The 35th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2018 — official reference solutions

Equation numbers (1), (2), … in each problem follow the circled numbers of the Chinese original. Passages that the original encloses in square brackets (alternative methods and remarks) are kept as such. Symbols and formulas lost in the source file are reconstructed and set in square brackets [·]. *Per-step marks are given only where the source prints them.*

Problem 1 (ball rolling off a hemisphere on a smooth table)

Solution

(1) (Method 1) The system formed by the hemisphere and the ball is acted on by no external force in the horizontal direction, so the horizontal momentum of the system is conserved:

$$-MV_M + m\big[(R+r)\dot\theta\cos\theta - V_M\big] = 0. \tag{1}$$

Let ω be the magnitude of the angular velocity of the ball. The ball rolls without slipping, so

$$r\omega = (R+r)\dot\theta. \tag{2}$$

No dissipative force does work, so the mechanical energy of the system is conserved:

$$\begin{aligned} mg(R+r)(1-\cos\theta) = \tfrac12 MV_M^2 + \tfrac12 m\Big\{\big[(R+r)\dot\theta\cos\theta - V_M\big]^2 \\ + \big[(R+r)\dot\theta\sin\theta\big]^2\Big\} + \tfrac12 I\omega^2, \end{aligned} \tag{3}$$

where $I = \frac25 mr^2$. Combining (1), (2) and (3), the speed of the hemisphere when the ball has reached the angular position θ_1 is

$$V_M = \sqrt{\frac{10m^2(R+r)g(1-\cos\theta_1)\cos^2\theta_1}{\big[7(M+m)-5m\cos^2\theta_1\big](M+m)}}, \tag{4}$$

or

$$V_M^2 = \frac{10m^2 g(R+r)(1-\cos\theta_1)\cos^2\theta_1}{\big[7M+(5\sin^2\theta_1+2)m\big](m+M)}.$$

Differentiating both sides of the last equation with respect to the time t,

$$\begin{aligned} 2V_M a_M = \frac{10mg(-2\cos\theta+3\cos^2\theta)\big[7(M+m)-5m\cos^2\theta\big] - 100m^2 g(1-\cos\theta)\cos^3\theta}{\big[7(M+m)-5m\cos^2\theta\big]^2} \\ \cdot\frac{m(R+r)\sin\theta\cdot\dot\theta}{M+m}. \end{aligned}$$

From (1),

$$\frac{m(R+r)\dot\theta}{M+m} = \frac{V_M}{\cos\theta}.$$

The last two equations give the magnitude of the acceleration of the hemisphere when the ball is at the angular position θ_1:

$$a_M(\theta_1) = -\frac{5mg\sin\theta_1\left[14(M+m) - 21(M+m)\cos\theta_1 + 5m\cos^3\theta_1\right]}{\left[7(M+m) - 5m\cos^2\theta_1\right]^2}. \tag{5}$$

[**(Method 2)** See the answer to part (2) below. Writing down the dynamical equations together with the kinematic constraint also gives the magnitude $a_M(\theta_1)$ of the acceleration of the hemisphere.]

(2) When the rolling ball is at the angular position θ ($\theta \le \theta_2$), let N and f be the magnitudes of the normal force and of the friction force exerted by the ball on the hemisphere. Newton's second law for the hemisphere gives

$$N\sin\theta - f\cos\theta = Ma_M. \tag{6}$$

In the frame of the hemisphere, the centre-of-mass motion theorem applied to the ball gives

$$mg\cos\theta - N - ma_M\sin\theta = m\frac{v_\mathrm{C}^2}{R+r}, \tag{7}$$

$$mg\sin\theta + ma_M\cos\theta - f = m\frac{\mathrm{d}v_\mathrm{C}}{\mathrm{d}t}, \tag{8}$$

where v_C is the speed of the centre of mass of the ball in the frame of the hemisphere:

$$v_\mathrm{C} = r\omega = (R+r)\dot{\theta} = \frac{M+m}{m\cos\theta}V_M. \tag{9}$$

The last equality in (9) has used (1). In the centre-of-mass frame of the ball, the theorem of rotation applied to the ball gives

$$fr = I\frac{\mathrm{d}\omega}{\mathrm{d}t}. \tag{10}$$

Equations (6)–(10) give

$$f = \frac{2m}{7}\left(g\sin\theta + a_M\cos\theta\right), \tag{11}$$

$$N = mg\cos\theta - ma_M\sin\theta - m\frac{v_\mathrm{C}^2}{R+r}. \tag{12}$$

Rolling without slipping requires

$$f \le \mu N.$$

When the rolling ball reaches the angular position θ_2 it starts to slip relative to the hemisphere, so the equality holds there. Substituting (11) and (12) into $f = \mu N$,

$$\frac{2m}{7}\left[g\sin\theta_2 + a_M(\theta_2)\cos\theta_2\right] = \mu\left[mg\cos\theta_2 - ma_M(\theta_2)\sin\theta_2 - m\frac{v_\mathrm{C}^2(\theta_2)}{R+r}\right].$$

Substituting (9) into the last equation, the equation satisfied by θ_2 is

$$\frac{2}{7}g\sin\theta_2 - \mu g\cos\theta_2 + a_M(\theta_2)\left(\frac{2}{7}\cos\theta_2 + \mu\sin\theta_2\right) + \frac{\mu(M+m)^2V_M^2(\theta_2)}{(R+r)m^2\cos^2\theta_2} = 0, \tag{13}$$

where $V_M(\theta_2)$ and $a_M(\theta_2)$ are given by (4) and (5) with $\theta_1 \to \theta_2$.

(3) At the instant at which the ball just leaves the hemisphere at the angular position θ_3,

$$N = 0. \tag{14}$$

The acceleration of the hemisphere is then zero. Hence, at the instant at which the ball leaves the hemisphere, the magnitude $v_m(\theta_3)$ of the velocity of the centre of mass of the ball relative to the hemisphere satisfies

$$mg\cos\theta_3 = m\frac{v_C'^2}{R+r}, \tag{15}$$

from which

$$v_C' = \sqrt{(R+r)g\cos\theta_3}. \tag{16}$$

Problem 2 (conducting plate bouncing between capacitor plates)

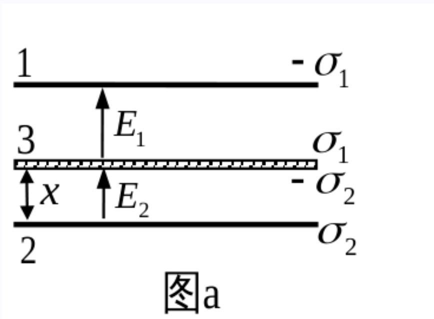
Solution

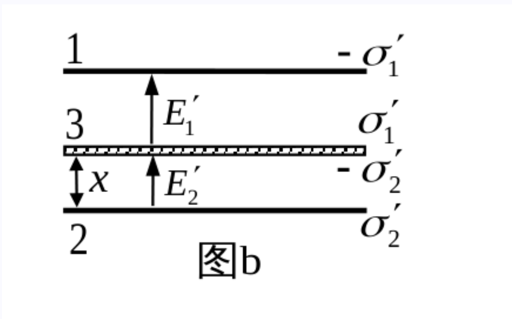
(1) Before plate 3 leaves plate 2, the charge on plate 3 is

$$Q = C_0 U = \frac{\varepsilon_0 S}{d} U.$$

After plate 3 has left plate 2, let the surface charge densities of the plates be as shown in Figure a. Conservation of charge gives

$$\sigma_1 - \sigma_2 = \sigma \equiv \frac{Q}{S} = \frac{\varepsilon_0}{d} U. \tag{1}$$





Labels: 图*a = Figure a;* 图*b = Figure b.*

Let E_1 and E_2 be the field strengths between the plates of the upper and of the lower capacitor (see Figure a):

$$E_1 = \frac{\sigma_1}{\varepsilon_0}, \qquad E_2 = \frac{\sigma_2}{\varepsilon_0}.$$

Substituting this into (1),

$$\varepsilon_0 E_1 - \varepsilon_0 E_2 = \frac{\varepsilon_0}{d} U,$$

that is,

$$E_1 - E_2 = \frac{U}{d}. \tag{2}$$

In addition, the total potential difference across the two capacitors in series is U, so

$$E_2 x + E_1(d - x) = U. \tag{3}$$

Combining (2) and (3),

$$E_1 = \frac{U}{d}\,\frac{d+x}{d}, \tag{4}$$

$$E_2 = \frac{U}{d}\,\frac{x}{d}. \tag{5}$$

From (4) and (5), the surface charge densities on plates 1 and 2 are

$$\sigma_1 = \varepsilon_0 E_1, \qquad \sigma_2 = \varepsilon_0 E_2.$$

The electric force on plate 3 (upward is taken as positive here and below) is

$$\begin{aligned} F_\mathrm{e} &= -\sigma_2 S\cdot\frac{E_2}{2} + \sigma_1 S\cdot\frac{E_1}{2} = \frac{\varepsilon_0 S}{2}\left(E_1^2 - E_2^2\right) = \varepsilon_0 S(E_1 - E_2)\left(\frac{E_1+E_2}{2}\right) \\ &= \varepsilon_0 S\,\frac{U}{d}\cdot(E_1+E_2)/2 = \varepsilon_0 S\,\frac{U}{d}\cdot\frac{U}{d}\left(\frac{2x+d}{2d}\right) = \frac{\varepsilon_0 S U^2}{2d^3}(2x+d). \end{aligned} \tag{6}$$

The resultant vertical force on plate 3 is

$$F_\text{total} = F_\mathrm{e} - mg = \frac{\varepsilon_0 S U^2}{2d^3}(2x+d) - mg = \left(\frac{\varepsilon_0 S U^2}{2d^2} - mg\right) + \frac{\varepsilon_0 S U^2}{d^3}x.$$

Hence the acceleration of plate 3 at the position shown in Figure a is

$$a = \frac{F_\text{total}}{m} = \left(\frac{\varepsilon_0 S U^2}{2md^2} - g\right) + \frac{\varepsilon_0 S U^2}{md^3}x. \tag{7}$$

For plate 3 to move upward, the condition

$$\frac{\varepsilon_0 S U^2}{2md^2} \geq g$$

must hold; moreover the acceleration stays positive once the motion has started, so plate 3 will finally strike plate 2.[a] The above condition means that the electromotive force U of the source must be at least

$$U_\text{min} = \sqrt{\frac{2md^2 g}{\varepsilon_0 S}}. \tag{8}$$

(2) From (7),

$$a = a_0 + Bx,$$

where

$$a_0 = \tfrac{1}{2}Bd - g, \qquad B = \frac{\varepsilon_0 S U^2}{md^3}.$$

Therefore

$$a = \frac{v\,\mathrm{d}v}{\mathrm{d}x} = a_0 + Bx,$$

i.e.

$$v\,\mathrm{d}v = (a_0 + Bx)\,\mathrm{d}x.$$

Integrating both sides,

$$\int_0^v v'\,\mathrm{d}v' = \int_0^x (a_0 + Bx')\,\mathrm{d}x',$$

which gives

$$\tfrac{1}{2}v^2 = a_0 x + \tfrac{1}{2}Bx^2,$$

or

$$v = \sqrt{2a_0 x + Bx^2},$$

where v is the speed of plate 3 when it is at the distance x from plate 2. The last equation is

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{2a_0 x + Bx^2}}. \tag{9}$$

Integrating both sides, the time interval in which plate 3 travels from plate 2 to plate 1 (a displacement d) is

$$t_1 = \int_0^{t_1} \mathrm{d}t = \int_0^d \frac{\mathrm{d}x}{\sqrt{2a_0 x + Bx^2}}.$$

Carrying out the integration,

$$t_1 = \sqrt{\frac{1}{B}} \ln\left[\frac{(3Bd - 2g) + \sqrt{8Bd(Bd - g)}}{Bd - 2g}\right].$$

Substituting $B = \varepsilon_0 SU^2/(md^3)$,

$$t_1 = \frac{d}{U}\sqrt{\frac{md}{\varepsilon_0 S}} \ln\left[\frac{(3\varepsilon_0 SU^2 - 2mgd^2) + 2U\sqrt{2\varepsilon_0 S(\varepsilon_0 SU^2 - mgd^2)}}{\varepsilon_0 SU^2 - 2mgd^2}\right]. \tag{10}$$

When plate 3 reaches plate 1, the positive charge on its upper face and the negative charge on plate 1 are exchanged and cancel each other; the charge on its lower face is

$$-Q_2 = -\varepsilon_0 E_2 S = -Q = -\frac{\varepsilon_0 S}{d}U,$$

and the charge on plate 2 is

$$Q_2 = \varepsilon_0 E_2 S = Q = \frac{\varepsilon_0 S}{d}U.$$

After the collision with plate 1 the speed of plate 3 is zero; under gravity and the electric force it moves downward again and undergoes a perfectly inelastic collision with plate 2. During the downward motion of plate 3 its total charge is

$$-Q = -\frac{\varepsilon_0 S}{d}U.$$

After plate 3 has left plate 1, let the surface charge densities of the plates be as shown in Figure b. Conservation of charge gives

$$\sigma_1' - \sigma_2' = -\frac{\varepsilon_0}{d}U. \tag{11}$$

The field strengths E_1' and E_2' between the plates of the upper and of the lower capacitor (see Figure b) are

$$E_1' = \frac{\sigma_1'}{\varepsilon_0}, \qquad E_2' = \frac{\sigma_2'}{\varepsilon_0}.$$

Substituting this into (11),

$$E_1' - E_2' = -\frac{U}{d}. \tag{12}$$

In addition, the total potential difference across the two capacitors in series is U, so

$$E_2' x + E_1'(d - x) = U. \tag{13}$$

Combining (12) and (13),

$$E_1' = \frac{U}{d}\frac{(d-x)}{d}, \tag{14}$$

$$E_2' = \frac{U}{d}\frac{2d-x}{d}. \tag{15}$$

Hence the surface charge densities are

$$\sigma_1' = \varepsilon_0 E_1', \qquad \sigma_2' = \varepsilon_0 E_2',$$

and the electric force on plate 3 is

$$\begin{aligned} F_{\mathrm{e}}' &= -\sigma_2' S\frac{E_2'}{2} + \sigma_1' S\frac{E_1'}{2} = \frac{\varepsilon_0 S}{2}\left(E_1'^2 - E_2'^2\right) = \varepsilon_0 S(E_1' - E_2')\frac{E_1' + E_2'}{2} \\ &= \varepsilon_0 S\left(-\frac{U}{d}\right)\cdot\frac{E_1'+E_2'}{2} = \varepsilon_0 S\left(-\frac{U}{d}\right)\frac{U}{d}\left(\frac{3d-2x}{2d}\right) = -\frac{\varepsilon_0 S U^2}{2d^3}(3d-2x). \end{aligned} \tag{16}$$

The resultant vertical force on plate 3 is

$$F_{\mathrm{total}}' = F_{\mathrm{e}}' - mg = -\frac{\varepsilon_0 S U^2}{2d^3}(3d-2x) - mg = -\left(\frac{3\varepsilon_0 S U^2}{2d^2} + mg\right) + \frac{\varepsilon_0 S U^2}{d^3}x.$$

Hence the acceleration of plate 3 at the position shown in Figure b is

$$a' = \frac{F_{\mathrm{total}}'}{m} = -\left(\frac{3\varepsilon_0 S U^2}{2md^2} + g\right) + \frac{\varepsilon_0 S U^2}{md^3}x. \tag{17}$$

Because $\dfrac{\varepsilon_0 S U^2}{2md^2} + g \geq 0$, we have $a' < 0$, so plate 3 can accelerate downward all the way. Put

$$a' = -a_0' - B'(d-x),$$

where

$$a_0' = \tfrac{1}{2}B'd + g, \qquad B' = B = \frac{\varepsilon_0 S U^2}{md^3}.$$

Repeating the treatment used above for the upward motion of plate 3,

$$\tfrac{1}{2}v^2 = \left[a_0'(d-x) + \tfrac{1}{2}B(d-x)^2\right],$$

where v is the speed of plate 3 after it has left plate 1, when its coordinate is x (the origin is at plate 2). The last equation is

$$\mathrm{d}t = -\frac{\mathrm{d}x}{\sqrt{2a_0'(d-x) + B(d-x)^2}}. \tag{18}$$

Integrating both sides, the time t_2 in which plate 3 travels from plate 1 to plate 2 (a displacement $-d$) is

$$t_2 = \int_0^{t_2}\mathrm{d}t = -\int_d^0 \frac{\mathrm{d}x}{\sqrt{2a_0'(d-x) + B(d-x)^2}}.$$

Carrying out the integration,

$$t_2 = \sqrt{\frac{1}{B}}\,\ln\left[\frac{(3Bd+2g) + \sqrt{8Bd(Bd+g)}}{Bd+2g}\right].$$

Substituting $B = \varepsilon_0 SU^2/(md^3)$,[b]

$$t_2 = \frac{d}{U}\sqrt{\frac{md}{\varepsilon_0 S}}\ln\left[\frac{(3\varepsilon_0 SU^2 + 2mgd^2) + 2U\sqrt{\varepsilon_0 S(2\varepsilon_0 SU^2 + mgd^2)}}{\varepsilon_0 SU^2 + 2mgd^2}\right]. \tag{19}$$

After plate 3 has travelled from plate 1 to plate 2 it undergoes a perfectly inelastic collision with plate 2 and its speed becomes zero. The negative charge on its lower face and the positive charge on plate 2 are exchanged and cancel each other; the charge on its upper face is

$$Q_1' = \varepsilon_0 E_1' S = Q = \frac{\varepsilon_0 S}{d}U,$$

and the charge on plate 1 is

$$-Q_1' = -\varepsilon_0 E_1' S = -Q = -\frac{\varepsilon_0 S}{d}U.$$

The state of the system is now exactly the same as the initial state; plate 3 has completed one full period of its motion, and afterwards it repeats the same up-and-down motion. The period of the motion of the conducting plate 3 is therefore

$$\begin{aligned} T &= t_1 + t_2 \\ &= \frac{d}{U}\sqrt{\frac{md}{\varepsilon_0 S}}\left\{\ln\left[\frac{(3\varepsilon_0 SU^2 - 2mgd^2) + 2U\sqrt{2\varepsilon_0 S(\varepsilon_0 SU^2 - mgd^2)}}{\varepsilon_0 SU^2 - 2mgd^2}\right]\right. \\ &\qquad \left. + \ln\left[\frac{(3\varepsilon_0 SU^2 + 2mgd^2) + 2U\sqrt{\varepsilon_0 S(2\varepsilon_0 SU^2 + mgd^2)}}{\varepsilon_0 SU^2 + 2mgd^2}\right]\right\}, \end{aligned} \tag{20}$$

where (10) and (19) have been used.

[a] *Translator's note:* The Chinese original prints "plate 2" here; it should read "plate 1", since plate 3 starts on plate 2 and moves upwards.

[b] *Translator's note:* Carrying out the substitution gives $2U\sqrt{2\varepsilon_0 S(\varepsilon_0 SU^2 + mgd^2)}$ for the square root in (19) (and likewise inside (20)); the source prints $2U\sqrt{\varepsilon_0 S(2\varepsilon_0 SU^2 + mgd^2)}$, which is a misprint.

Problem 3 (rope driven by a rotating pulley)

Solution

(1) Consider the vertical parts of the rope on the left and on the right of the pulley and an arbitrary small element of the rope on the pulley. For the vertical part of the rope on the left of the pulley, take downward as positive. In the time $\mathrm{d}t$ a small element $\mathrm{d}x = v\,\mathrm{d}t$ is fed in from the pulley, while another element of the same length falls onto the ground; the momentum brought in from the ground is zero, so the change of momentum is $\lambda L\,\mathrm{d}v$. The part of the soft rope that has fallen onto the ground exerts no reaction on the rope, so the force acting on this part is $\lambda Lg - T_1$ and its impulse is $(\lambda Lg - T_1)\mathrm{d}t$. The momentum theorem gives

$$-T_1 + \lambda Lg = \lambda L\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{1}$$

For the vertical part of the rope on the right, take upward as positive. First consider the small element $\mathrm{d}x = v\,\mathrm{d}t$ lifted from the ground in the time $\mathrm{d}t$; its momentum changes from 0 to $\lambda\,\mathrm{d}x\,v = \lambda v^2\mathrm{d}t$, and the vertical part of the rope on the right acts on it with the force

$T' - \lambda\,\mathrm{d}x\,g$, where T' is the tension of the rope at the ground. The momentum theorem gives

$$T' - \lambda\,\mathrm{d}x\,g = \lambda v^2.$$

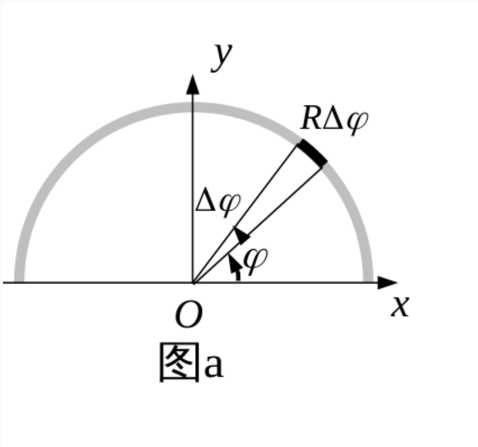
Dropping the infinitesimal,

$$T' = \lambda v^2.$$

The change of momentum of the vertical part of the rope on the right is $\lambda L\,\mathrm{d}v$, and the force acting on it is $T_2 - \lambda L g - T' = T_2 - \lambda L g - \lambda v^2$. The momentum theorem gives

$$T_2 - \lambda v^2 - \lambda L g = \lambda L \frac{\mathrm{d}v}{\mathrm{d}t}. \tag{2}$$

For the small element $R\Delta\varphi$ of the soft rope on the pulley at the angle φ (counterclockwise is taken as positive; see Figure a), the mass is $\lambda R\Delta\varphi$, the tensions at its two ends are $T(\varphi+\Delta\varphi)$ and $T(\varphi)$, and the supporting force of the pulley on it is $NR\Delta\varphi$, where N is the supporting force per unit length of the rope at φ. Newton's second law along the tangential and along the normal direction gives



Labels: 图*a = Figure a.*

$$\begin{aligned}
&T(\varphi+\Delta\varphi)\cos\frac{\Delta\varphi}{2} - T(\varphi)\cos\frac{\Delta\varphi}{2} + \mu N R\Delta\varphi - \lambda R\Delta\varphi\, g\cos\varphi \\
&\qquad = \lambda R\Delta\varphi \frac{\mathrm{d}v}{\mathrm{d}t}, \\
&T(\varphi+\Delta\varphi)\sin\frac{\Delta\varphi}{2} + T(\varphi)\sin\frac{\Delta\varphi}{2} - N R\Delta\varphi + \lambda R\Delta\varphi\, g\sin\varphi \\
&\qquad = \lambda R\Delta\varphi \frac{v^2}{R}.
\end{aligned}$$

When $\Delta\varphi \to 0$ these two equations become

$$\frac{\mathrm{d}T}{\mathrm{d}\varphi} + \mu N R - \lambda R g\cos\varphi = \lambda R \frac{\mathrm{d}v}{\mathrm{d}t}, \tag{3}$$

$$T - NR + \lambda R g \sin\varphi = \lambda R \frac{v^2}{R}. \tag{4}$$

Eliminating N from (3) and (4),

$$\frac{\mathrm{d}T}{\mathrm{d}\varphi} + \mu T - \lambda R g\big(\cos\varphi - \mu\sin\varphi\big) = \lambda R\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{5}$$

Equations (1), (2), (3), (4), or equivalently (1), (2), (5), form the set of dynamical equations satisfied by the motion of the soft rope.

(2) Depending on the value of ω there are two possible cases. In the first, ω is large enough that the rope is still slipping on the pulley when the rope reaches its maximum speed. In the second, ω is small, and when the rope reaches its maximum speed there is no slipping between the rope and the pulley, so the maximum speed of the rope is just $R\omega$.

(Method 1) Note that only T_1 and T_2 appear in (1) and (2), so we only need the relation between T_1 and T_2 obtained from (5); $T(\varphi)$ itself is not needed. With the help of the relation

$$\frac{\mathrm{d}T}{\mathrm{d}\varphi} + \mu T = \mathrm{e}^{-\mu\varphi}\frac{\mathrm{d}(T\mathrm{e}^{\mu\varphi})}{\mathrm{d}\varphi}, \tag{6}$$

equation (5) can be written as

$$\frac{\mathrm{d}(T\mathrm{e}^{\mu\varphi})}{\mathrm{d}\varphi} = \lambda Rg\,\mathrm{e}^{\mu\varphi}(\cos\varphi - \mu\sin\varphi) + \mathrm{e}^{\mu\varphi}\lambda R\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{7}$$

Integrating both sides,

$$T_1\mathrm{e}^{\mu\pi} - T_2 = \lambda Rg\left(\int_0^\pi \mathrm{e}^{\mu\varphi}\cos\varphi\,\mathrm{d}\varphi - \mu\int_0^\pi \mathrm{e}^{\mu\varphi}\sin\varphi\,\mathrm{d}\varphi\right) + \lambda R\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right)\int_0^\pi \mathrm{e}^{\mu\varphi}\mathrm{d}\varphi, \tag{8}$$

that is,

$$T_2 - T_1\mathrm{e}^{\mu\pi} = \lambda Rg\,\frac{2\mu}{1+\mu^2}\left(\mathrm{e}^{\mu\pi}+1\right) - \frac{\lambda R}{\mu}\left(\mathrm{e}^{\mu\pi}-1\right)\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{9}$$

[**(Method 2)** Put

$$T = \bar{T} + C_1\sin\varphi + C_2\cos\varphi + C_3. \tag{6$'$}$$

Substituting this into (5) and requiring the coefficients of the trigonometric terms and the constant terms on the two sides to be equal,

$$C_1 = \lambda Rg\,\frac{1-\mu^2}{1+\mu^2}, \qquad C_2 = \lambda Rg\,\frac{2\mu}{1+\mu^2}, \qquad C_3 = \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{7$'$}$$

The equation becomes

$$\frac{\mathrm{d}\bar{T}}{\bar{T}} = -\mu\,\mathrm{d}\varphi,$$

and integration gives

$$\bar{T} = \bar{T}_0\mathrm{e}^{-\mu\varphi}, \tag{8$'$}$$

that is,

$$T = \bar{T}_0\mathrm{e}^{-\mu\varphi} + \lambda Rg\frac{1-\mu^2}{1+\mu^2}\sin\varphi + \lambda Rg\frac{2\mu}{1+\mu^2}\cos\varphi + \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right).$$

At the two points where the soft rope is tangent to the pulley, corresponding to $\varphi = 0$ and $\varphi = \pi$, the tensions are $T_2 = T(0)$ and $T_1 = T(\pi)$:

$$T_2 = \bar{T}_0 + \lambda Rg\frac{2\mu}{1+\mu^2} + \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right),$$

$$T_1 = \bar{T}_0\mathrm{e}^{-\mu\pi} - \lambda Rg\frac{2\mu}{1+\mu^2} + \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right).$$

Hence

$$T_2 - T_1 e^{\mu\pi} = \lambda R g \frac{2\mu}{1+\mu^2}\left(e^{\mu\pi}+1\right) - \frac{\lambda R}{\mu}\left(e^{\mu\pi}-1\right)\left(\frac{dv}{dt} + \mu\frac{v^2}{R}\right). \tag{9'}$$

]
From (1) and (2),

$$T_2 - T_1 e^{\mu\pi} = \lambda v^2 - \lambda L g\left(e^{\mu\pi}-1\right) + \lambda L\left(e^{\mu\pi}+1\right)\frac{dv}{dt}. \tag{10}$$

From (9) and (10),

$$Lg\left(e^{\mu\pi}-1\right) + Rg\frac{2\mu}{1+\mu^2}\left(e^{\mu\pi}+1\right) - e^{\mu\pi}v^2 = \left(L\left(e^{\mu\pi}+1\right) + \frac{R}{\mu}\left(e^{\mu\pi}-1\right)\right)\frac{dv}{dt}. \tag{11}$$

Assume that the rope always slips on the pulley. When $\frac{dv}{dt} = 0$ the rope has reached its maximum speed $v_{\max}$, and (11)[a] gives

$$v_{\max}^2 = Lg\left(1-e^{-\mu\pi}\right) + Rg\frac{2\mu}{1+\mu^2}\left(1+e^{-\mu\pi}\right),$$

that is,

$$v_{\max} = \sqrt{Lg\left(1-e^{-\mu\pi}\right) + Rg\frac{2\mu}{1+\mu^2}\left(1+e^{-\mu\pi}\right)}. \tag{12}$$

In this case one must have

$$R\omega > \sqrt{Lg\left(1-e^{-\mu\pi}\right) + Rg\frac{2\mu}{1+\mu^2}\left(1+e^{-\mu\pi}\right)}. \tag{13}$$

If this inequality is not satisfied, then there is no longer relative slipping between the rope and the pulley when the rope reaches its maximum speed, which corresponds to

$$v_{\max} = R\omega. \tag{14}$$

[In part (2), since only the maximum speed is asked for, one may set $\frac{dv}{dt} = 0$ in (1), (2) and (5) right at the start.]

[a] *Translator's note:* The source prints "(12)" here; the equation actually used is (11).

Problem 4 (forced vibrations and standing waves on a string)

Solution

(1) (i) In the steady state, let the wave travelling to the right on the string be

$$y_R(x,t) = A_1 e^{-\gamma x}\cos\left(\omega t - \frac{\omega x}{u} + \varphi_1\right), \tag{1}$$

where A_1 and φ_1 are coefficients to be determined. Since the right end of the string is fixed, the wave reflected at the right end is

$$y_L(x,t) = A_1 e^{-2\gamma L + \gamma x}\cos\left(\omega t + \frac{\omega x}{u} - 2\frac{\omega L}{u} + \pi + \varphi_1\right). \tag{2}$$

Therefore the transverse vibration on the string in the steady state is

$$y(x,t) = y_{\mathrm{L}}(x,t) + y_{\mathrm{R}}(x,t) = A(x)\cos\left[\omega t + \varphi(x)\right],$$

where

$$A(x) = A_1\sqrt{\mathrm{e}^{-2\gamma x} + \mathrm{e}^{-4\gamma L+2\gamma x} - 2\mathrm{e}^{-2\gamma L}\cos\left(2\frac{\omega}{u}x - 2\frac{\omega L}{u}\right)}. \tag{3}$$

Since the vibration of the left end of the string in the steady state is known,

$$A(x=0) = A_1\sqrt{1 + \mathrm{e}^{-4\gamma L} - 2\mathrm{e}^{-2\gamma L}\cos\left(2\frac{\omega L}{u}\right)} = A_0.$$

Comparing the last two equations,

$$A_1 = \frac{A_0}{\sqrt{1 + \mathrm{e}^{-4\gamma L} - 2\mathrm{e}^{-2\gamma L}\cos\left(2\frac{\omega L}{u}\right)}}. \tag{4}$$

Hence the amplitude of the vibration at every point of the string in the steady state is

$$\begin{aligned} A(x) = {}& \frac{A_0}{\sqrt{1 + \mathrm{e}^{-4\gamma L} - 2\mathrm{e}^{-2\gamma L}\cos\left(2\frac{\omega L}{u}\right)}} \\ & \times \sqrt{\mathrm{e}^{-2\gamma x} + \mathrm{e}^{-4\gamma L+2\gamma x} - 2\mathrm{e}^{-2\gamma L}\cos\left(2\frac{\omega}{u}x - 2\frac{\omega L}{u}\right)}. \end{aligned} \tag{5}$$

(ii) Neglecting the decay of the amplitude along the propagation direction, the standing wave excited on the string can be written as

$$y(x,t) = A\sin\left(\frac{\omega x}{u} + \varphi\right)\cos(\omega t + \phi). \tag{6}$$

The right end of the string is fixed, i.e.

$$y(x=L,t) = 0.$$

The last two equations give

$$\frac{\omega L}{u} + \varphi = m\pi, \qquad m = 0, 1, 2, \ldots$$

We may take $m = 0$, so that

$$\varphi = -\frac{\omega L}{u}. \tag{7}$$

Other values of m give results that differ only in the corresponding value of ϕ; they are in fact equivalent. Using the vibration of the left end of the string, $y(x=0,t) = A_0\cos(\omega t)$,

$$A\sin\left(-\frac{\omega L}{u}\right)\cos(\omega t + \phi) = A_0\cos\omega t.$$

Since the last equation holds at every time t,

$$A = -\frac{A_0}{\sin\left(\frac{\omega L}{u}\right)}, \tag{8}$$

$$\phi = 0. \tag{9}$$

Substituting (8) and (9) into (6), the standing wave on the string is

$$y(x,t) = -\frac{A_0}{\sin\left(\dfrac{\omega L}{u}\right)}\sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right)\cos(\omega t). \tag{10}$$

From (10), the amplitude $A(x)$ of the vibration of the point at the coordinate x is

$$A(x) = \left|\frac{A_0}{\sin\left(\dfrac{\omega L}{u}\right)}\sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right)\right|.$$

The positions x_n of the nodes satisfy

$$\left|A(x_n)\right| = 0,$$

that is,

$$\sin\left(\frac{\omega x_n}{u} - \frac{\omega L}{u}\right) = 0,$$

from which

$$x_n = L - n\pi\frac{u}{\omega}, \tag{11}$$

where n is an integer satisfying

$$0 \le n < \frac{L\omega}{\pi u}. \tag{12}$$

At an antinode x_a the amplitude $A(x)$ is a maximum, which gives

$$x_a = L - \pi\frac{u}{2\omega} - l\pi\frac{u}{\omega}, \tag{13}$$

where l is an integer satisfying

$$0 < l < \frac{L\omega}{\pi u}. \tag{14}$$

(2) Decompose the vibration of the string into the superposition of the two cases

- A: the left end of the string vibrates, the right end is fixed;
- B: the right end of the string vibrates, the left end is fixed.

Let the vibration of the points of the string in case A be $y_1(x,t)$ and in case B be $y_2(x,t)$. Using the result of (1.ii), the standing wave on the string is

$$y_1(x,t) = -\frac{A_0}{\sin\left(\dfrac{\omega L}{u}\right)}\sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right)\cos(\omega t).$$

Using the result of (1.ii) again, the standing wave on the string is

$$y_2(x,t) = \frac{A_0}{\sin\left(\dfrac{\omega L}{u}\right)}\sin\left(\frac{\omega x}{u}\right)\cos(\omega t + \varphi_0). \tag{15}$$

When both the left and the right end of the string vibrate, the resultant vibration at every point of the string is the superposition of the last two expressions:

$$y(x,t) = y_1(x,t) + y_2(x,t) = \frac{A_0}{\sin\left(\dfrac{\omega L}{u}\right)}\left[\sin\left(\frac{\omega L}{u} - \frac{\omega x}{u}\right)\cos\omega t + \sin\left(\frac{\omega x}{u}\right)\cos(\omega t + \varphi_0)\right].$$

When $\varphi_0 = 0$, the resultant vibration at every point of the string is

$$y(x,t) = \frac{A_0}{\cos\left(\frac{\omega L}{2u}\right)} \cos\left(\frac{\omega L}{2u} - \frac{\omega x}{u}\right) \cos\omega t. \tag{16}$$

At resonance

$$\cos\left(\frac{\omega L}{2u}\right) = 0,$$

from which

$$\omega = \frac{(2n_1+1)\pi u}{L}, \qquad n_1 = 0, 1, 2, 3 \ldots \tag{17}$$

When $\varphi_0 = \pi$, the resultant vibration at every point of the string is

$$y(x,t) = \frac{A_0}{\sin\left(\frac{\omega L}{2u}\right)} \sin\left(\frac{\omega L}{2u} - \frac{\omega x}{u}\right) \cos\omega t. \tag{18}$$

At resonance

$$\sin\left(\frac{\omega L}{2u}\right) = 0,$$

from which

$$\omega = \frac{2n_2\pi u}{L}, \qquad n_2 = 1, 2, 3 \ldots \tag{19}$$

Problem 5 (gas leaking from a container in outer space)

Solution

(1) Let the hole be perpendicular to the x direction. The number of gas molecules leaving the hole in the time $\mathrm{d}t$ equals the number of gas molecules, in the cylinder of volume $v_x S\,\mathrm{d}t$ just inside the hole, whose velocity component perpendicular to the hole lies between v_x and $v_x + \mathrm{d}v_x$. The number of such molecules in that cylinder is $n_{v_x} S v_x \mathrm{d}t\,\mathrm{d}v_x$, where n_{v_x} is the number density of gas molecules with velocity v_x; it is related to the number density n by

$$n_{v_x} = n f(v_x) = n\sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv_x^2}{2kT}\right). \tag{1}$$

The average number of gas molecules leaving the hole per unit time and per unit area is

$$N_{\text{leak}} = \int_0^\infty n_{v_x} v_x\,\mathrm{d}v_x = \sqrt{\frac{m}{2\pi kT}} \int_0^\infty v_x\, n \exp\left(-\frac{mv_x^2}{2kT}\right) \mathrm{d}v_x = n\sqrt{\frac{kT}{2\pi m}}. \tag{2}$$

(2) When the temperature of the gas is T, the average kinetic energy of the molecules leaving the hole is

$$\bar{E}_{\text{k}} = \frac{1}{N_{\text{leak}}} \int_0^\infty \tfrac{1}{2} m v_x^2\, n_{v_x} v_x\,\mathrm{d}v_x + kT, \tag{3}$$

where kT is the contribution to the average kinetic energy of the velocity components perpendicular to the x direction. Substituting (1) and (2) into (3),

$$\begin{aligned}\bar{E}_{\mathrm{k}} &= \frac{1}{n\sqrt{\dfrac{kT}{2\pi m}}}\int_0^\infty \tfrac{1}{2}m\sqrt{\frac{m}{2\pi kT}}\,v_x^3\exp\!\left(-\frac{mv_x^2}{2kT}\right)\mathrm{d}v_x + kT \\ &= \frac{1}{\sqrt{\dfrac{kT}{2\pi m}}}\cdot\tfrac{1}{2}m\sqrt{\frac{m}{2\pi kT}}\cdot\frac{1}{2(m/2kT)^2} + kT = 2kT.\end{aligned} \tag{4}$$

(3) Since the hole is perpendicular to the x direction, symmetry shows that after the gas has leaked out the container moves along the negative x direction. Let the number of gas molecules in the container at the time t be N and the speed of the container as a whole be u; at the time $t+\mathrm{d}t$ let the number of gas molecules in the container be $N+\mathrm{d}N$ and the speed of the container as a whole be $u+\mathrm{d}u$. The momentum of the whole system is conserved in this process:

$$(M+Nm)u = \big[(M+Nm+m\,\mathrm{d}N)(u+\mathrm{d}u)\big] - m\,\mathrm{d}N\,(u-\bar{v}_x), \tag{5}$$

where $\bar{v}_x$ is the average speed in the x direction of the gas molecules that leave in the time $\mathrm{d}t$; note that $\mathrm{d}N$ is negative. Simplifying and dropping the higher-order infinitesimals,

$$(M+Nm)\,\mathrm{d}u = -m\bar{v}_x\,\mathrm{d}N. \tag{6}$$

The energy of the system is conserved in this process, so

$$\begin{aligned}\tfrac{1}{2}(M+Nm)u^2 + \tfrac{3}{2}NkT = \tfrac{1}{2}(M+Nm+\mathrm{d}N\,m)(u+\mathrm{d}u)^2 + \tfrac{3}{2}(N+\mathrm{d}N)k(T+\mathrm{d}T) \\ -\tfrac{1}{2}\mathrm{d}N\,m\,\overline{(u-v_x)^2} - \mathrm{d}N\,kT.\end{aligned} \tag{7}$$

The two terms on the left are the translational kinetic energy of the system as a whole and the internal energy before $\mathrm{d}t$. The first two terms on the right are the translational kinetic energy of the system as a whole and the internal energy after $\mathrm{d}t$; the last two terms correspond to the average kinetic energy of the gas molecules that have leaked out. Simplifying and dropping the higher-order infinitesimals,

$$(M+Nm)u\,\mathrm{d}u + \tfrac{3}{2}Nk\,\mathrm{d}T + \tfrac{1}{2}kT\,\mathrm{d}N + mu\bar{v}_x\,\mathrm{d}N - \tfrac{1}{2}m\overline{v_x^2}\,\mathrm{d}N = 0. \tag{8}$$

Substituting (6) into (8),

$$\left(\tfrac{1}{2}m\overline{v_x^2} - \tfrac{1}{2}kT\right)\mathrm{d}N = \tfrac{3}{2}Nk\,\mathrm{d}T. \tag{9}$$

From (4),

$$\tfrac{1}{2}m\overline{v_x^2} + kT = 2kT,$$

that is,

$$\tfrac{1}{2}m\overline{v_x^2} = kT.$$

Substituting the last equation into (9),

$$\tfrac{1}{2}kT\,\mathrm{d}N = \tfrac{3}{2}Nk\,\mathrm{d}T,$$

that is,

$$\frac{\mathrm{d}N}{N} = 3\frac{\mathrm{d}T}{T}. \tag{10}$$

Integrating both sides,

$$T = CN^{1/3}, \tag{11}$$

where C is a constant of integration. Inserting the initial conditions,

$$T_0 = CN_0^{1/3}, \qquad C = \frac{T_0}{N_0^{1/3}}. \tag{12}$$

By (2), the number of gas molecules leaking out in the time $\mathrm{d}t$ is

$$\mathrm{d}N = -\frac{N}{V} S \sqrt{\frac{kT}{2\pi m}}\,\mathrm{d}t = -\frac{N^{7/6}}{V} S \sqrt{\frac{kC}{2\pi m}}\,\mathrm{d}t. \tag{13}$$

Integrating,

$$N = N_0 \left(1 + \frac{S}{6V}\sqrt{\frac{kT_0}{2\pi m}}\,t\right)^{-6}. \tag{14}$$

Hence the temperature of the gas in the container at the time t is

$$T = CN^{1/3} = T_0 \left(1 + \frac{S}{6V}\sqrt{\frac{kT_0}{2\pi m}}\,t\right)^{-2}. \tag{15}$$

(4) From (6),

$$\mathrm{d}u = -\frac{m\bar{v}_x\,\mathrm{d}N}{M + Nm}, \tag{16}$$

where $\bar{v}_x$ is the average speed of the gas molecules that leave:

$$\begin{aligned} \bar{v}_x &= \frac{1}{N_{\text{leak}}}\int_0^\infty v_x\, n_{v_x} v_x\,\mathrm{d}v_x = \frac{1}{\sqrt{\dfrac{kT}{2\pi m}}}\int_0^\infty \sqrt{\frac{m}{2\pi kT}}\, v_x^2 \exp\left(-\frac{mv_x^2}{2kT}\right)\mathrm{d}v_x \\ &= \frac{1}{4}\frac{1}{\sqrt{\dfrac{kT}{2\pi m}}}\sqrt{\frac{m}{2\pi kT}}\sqrt{\frac{\pi}{(m/2kT)^3}} = \sqrt{\frac{\pi kT}{2m}}. \end{aligned} \tag{17}$$

Substituting (17) into (16),

$$u(t) = -\int_{N_0}^{N(t)} \sqrt{\frac{\pi m kT}{2}}\,\frac{\mathrm{d}N}{M + Nm} = -\sqrt{\frac{\pi m k T_0}{2N_0^{1/3}}}\int_{N_0}^{N(t)} \frac{N^{1/6}\,\mathrm{d}N}{M + Nm}. \tag{18}$$

When the mass of the container is much larger than the mass of the gas inside it ($M \gg N_0 m$), the last expression may be approximated by

$$u(t) \approx -\frac{1}{M}\sqrt{\frac{\pi m k T_0}{2N_0^{1/3}}}\int_{N_0}^{N(t)} N^{1/6}\,\mathrm{d}N = -\frac{6}{7M}\sqrt{\frac{\pi m k T_0}{2N_0^{1/3}}}\left(N(t)^{7/6} - N_0^{7/6}\right). \tag{19}$$

Substituting (14) into (19), the magnitude of the velocity of the container at the time t is

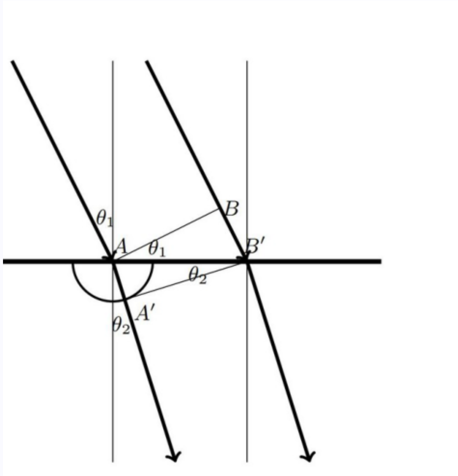
$$u(t) \approx \frac{6N_0}{7M}\sqrt{\frac{\pi m k T_0}{2}}\left[1 - \left(1 + \frac{S}{6V}\sqrt{\frac{kT_0}{2\pi m}}\,t\right)^{-7}\right]. \tag{20}$$

Problem 6 (refraction and imaging with a negative-index medium)

Solution

(1) When $n_1 > 0$ and $n_2 > 0$, as shown in Figure 1a, AB is a surface of equal optical path (equal phase) of the incident light. When the point B arrives at B', the wavefront of the wavelet emitted from the point A lies, in medium 2, on a hemisphere centred at A; in the plane of incidence of the light this sphere is a semicircle of radius r, and the radius r is determined by

$$n_2 r = n_1 \left|BB'\right|. \tag{1}$$



Labels: 图 1a = Figure 1a.

Draw the straight line $B'A'$ tangent to the circle of the wavelet phase surface at the point A'; then $B'A'$ is a surface of equal optical path (equal phase). From the geometry,

$$\left|BB'\right| = \left|AB'\right| \sin\theta_1,$$

$$r = \left|AB'\right| \sin\theta_2.$$

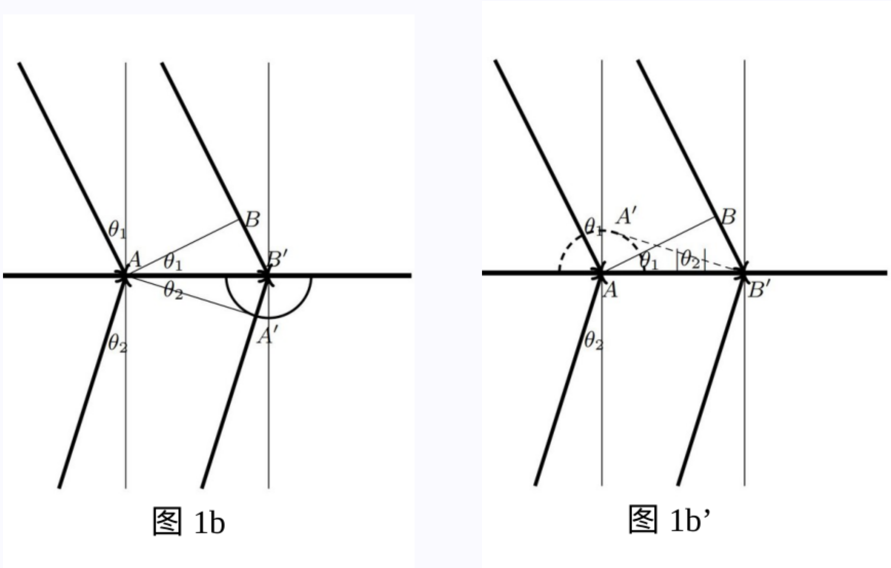
The above equations give

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{2}$$

When $n_1 > 0$ and $n_2 < 0$, the optical path of the light propagating in medium 2 is negative.

(Method 1) As shown in Figure 1b, the wavefront of the wavelet emitted from B' is a semicircle of radius r, corresponding to the optical path $n_2 r < 0$. Therefore the surface in medium 2 that is in phase with the point A is determined by the line through A tangent to the circle, with the point of tangency A'. The radius of the circle is determined by

$$n_1 \left|BB'\right| + n_2 \left|B'A'\right| = 0. \tag{3}$$



Labels: 图 *1b = Figure 1b;* 图 *1b′ = Figure 1b′.*

From the geometry,

$$\left|BB'\right| = \left|AB'\right| \sin\theta_1, \qquad \left|B'A'\right| = \left|AB'\right| \sin\left|\theta_2\right|.$$

The above equations give

$$n_1\left|AB'\right| \sin\theta_1 = -n_2\left|AB'\right| \sin\left|\theta_2\right| = n_2\left|AB'\right| \sin\theta_2, \tag{4}$$

that is,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{5}$$

[**(Method 2)** As shown in Figure 1b′, AB is a surface of equal optical path (equal phase) of the incident light. When the point B arrives at B', the wavefront of the wavelet emitted from the point A is, in the plane of incidence in medium 2, a circle centred at A with radius r, corresponding to the optical path $n_2 r < 0$. The radius r is determined by

$$\left|n_2\right| r = n_1\left|BB'\right|. \tag{3′}$$

Because the optical path is negative, this is equivalent to moving the surface of equal optical path in the opposite direction, so that it becomes the semicircle drawn as a dashed line in medium 1. Draw the straight line $B'A'$ tangent to the dashed circle of the wavelet phase surface at the point A'; then $B'A'$ is a surface of equal optical path (equal phase). From the geometry,

$$\left|BB'\right| = \left|AB'\right| \sin\theta_1, \qquad r = \left|AB'\right| \sin\left|\theta_2\right|. \tag{4′}$$

The above equations give

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{5′}$$

]

(2) Adopt the convention that the object distance s_1 on the object side and the image distance s_2 on the image side are positive, while an image distance on the object side and an object distance on the image side are negative. The angle of incidence θ_1 is positive and the angle of refraction θ_2 is negative. All the other angles marked in the figure are taken as positive. From Figure c,

$$\theta_1 = \alpha_1 + \beta, \qquad -\theta_2 = \alpha_2 - \beta. \tag{6}$$

In the paraxial condition all these angles are small, so

$$\sin\theta_1 = \theta_1, \qquad \sin\theta_2 = \theta_2,$$

and

$$\tan\alpha_1 = \sin\alpha_1 = \alpha_1 = \frac{h}{s_1}, \qquad \tan\alpha_2 = \sin\alpha_2 = \alpha_2 = \frac{h}{s_2}, \qquad \tan\beta = \sin\beta = \beta = \frac{h}{R}, \tag{7}$$

where h is the distance from the refraction point M to the optical axis. By the law of refraction,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

that is,

$$n_1\left(\frac{h}{s_1} + \frac{h}{R}\right) = -n_2\left(\frac{h}{s_2} - \frac{h}{R}\right). \tag{8}$$

Rearranging,

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}. \tag{9}$$

Note that $n_2 < 0$ and $n_1 > 0$, so the right-hand side is negative.
To calculate the transverse magnification, consider the object Q_1P_1 as shown in Figure 2, and let its image be Q_2P_2. Imagine the optical axis rotated about the point C by a small angle, so that the point P_1 lies on the optical axis; then Q_1 moves to Q_1' and Q_2 to Q_2', and the image of P_1 is P_2. In the paraxial approximation the rotation angle is very small, so

$$Q_1P_1 = Q_1Q_1', \qquad Q_2P_2 = Q_2Q_2'. \tag{10}$$

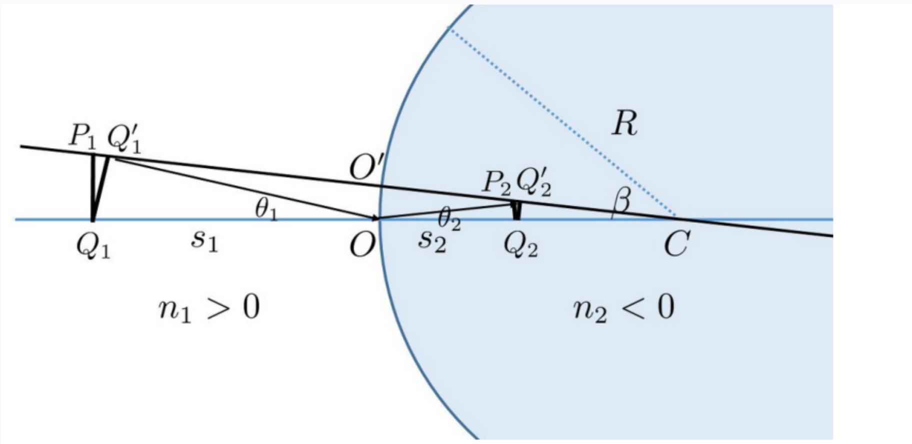


图 2

Labels: 图 2 = Figure 2.

From the geometry of Figure 2,

$$|Q_1P_1| = s_1\theta_1, \tag{11}$$

$$|Q_2P_2| = -s_2\theta_2. \tag{12}$$

Hence the transverse magnification formula is

$$\frac{|Q_2P_2|}{|Q_1P_1|} = \frac{-s_2\theta_2}{s_1\theta_1} = -\frac{s_2 n_1}{s_1 n_2}. \tag{13}$$

(3) *(12 points)* As shown in Figure 3, if medium 2 were not separated off by the spherical surface, the parallel light would converge at the focus after passing through the thin convex lens, as shown by the dashed lines in the figure. After medium 2 has been introduced, the convergence point shown by the dashed lines is a virtual object of the spherical surface, and the corresponding object distance is

$$s_1 = -(f-d). \tag{14}$$

If $n_2 > 0$, the imaging formula of the spherical surface is known to be

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}, \tag{15}$$

which is formally exactly the same as the imaging formula obtained in part (2). For both $n_2 > 0$ and $n_2 < 0$ the image distance s_2 can therefore be solved as

$$s_2 = \frac{n_2 s_1 R}{n_2 s_1 - n_1(s_1 + R)} = \frac{n_2 R(f-d)}{(n_2 - n_1)(f-d) + n_1 R}. \tag{16}$$

For the data given in the problem, the corresponding values of s_2 are listed in the table below:

No.	n_2	d	s_2
1	1.5	$0.35R$	$1.10R$
2	1.5	$0.85R$	$0.74R$
3	-1.5	$0.35R$	$0.92R$
4	-1.5	$0.85R$	$1.56R$

According to the results in the table, the ray diagram is the one shown in Figure 3.

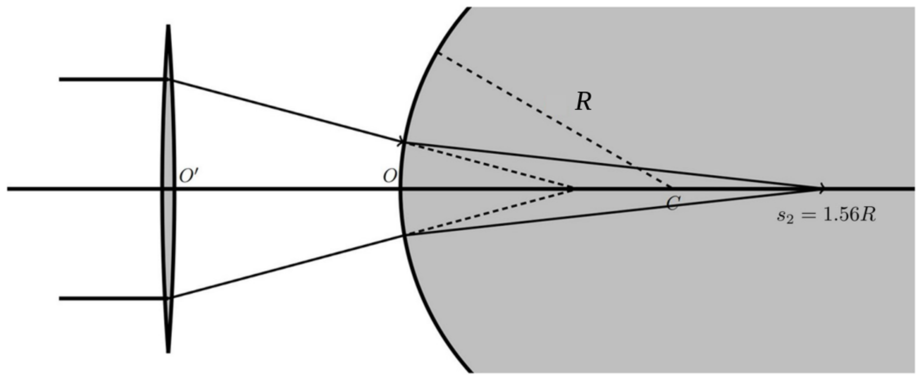


图 3

Labels: 图 3 = Figure 3.

Problem 7 (quasi-particles in a two-dimensional interface structure)

Solution

(1) By the work–energy theorem,

$$\mathrm{d}K = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\cdot\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{p}. \tag{1}$$

Using the expression for the kinetic energy of a real free particle,

$$\mathrm{d}K = \frac{\boldsymbol{p}}{m}\cdot\mathrm{d}\boldsymbol{p}. \tag{2}$$

Equations (1) and (2) give

$$\left(\frac{\boldsymbol{p}}{m} - \boldsymbol{v}\right)\cdot\mathrm{d}\boldsymbol{p} = 0.$$

Since the last equation holds for arbitrary $\mathrm{d}\boldsymbol{p}$,

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{m}. \tag{3}$$

(2) Using the expression given in the problem for the kinetic energy of the quasi-particle,

$$\mathrm{d}K = \frac{\boldsymbol{p}}{m}\cdot\mathrm{d}\boldsymbol{p} + \alpha\frac{\boldsymbol{p}}{p}\cdot\mathrm{d}\boldsymbol{p}. \tag{4}$$

Equations (1) and (4) give

$$\left(\frac{\boldsymbol{p}}{m} + \alpha\frac{\boldsymbol{p}}{p} - \boldsymbol{v}\right)\cdot\mathrm{d}\boldsymbol{p} = 0.$$

Since the last equation holds for arbitrary $\mathrm{d}\boldsymbol{p}$,

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{m} + \alpha\frac{\boldsymbol{p}}{p}. \tag{5}$$

(3) From (5),

$$v^2 = \frac{p^2}{m^2} + 2\alpha\frac{p}{m} + \alpha^2,$$

and therefore

$$v = \frac{p}{m} + \alpha. \tag{6}$$

The momentum of a quasi-particle with energy K is

$$p = -m\alpha + \sqrt{m^2\alpha^2 + 2mK}. \tag{7}$$

Equations (6) and (7) give

$$v = \sqrt{\alpha^2 + \frac{2K}{m}}. \tag{8}$$

(4) In a uniform magnetic field the quasi-particle performs uniform circular motion,

$$v = \omega R,$$

and the magnitude of the rate of change of its momentum is

$$\left|\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right| = \omega p.$$

The last two equations give

$$\left|\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right| = \frac{v}{R}\,p. \tag{9}$$

The dynamical equation for the uniform circular motion of the quasi-particle is

$$p\frac{v}{R} = qvB. \tag{10}$$

Using (7),

$$R = \frac{p}{qB} = \frac{-m\alpha + \sqrt{m^2\alpha^2 + 2mK}}{qB}. \tag{11}$$

The period of the uniform circular motion of the quasi-particle is

$$T = \frac{2\pi R}{v} = \frac{2\pi\left(-m\alpha + \sqrt{m^2\alpha^2 + 2mK}\right)}{qB\sqrt{\alpha^2 + \dfrac{2K}{m}}}. \tag{12}$$

The magnitude of the angular momentum of the quasi-particle is

$$L = pR. \tag{13}$$

Substituting (7) into (13),

$$L = \frac{2m^2\alpha^2 + 2mK - 2m\alpha\sqrt{m^2\alpha^2 + 2mK}}{qB}. \tag{14}$$

(5) (Method 1) From (5),

$$\boldsymbol{p} = m\boldsymbol{v} - m\alpha\frac{\boldsymbol{v}}{v}. \tag{15}$$

The equation of motion of the quasi-particle in the electric field is

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{E}, \tag{16}$$

so the acceleration of the quasi-particle satisfies

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{v}}{v}\right) + \frac{q}{m}\boldsymbol{E}. \tag{17}$$

In component form,[a]

$$a_x = \frac{\alpha}{v}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{\alpha v_x}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t} + qE,$$

$$a_y = \frac{\alpha}{v}\frac{\mathrm{d}v_y}{\mathrm{d}t} - \frac{\alpha v_y}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t}.$$

Using the relation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v_x}{v}a_x + \frac{v_y}{v}a_y, \tag{18}$$

equation (17) can be rewritten as

$$\begin{cases} \left(1 - \dfrac{\alpha}{v} + \dfrac{\alpha v_x^2}{v^3}\right)a_x + \dfrac{\alpha v_x v_y}{v^3}a_y = \dfrac{qE}{m} \\ \left(1 - \dfrac{\alpha}{v} + \dfrac{\alpha v_y^2}{v^3}\right)a_y + \dfrac{\alpha v_x v_y}{v^3}a_x = 0 \end{cases} \tag{19}$$

or, equivalently,

$$\left(1-\frac{\alpha\sin^2\theta}{v}\right)a_x+\frac{\alpha\sin\theta\cos\theta}{v}a_y=\frac{qE}{m},$$

$$\left(1-\frac{\alpha\cos^2\theta}{v}\right)a_y+\frac{\alpha\sin\theta\cos\theta}{v}a_x=0.$$

Solving this pair of linear equations,

$$\begin{cases} a_x=\dfrac{qE}{m}+\dfrac{qE}{m}\dfrac{\alpha\sin^2\theta}{v-\alpha} \\ a_y=-\dfrac{qE}{m}\dfrac{\alpha\cos\theta\sin\theta}{v-\alpha} \end{cases} \tag{20}$$

[**(Method 2)** From (5),

$$\boldsymbol{p}=m\boldsymbol{v}-m\alpha\frac{\boldsymbol{v}}{v}. \tag{15$'$}$$

The equation of motion of the quasi-particle in the electric field is

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=q\boldsymbol{E}. \tag{16$'$}$$

In Cartesian coordinates,

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t}\cos\theta-p\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=qE \\ \dfrac{\mathrm{d}p}{\mathrm{d}t}\sin\theta+p\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=0 \end{cases} \tag{17$'$}$$

from which

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t}=qE\cos\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t}=-\dfrac{qE}{p}\sin\theta \end{cases} \tag{18$'$}$$

From (5),

$$\begin{cases} a_x=\dfrac{\mathrm{d}v_x}{\mathrm{d}t}=\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dfrac{p}{m}+\alpha\right)\cos\theta\right]=\dfrac{1}{m}\cos\theta\dfrac{\mathrm{d}p}{\mathrm{d}t}-\left(\dfrac{p}{m}+\alpha\right)\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \\ a_y=\dfrac{\mathrm{d}v_y}{\mathrm{d}t}=\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dfrac{p}{m}+\alpha\right)\sin\theta\right]=\dfrac{1}{m}\sin\theta\dfrac{\mathrm{d}p}{\mathrm{d}t}+\left(\dfrac{p}{m}+\alpha\right)\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \end{cases} \tag{19$'$}$$

Substituting (18$'$) into (19$'$) and using (6),

$$\begin{cases} a_x=\dfrac{qE}{m}+\dfrac{qE}{m}\dfrac{\alpha\sin^2\theta}{v-\alpha} \\ a_y=-\dfrac{qE}{m}\dfrac{\alpha\cos\theta\sin\theta}{v-\alpha} \end{cases} \tag{20$'$}$$

(Method 3) From (5),

$$\boldsymbol{p}=m\boldsymbol{v}-m\alpha\frac{\boldsymbol{v}}{v}. \tag{15$''$}$$

The equation of motion of the quasi-particle in the electric field is

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=q\boldsymbol{E}. \tag{16$''$}$$

In Cartesian coordinates, (15″) can be written in component form as

$$\begin{cases} p_x = m(v-\alpha)\cos\theta \\ p_y = m(v-\alpha)\sin\theta \end{cases} \tag{17″}$$

Equations (16″) and (17″) give

$$\begin{cases} -m(v-\alpha)\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} + m\cos\theta\dfrac{\mathrm{d}v}{\mathrm{d}t} = qE \\ m(v-\alpha)\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} + m\sin\theta\dfrac{\mathrm{d}v}{\mathrm{d}t} = 0 \end{cases} \tag{18″}$$

from which

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{qE}{m}\cos\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{qE}{m}\dfrac{\sin\theta}{v-\alpha} \end{cases} \tag{19″}$$

Substituting (19″) into[b]

$$\begin{cases} a_x = \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}t}\cos\theta - \sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \\ a_y = \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}t}\sin\theta + \cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \end{cases}$$

gives

$$\begin{cases} a_x = \dfrac{qE}{m} + \dfrac{qE}{m}\dfrac{\alpha\sin^2\theta}{v-\alpha} \\ a_y = -\dfrac{qE}{m}\dfrac{\alpha\cos\theta\sin\theta}{v-\alpha} \end{cases} \tag{20″}$$

]

[a] *Translator's note:* The source prints $+qE$ in the equation for a_x; by (17) it should read $+qE/m$, as the next display confirms.

[b] *Translator's note:* As printed, the factor v is missing in the two terms of the next display; they should read $-v\sin\theta\,\mathrm{d}\theta/\mathrm{d}t$ and $+v\cos\theta\,\mathrm{d}\theta/\mathrm{d}t$, since $v_x = v\cos\theta$ and $v_y = v\sin\theta$. With the factor v restored the stated result (20) does follow.

Problem 8 (heat engine driven by the radiation pressure on a moving mirror)

Solution

(1) Take as the system under study the uniformly moving ideal mirror together with all the photons that collide with it per unit time. Let the total energies of these photons before and after the reflection be ε_1 and ε_2, and let the magnitude of the resistance force on the mirror be F. For a uniformly moving ideal mirror the total energy (kinetic plus rest energy) and the momentum are unchanged by the collision, and the ratio of the energy to the momentum of a photon is a constant independent of the frequency — the speed of light in vacuum c. The momentum theorem and the energy theorem for the system give

$$-\frac{\varepsilon_2}{c} - \frac{\varepsilon_1}{c} = -F, \tag{1}$$

$$\varepsilon_2 - \varepsilon_1 = -Fv. \tag{2}$$

The efficiency with which the photon energy is used to do work against the resistance force is

$$\eta = \frac{Fv}{\varepsilon_1}. \tag{3}$$

Combining (1) and (2),

$$\eta = \frac{2v}{c+v}. \tag{4}$$

(2) Because of the relativistic Doppler effect, the incident radiation of frequency ν_1 in the laboratory frame S is red-shifted to the frequency ν_1' in the frame S' of the mirror:

$$\frac{\nu_1'}{\nu_1} = \sqrt{\frac{c-v}{c+v}}. \tag{5}$$

This frequency ratio does not depend on the incident frequency ν_1; it depends only on the speed v of the mirror.
Let a piece of the electromagnetic wave that passes, in the time interval $\mathrm{d}t_1$ in the frame S, through some cross-section on the incident path be received by the mirror surface in the frame S' during the time interval $\mathrm{d}t_1'$. Since the number of wave crests that pass through the cross-section equals the number received by the mirror surface,

$$\nu_1 \mathrm{d}t_1 = \nu_1' \mathrm{d}t_1'. \tag{6}$$

Let the incident energy-flux spectra in the frames S and S' be $\varphi_1(\nu_1)$ and $\varphi_1'(\nu_1')$. The photons that are incident in the frame S during the time interval $\mathrm{d}t_1$ in the frequency interval $[\nu_1, \nu_1 + \mathrm{d}\nu_1]$ all arrive at the mirror surface in the frame S' during the time interval $\mathrm{d}t_1'$, so

$$\frac{\varphi_1(\nu_1)\mathrm{d}\nu_1\mathrm{d}t_1}{h\nu_1} = \frac{\varphi_1'(\nu_1')\mathrm{d}\nu_1'\mathrm{d}t_1'}{h\nu_1'}. \tag{7}$$

Equations (5), (6) and (7) give

$$\frac{\varphi_1(\nu_1)}{\nu_1} = \frac{\varphi_1'(\nu_1')}{\nu_1'}. \tag{8}$$

In the frame S' the reflected radiation has the same frequency as the incident radiation, i.e.

$$\frac{\nu_2'}{\nu_1'} = 1. \tag{9}$$

Using the fact that in this frame, for an electromagnetic wave of any frequency, the number of incident wave crests equals the number of reflected wave crests and the number of incident photons equals the number of reflected photons (an ideal mirror absorbs no photons), a derivation similar to that of (8) gives the relation between the incident and the reflected energy-flux spectra $\varphi_1'(\nu_1')$ and $\varphi_2'(\nu_2')$:

$$\frac{\varphi_1'(\nu_1')}{\nu_1'} = \frac{\varphi_2'(\nu_2')}{\nu_2'}. \tag{10}$$

Let the temperature of the one-dimensional blackbody radiation field of the incident radiation be T_1, i.e.

$$\varphi_1 = \frac{2h\nu_1}{\mathrm{e}^{h\nu_1/kT_1} - 1}. \tag{11}$$

From (5), (8), (9), (10) and (11), the incident and the reflected energy-flux spectra in the frame S' have the following one-dimensional blackbody form:

$$\varphi_1' = \frac{2h\nu_1'}{\mathrm{e}^{h\nu_1'/kT_1'} - 1}, \tag{12}$$

(1 point)

$$\varphi_2' = \frac{2h\nu_2'}{\mathrm{e}^{h\nu_2'/kT_2'} - 1}, \tag{13}$$

(1 point)

with the corresponding temperatures

$$T_1' = \frac{\nu_1'}{\nu_1} T_1 = \sqrt{\frac{c-v}{c+v}}\, T_1, \tag{14}$$

(2 points)

$$T_2' = \frac{\nu_2'}{\nu_1'} T_1' = T_1'. \tag{15}$$

(1 point)

By Carnot's theorem the efficiency of the ideal heat engine is

$$\eta' = 1 - \frac{T_2'}{T_1'} = 0. \tag{16}$$

(1 point)

This result agrees with the one obtained from the dynamical point of view. In fact, in the frame of the mirror the displacement of the mirror is zero, so the work it does on the surroundings is zero, and the efficiency with which the mirror uses the photon energy to do work against the resistance force must also be zero.

(3) The numerical values of work and of energy both depend on the reference frame chosen, so the efficiencies in the frame of the mirror and in the laboratory frame are naturally different. In fact, the reflected radiation of frequency ν_2' in the frame S' has, in the frame S, the Doppler-red-shifted frequency ν_2, with

$$\frac{\nu_2}{\nu_2'} = \sqrt{\frac{c-v}{c+v}}. \tag{17}$$

A derivation similar to that of (12) gives the reflected energy-flux spectrum $\varphi_2(\nu_2)$ in the frame S:

$$\varphi_2 = \frac{2h\nu_2}{\mathrm{e}^{h\nu_2/kT_2} - 1}. \tag{18}$$

This is exactly the one-dimensional blackbody radiation at the temperature

$$T_2 = \frac{\nu_2}{\nu_2'} T_2' = \frac{c-v}{c+v}\, T_1, \tag{19}$$

where (14), (15) and (17) have been used in the derivation. The efficiency of the ideal heat engine is

$$\eta = 1 - \frac{T_2}{T_1} = \frac{2v}{c+v}. \tag{20}$$

This also agrees with the result obtained from the dynamical point of view.