{u} + \varphi\Big),$$

where A and φ are the amplitude and the initial phase of the vibration at $x = 0$.)

(ii) Neglect the decay of the amplitude of the wave along the propagation direction. Find the expression for the standing wave on the string, and determine the x coordinates of its antinodes and nodes.

(2) Connect both P_1 and P_2 to vibration sources, so that the vibrations at P_1 and P_2 are $y(x = 0,t) = A_0\cos\omega t$ and $y(x = L,t) = A_0\cos(\omega t + \varphi_0)$ respectively, where φ_0 is a constant. Neglect the decay of the amplitude of the wave along the propagation direction. For the two cases $\varphi_0 = 0$ and $\varphi_0 = \pi$, calculate separately the vibration at every point of the string and the condition that the angular frequency ω must satisfy at resonance.

Problem 5

A container with adiabatic thin walls and mass M is in outer space, far from any other celestial body (space may be regarded as vacuum). Observed in a certain inertial frame, the initial velocity of the container is zero. The volume of the container is V; it is filled with a monatomic ideal gas whose initial number of molecules is N_0 and whose molecular mass is m; the initial temperature of the gas is T_0. At $t = 0$ a small hole of area S appears in the wall of the container. Because gas leaks out of the hole the container begins to move, but it does not rotate. Assume the hole is small enough that the gas in the container stays in equilibrium throughout the leaking process. The Maxwell distribution function of the component v_x of the molecular velocity along the x direction is

$$f(v_x) = \sqrt{\frac{m}{2\pi kT}}\,\exp\Big(-\frac{mv_x^2}{2kT}\Big)$$

(k is the Boltzmann constant). During the leaking process, find:

(1) the number of gas molecules leaving the hole per unit time and per unit area when the number density of the gas is n and its temperature is T;

(2) the average kinetic energy, relative to the container, of the gas molecules leaving the hole when the temperature of the gas in the container is T;

(3) the temperature of the gas in the container at time t;

(4) the magnitude of the velocity of the container at time t (assume $M \gg N_0 m$).

The following integral formulas are given:

$$\int_0^\infty x\,\mathrm{e}^{-Ax^2}\mathrm{d}x = \frac{1}{2A}, \qquad \int_0^\infty x^2\mathrm{e}^{-Ax^2}\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{A^3}}, \qquad \int_0^\infty x^3\mathrm{e}^{-Ax^2}\mathrm{d}x = \frac{1}{2A^2}.$$

Problem 6

The refractive index n of a medium can be greater than 0 and can also be less than 0. A medium with $n < 0$ is called a negative-refraction medium. When light propagates inside a negative-refraction medium its optical path is negative (the way the phase changes with the propagation distance is opposite to that in a medium with positive refractive index). If we define the refraction angle to be negative when the refracted ray and the incident ray lie on the same side of the normal to the interface, then it can be proved that the law of refraction still holds when there is a negative-refraction medium on either side of the interface, i.e.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where both n_1 and n_2 may be greater than 0 or less than 0, and θ_2 is the refraction angle.

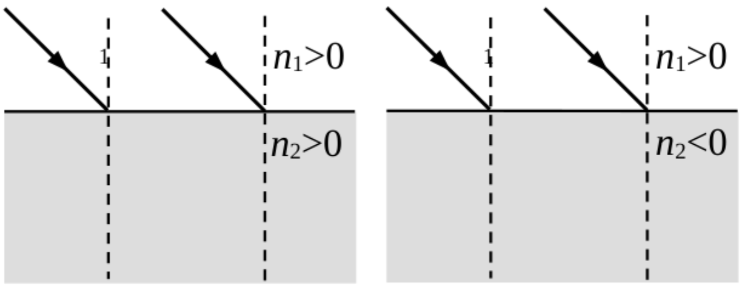


图 a 图 b

Labels: 图 *a = Figure a;* 图 *b = Figure b.*

(1) Imagine a parallel beam of light incident on the interface. Using Huygens' principle, draw on your answer sheet the rays entering medium 2 and the corresponding wavelets for the situations of Figure a and Figure b, and use them to prove that the law of refraction holds.

(2) As shown in Figure c, a spherical surface of radius R divides space into two regions whose refractive indices are n_1 ($n_1 > 0$) and n_2 ($n_2 < 0$). The point C is the centre of the sphere; take the intersection point O of a chosen optical axis with the sphere as origin. One incident ray and the corresponding refracted ray are already drawn for this case; x and y are the coordinates of the intersections of the incident ray and of the refracted ray with the optical axis. Denote the object distance by s_1 and the image distance by s_2. In the paraxial approximation, derive the imaging formula and the transverse magnification formula of the spherical surface. State clearly the sign convention adopted for every quantity in the final results.

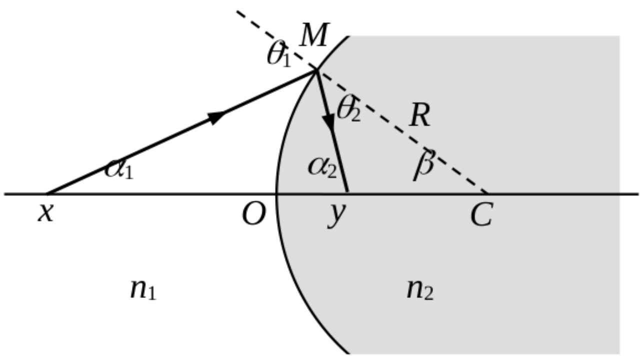


图 c

Labels: 图 *c = Figure c.*

(3) Let medium 1 be air, i.e. $n_1 \approx 1$; n_2 may be greater than 0 or less than 0. In front of the spherical surface (see Figure c) an ordinary thin convex lens is placed; the optical axis of

the lens passes through the centre C, and its focus lies inside the region of the negative-refraction medium. The focal length of the lens is $f = 1.5R$, and the distance between the centre O' of the lens and the point O is d. A parallel beam of light travelling along the optical axis is incident on the thin lens. For each of the four sets of parameters in the table, calculate the distance from O of the point on the optical axis at which the incident light converges, and draw on your answer sheet the ray diagram for case No. 4.

No.	n_2	d
1	1.5	$0.35R$
2	1.5	$0.85R$
3	-1.5	$0.35R$
4	-1.5	$0.85R$

Problem 7

In solid materials the physical properties can be studied, once the interactions are taken into account, with the concept of a "quasi-particle". The relation between the kinetic energy and the momentum of a quasi-particle may differ from that of a real particle. When an external electric or magnetic field is applied, the motion of a quasi-particle can often be treated by the methods of classical mechanics. In a certain two-dimensional interface structure there are quasi-particles of charge q and effective mass m, which can move only in the x–y plane; the relation between their kinetic energy K and the magnitude p of their momentum is

$$K = \frac{p^2}{2m} + \alpha p,$$

where α is a positive constant.

(1) For a real free particle of mass m the relation between the kinetic energy K and the magnitude p of the momentum is $K = p^2/2m$. Starting from the work–energy theorem, derive the relation between the velocity $\boldsymbol{v}$ and the momentum $\boldsymbol{p}$ of this particle.

(2) Following the method of part (1), derive the relation between the velocity $\boldsymbol{v}$ and the momentum $\boldsymbol{p}$ of the quasi-particle.

(3) Express the speed of the quasi-particle in terms of its kinetic energy.

(4) Place this two-dimensional interface structure in a uniform magnetic field along the positive z direction with magnetic induction of magnitude B. Find the radius, the period and the magnitude of the angular momentum of the uniform circular motion of a quasi-particle with kinetic energy K.

(5) Place this two-dimensional interface structure in a uniform electric field. The quasi-particle may acquire an acceleration in the direction perpendicular to the electric field. Let the electric field be along the positive x direction with magnitude E. When the speed of the quasi-particle is v ($v \neq \alpha$) and its direction makes the angle θ with the positive x direction, find the components a_x and a_y of its acceleration.

Problem 8

Thermal radiation falls on a mirror; the mirror can use the radiation pressure of the thermal radiation to do work on the surroundings. This process can be studied by dynamics or by thermodynamics. For simplicity, treat the thermal radiation as one-dimensional blackbody radiation falling normally on a plane ideal (totally) reflecting mirror. The radiation pressure on the mirror is balanced by an external resistance force, so that the mirror moves uniformly with velocity v in the same direction as the incident radiation. In the laboratory frame the one-dimensional blackbody radiation spectrum at temperature T (the radiation energy emitted per unit time per unit frequency interval near the frequency ν) is

$$\varphi(\nu, T) = \frac{2h\nu}{\mathrm{e}^{h\nu/kT} - 1},$$

where h is the Planck constant and k is the Boltzmann constant. The speed of light in vacuum is c.

(1) Starting from the dynamical picture of the collisions between the photons of the one-dimensional blackbody radiation and the moving ideal mirror, calculate, in the laboratory frame, the efficiency η with which the mirror uses the photon energy to do work against the resistance force.

(2) From the thermodynamic point of view, the incidence of the radiation and the reflection that follows it can be regarded as one small ideal heat-engine cycle undergone by the mirror as the working substance: the incidence is the absorption of heat by the mirror from a high-temperature reservoir, the reflection is the release of heat by the mirror to a low-temperature reservoir, and finally the mirror returns to its original state. On this basis, prove that in the frame of the mirror both the incident and the reflected radiation are one-dimensional blackbody radiation, and calculate the efficiency of this heat engine in the frame of the mirror.

(3) Calculate the efficiency of this heat engine in the laboratory frame.