The 36th Chinese Physics Olympiad

Final — Theoretical Examination (Problems)

2019

Translator's note: the source document ("Problems and reference solutions of the theoretical examination of the 36th CPhO finals", 26 September 2019) prints the problems and the official reference solutions together. This file contains the six problems only; the reference solutions are translated in the companion file. In problem 6 the original uses Chinese characters as subscripts; they have been replaced by Latin abbreviations: S_{SE} *= Sun–Earth distance (日地),* S_{SV} *= Sun–Venus distance (日金),* T_{E}*,* T_{V} *= orbital periods of the Earth and of Venus.*

Problem 1

Every new model of car must pass destructive testing before it leaves the factory. In one car test, one of the three spokes of a car wheel was knocked off; the cross section of the wheel is shown in the figure. The wheel may be regarded as made of a wheel disc with inner and outer radii $R_1 = \dfrac{4}{5}R_0$ and R_0, together with two spokes (assume that the disc may be treated as a uniform annular disc and each spoke as a uniform thin rod; each spoke has mass m, and the mass of the disc is $M = 8m$). The wheel is released from rest at the position shown in figure a; at the moment of release, the bisector of the angle spanned by the two spokes is exactly horizontal. After the release the wheel rolls without slipping on the horizontal ground. Find

(1) the angular acceleration of the wheel just after the release;

(2) the angular acceleration of the wheel, and the friction force and the normal force exerted by the ground on the wheel, at the instant when the spoke OB first reaches the vertical position after the release.

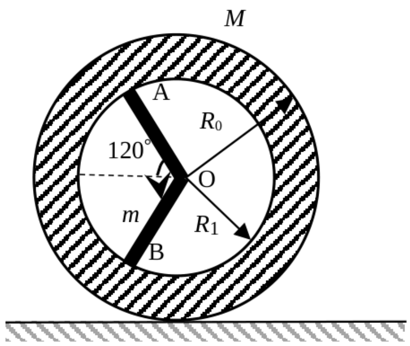

图 a

Problem 2

A half cylinder of length l, with inner radius a and outer radius b, is made of two different lossy dielectrics. Their relative permittivities and conductivities are ε_{r1} and σ_1 (in the region $0 \le \varphi < \theta_0$), and ε_{r2} and σ_2 (in the region $\theta_0 < \varphi \le \pi$). The cross section is shown in figure a. Metal films are deposited on the two bottom faces on the two sides of the half cylinder; a DC voltage V_0 is applied between the two metal films, and a steady state has been reached. The permittivity of vacuum is ε_0; the relative permittivities of the two dielectrics are large and their conductivities are small. Neglect edge effects.

(1) Find the distributions of the electric field strength and of the potential inside the dielectrics;

(2) find the total charge on the interface $\varphi = \theta_0$;

(3) compute separately the resistance and the capacitance of the two dielectric regions $0 \le \varphi < \theta_0$ and $\theta_0 < \varphi \le \pi$;

(4) the power supply is disconnected at $t = 0$; find how the voltage between the two metal films then varies with time (the process may be treated as quasi-steady), and draw the corresponding equivalent circuit.

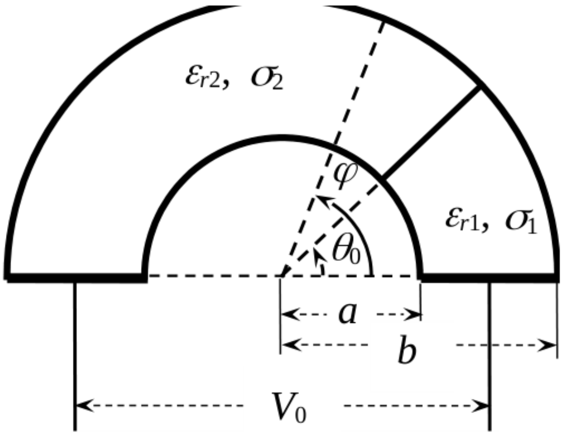

图 a

Problem 3

Long-term high-precision observation of the small variations of the gravitational acceleration at the surface of the Earth, and of their distribution, with a superconducting gravimeter yields information about the internal structure of the Earth — mantle motion, mass distribution, solid tides, groundwater and mineral distribution — as well as external data such as ocean tides and climate change. A superconducting gravimeter has high sensitivity (down to 10^{-9} of the surface gravitational acceleration) and good stability, and is at present the gravity-observing device most widely used in the earth sciences. As the number of superconducting gravimeters distributed over the globe increases, and as observation networks and data sharing are established, new opportunities are provided for the development of the earth sciences and for precise tests of the law of gravitation. Answer the following questions about the gravity change caused by the distribution of groundwater and about the principle of the superconducting gravimeter.

(1) Assume that the Earth may be regarded as a uniform sphere of radius 6370 km whose surface gravitational acceleration is $g_0 = 9.80\ \mathrm{m{\cdot}s^{-2}}$ (the rotation of the Earth is not taken into account). At a certain place there is, underground, a spherical underground lake of diameter 20 km filled with water, whose centre is 15 km below the ground surface. Find the small change Δg of the gravitational acceleration at the ground surface directly above the lake. The gravitational constant is $G = 6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2{\cdot}kg^{-2}}$ and the density of water is $\rho_\mathrm{w} = 1.0 \times 10^3\ \mathrm{kg{\cdot}m^{-3}}$.

(2) The core part of a superconducting gravimeter is a superconducting spherical shell made of the metal niobium (Nb), suspended in a magnetic field; for ease of calculation we simplify it here to a thin superconducting ring (see figure a). The levitating magnetic field is provided

by a quasi-Helmholtz coil pair, made of an upper and a lower coaxial thin superconducting coil, with αN and N turns respectively ($\alpha < 1$), connected in series; the radii of the two coils and the distance between their centres are all equal to R. Before the gravimeter is adjusted, the currents in the Helmholtz coils and in the superconducting ring are both zero, and the ring lies in the plane of the lower coil ($z = 0$), coaxial with the coils. When the adjustment starts, an external source slowly raises the current in the quasi-Helmholtz coils to i_0; during this process an induced current appears in the superconducting ring. The value of i_0 is adjusted so that the superconducting ring is levitated exactly in the plane of the upper coil ($z = R$), as shown in figure a. Because the resistances of the superconducting coils and of the superconducting ring are both zero (the temperature at their location is 4.2 K), after the ring has been levitated stably one can remove the source that supplied the current in the superconducting coils: the current i_0 in the superconducting coils, and the magnetic field it produces, then stay stable for a long time. The superconducting ring has mass m, diameter D ($D \ll 2R$) and self-inductance L. Near the plane of the upper coil the z component and the radial component of the magnetic induction B produced by the quasi-Helmholtz coils can be approximated by

$$\begin{cases} B_z = B_0\left[1 - \beta(z - R)\right], \\ B_r = \frac{1}{2}\beta B_0 r, \end{cases}$$

where B_0 is the magnetic induction at the centre of the upper coil, β is the radial coefficient, and r is the distance from the axis. Using the above known parameters of the coils and of the superconducting ring, and the externally supplied current i_0 in the coils, find

(i) the expressions for B_0 and β;

(ii) the expression for the induced current I_0 in the superconducting ring at the equilibrium position ($z = R$);

(iii) the expression for the gravitational acceleration g_0 at the place where the superconducting gravimeter is located.

(3) In order to measure small changes of the gravitational acceleration accurately, an AC bridge balance method is used in the superconducting gravimeter. To understand its principle and to simplify the calculation, regard the superconducting ring as a thin “plate” of area A, and fix, at positions symmetric about the ring, i.e. at $R+d/2$ and $R-d/2$, two metal plates of the same shape and also of area A. They form two capacitors, C_1 (upper plate–superconducting “plate”) and C_2 (lower plate–superconducting “plate”), as shown in figure b. Then C_1 and C_2 are connected with two identical sets of inductances L_0 and capacitances C_0 into a bridge, which is connected to an AC source (whose frequency is much higher than the rate of change of the gravitational acceleration), as shown in figure c. Suppose that at a certain moment the gravitational acceleration at the place of the gravimeter is g, the superconducting ring (the superconducting “plate”) is at its equilibrium position $z = R$, and the AC bridge is balanced; at this moment the same DC voltage V_C is applied to C_1 and to C_2. When the gravitational acceleration changes by Δg, the bridge becomes unbalanced; increasing the DC voltage on C_1 by ΔV and at the same time decreasing the DC voltage on C_2 by ΔV restores the balance of the bridge. Give the relation between Δg and ΔV (neglect the edge effects of the capacitors).

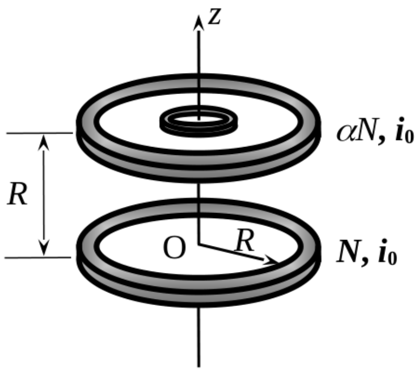

图 a

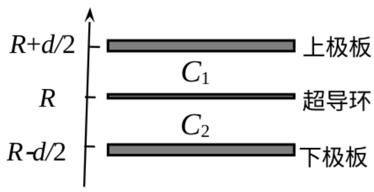

图 b

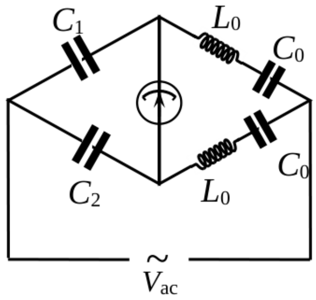

图 c

Labels in figure b: 上极板 *= upper plate;* 超导环 *= superconducting ring;* 下极板 *= lower plate.*

Problem 4

A car is powered by the combustion of the air in the engine cylinder with petrol; the output power of the engine is proportional to the mass of air that enters the cylinder. To raise the power, many cars are fitted with a turbocharger, which compresses the air before it enters the engine cylinder in order to increase the density of the air; to increase the air density still further, the air is also passed through an intercooler, so that it enters the cylinder after its temperature has been lowered. In a typical device the initial pressure of the air entering the turbocharger is one atmosphere, $p_0 = 1.01 \times 10^5$ Pa, and its temperature is $T_0 = 15$ °C. After the turbocharger the air is compressed adiabatically to $p_1 = 1.45 \times 10^5$ Pa; then it passes through the intercooler, where it is cooled isobarically back to the initial temperature T_0 and fills the cylinder. Assume that air is an ideal gas. The volume of the cylinder is $V_0 = 400$ cm^3, the molar mass of air is $M = 29.0$ g·mol^{-1}, the ratio of the molar heat capacities is $\gamma = 7/5$, and the universal gas constant is $R = 8.31$ J·K^{-1}·mol^{-1}.

(1) Find the mass of the air in the cylinder in this case. Compared with the case without supercharging and cooling, by what factor is the output power of the engine increased?

(2) If there were no intercooler, and the air were fed into the cylinder after being compressed by the turbocharger only, what would the corresponding mass of air be? Compared with the case without supercharging, by what factor is the output power of the engine increased?

(3) For the thermodynamic processes in the turbocharger and in the intercooler, find the work done by the surroundings on the gas that enters the cylinder, and the entropy change of the gas in the cylinder in each of these two processes.

Problem 5

Part of the 2018 Nobel Prize in Physics was awarded to the American scientist Arthur Ashkin for his pioneering work on optical trapping (optical tweezers). Optical tweezers use a focused laser beam to trap tiny particles; figure a is a sketch of one experimental optical-tweezers setup. A He–Ne laser emits light of wavelength $\lambda = 632.8$ nm, with beam diameter $D_1 = 1.5$ mm and output power $P = 30$ mW. To make the trapping more effective, a beam expander may be placed after the emitted laser beam. After reflection by a mirror, the laser beam enters a focusing objective of focal length $f = 4.0$ mm, which focuses the beam into the distilled water in the sample cell. In the liquid in the sample cell there is a certain number of polystyrene spheres; a polystyrene sphere may be trapped when it passes through the focused laser spot. The diameter of the spheres is $D_\mathrm{b} = 2.0\ \mu$m and their density is $\rho_\mathrm{b} = 1.0\times10^3$ kg·m^{-3}; the refractive index of the optical glass of the sample cell is $n_\mathrm{g} = 1.46$; the refractive index of the distilled water in the cell is $n_\mathrm{w} = 1.33$; the thickness of the glass at the bottom of the sample cell is $d = 1.0$ mm; the distance from the outer surface of the cell bottom to the centre of the focusing objective is $a = 2.0$ mm. The temperature during the experiment is $T = 300$ K, the refractive index of air is $n_\mathrm{a} = 1.0$, the gravitational acceleration is $g = 9.80$ m·s^{-2}, and the Boltzmann constant is $k_\mathrm{B} = 1.38 \times 10^{-23}$ J·K^{-1}.

(1) Find the diameter of the focused light spot produced in air by the focusing objective when no beam expander is used;

(2) to improve the trapping, the emitted laser beam is expanded; the beam expander used consists of a concave lens of focal length $f_1 = -6.3$ mm and a convex lens of focal length $f_2 = 25.2$ mm. Find the distance between the two lenses and the diameter D_2 of the expanded parallel laser beam;

(3) the expanded laser beam is focused by the focusing objective of figure a into the distilled water in the sample cell. In the paraxial approximation, find the distance from the centre of the focused spot to the inner surface of the bottom of the sample cell, and the diameter of the focused spot;

(4) figure b is a geometrical-optics model for the force acting on a sphere immersed in distilled water in a light-intensity gradient (the gradient is simplified to two beams of different intensity). The incident rays L_1 and L_2 travel along the x direction, and their optical powers are $p_1 = 1.0$ mW and $p_2 = 2.0$ mW respectively. After refraction by the sphere, the directions of the rays make with the x axis the angles shown in figure b. Neglecting absorption and reflection losses during the propagation, find the forces on the sphere in the x direction and in the y direction, and find the ratio of each of them to the weight of the sphere;

(5) find the root-mean-square speed of the spheres in the distilled water at room temperature (300 K);

(6) optical tweezers can also be understood through an optical-potential-well model: after a parallel laser beam is focused by a lens, the optical potential well in the radial direction $\hat{r}$ is

$$U(r) = \begin{cases} -U_0\left(1 - \dfrac{r^2}{b^2}\right), & r \le b, \\ 0, & r > b, \end{cases}$$

where $U_0 = 0.078$ eV, b is a constant and r is the distance from the symmetry axis. Up to what speed can a sphere still be trapped by this optical potential well?

(7) the speed distribution function of the polystyrene spheres can be written as

$$f(v) = 4\pi\left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} v^2\,\mathrm{e}^{-\frac{mv^2}{2k_\mathrm{B}T}},$$

where m and v are the mass and the speed of a sphere and T is the absolute temperature. Analyse and compare the probabilities that a polystyrene sphere of mass m and a polystyrene sphere of mass $2m$ are trapped by the optical potential well.

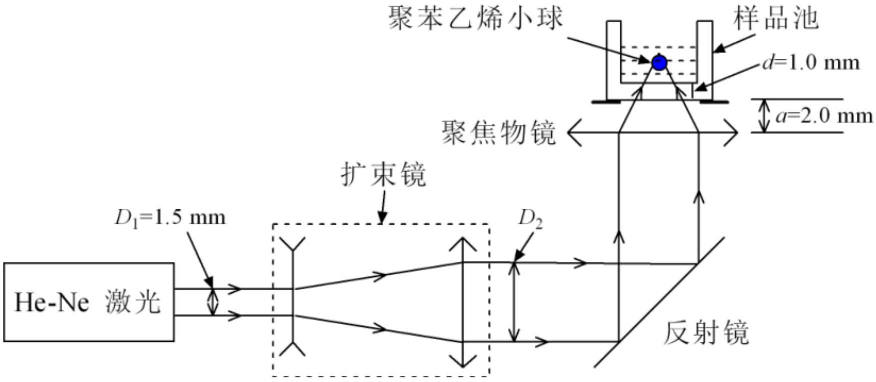

图 a

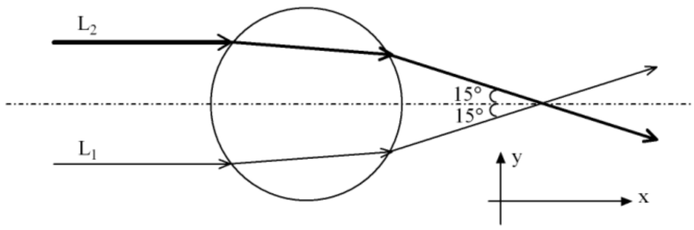

figure b. Simplified model for the analysis of the force on a sphere in a light-intensity gradient

Labels in figure a: 聚苯乙烯小球 = *polystyrene sphere;* 样品池 = *sample cell;* 聚焦物镜 = *focusing objective;* 扩束镜 = *beam expander;* 反射镜 = *mirror;* 激光 = *laser.*

Problem 6

Looking up at the starry sky can awaken boundless imagination. With the progress of science and technology, our understanding of the universe keeps deepening; the solution of the puzzle of the missing neutrinos, the direct detection of gravitational waves and the recent photograph of a black hole have aroused a wide public interest in exploring the universe. How can mankind, with limited equipment, observe and study the vast universe and one of its important constituents, the stars? Is it possible for members of the public with a high-school physics background to understand quantitatively the internal and external properties of a star, such as its size, mass, age, lifetime, composition and energy-production mechanism? This problem takes the star on which we depend — the Sun — as an example and, following the historical development, builds up a model of the Sun step by step from simple to complex. Since the interior of the Sun cannot be observed directly, the model that is built must be checked from many sides — age, radius, surface temperature, energy output, neutrino flux and so on — and all these must be mutually consistent within one and the same model.

The radius of the Earth is 6370 km; the gravitational constant is $G = 6.67 \times 10^{-11}$ N·m^2·kg^{-2}; the Stefan–Boltzmann constant is $\sigma = 5.67 \times 10^{-8}$ W·m^{-2}·K^{-4}; the Boltzmann constant is $k_{\mathrm{B}} = 1.38 \times 10^{-23}$ J·K^{-1}; the electron charge is $e = 1.602 \times 10^{-19}$ C; the electron mass is $m_{\mathrm{e}} = 0.51$ MeV/c^2; the proton mass is $m_{\mathrm{p}} = 938.3$ MeV/c^2; the deuteron mass is $m_{\mathrm{D}} = 1875.6$ MeV/c^2; the mass of the ^{3}He nucleus is $m_{^3\mathrm{He}} = 2808.4$ MeV/c^2; the mass of the ^{4}He nucleus is $m_{^4\mathrm{He}} = 3727.5$ MeV/c[1]; the speed of light in vacuum is $c = 3.00 \times 10^8$ m·s^{-1}.

[1] *Translator's note:* printed exactly so in the source; it should read MeV/c^2.

(1) The *Zhoubi Suanjing* (originally called *Zhoubi*), written more than two thousand years ago, describes a method for measuring the ratio of the diameter D of the Sun to the Sun–Earth distance S_{SE} (also called one astronomical unit): take a long bamboo pole and chisel away the internal nodes so that the inside is open all through; the inner diameter of the pole is d. Point the pole at the Sun and change the length L of the pole, so that the disc of the Sun exactly fills the inner tube of the pole.

(i) Give the relation between the angular size of the Sun ($\Phi = D/S_{\mathrm{SE}}$) and d and L;

(ii) to reduce the error, L must be long enough; what other methods can reduce the error?

(iii) the measurement gives $d = 1$ cun and $L = 8$ chi (10 cun make one chi); estimate the value of Φ.

(2) At certain special times (for instance on 6 June 2012) the Earth, Venus and the Sun lie almost on one straight line; then, observed from the Earth, Venus looks like a small black dot inlaid on the disc of the Sun and moving slowly across it. This is called a transit of Venus, shown in figure a. In 1716 the British astronomer Halley proposed that a transit of Venus be observed jointly from different places all over the world, in order to measure the angular size of the Sun and the Sun–Earth distance. The periods of revolution of the Earth and of Venus around the Sun (in the same sense) are 365.256 days and 224.701 days respectively, and the inclination between their orbital planes is very small. Do not take the rotation of the Earth into account.

(i) The orbits of the Earth and of Venus may both be regarded as circles; find the ratio r_{VE} of the Sun–Venus distance S_{SV} to the Sun–Earth distance S_{SE};

(ii) on 6 June 2012 the transit of Venus observed at a certain place is as shown in figure a; compute the ratio of the chord D_{P} swept by Venus over the solar disc to the Sun–Earth distance S_{SE}. If the distance from this chord to the centre of the disc is $h_{\mathrm{P}} = 5D/16$, what is the angular size Φ of the Sun?

(iii) to measure the Sun–Earth distance the transit of Venus must be observed from different points on the surface of the Earth, as shown in figure b. Assume that two observers P and P′, on the same meridian and separated by the straight-line distance H (measured in the projection perpendicular to the direction of observation), observe the transit at the same time. For P the path of Venus on the solar disc is AB and the observed transit time (the time interval during which the centre of Venus sweeps across the solar disc) is t_{P}; for P′ the path of Venus on the solar disc is A′B′ and the observed transit time is $t_{\mathrm{P'}}$. Find an expression for the Sun–Earth distance S_{SE} (in terms of Φ, r_{VE}, T_{E}, T_{V}, t_{P}, $t_{\mathrm{P'}}$ and H);

(iv) the latitudes of Beijing and of Hong Kong are 39.5° and 22.5°; on 6 June 2012 the transit times of Venus measured at Beijing and at Hong Kong were $t_{\mathrm{P}} = 6:21:57$ and $t_{\mathrm{P'}} = 6:19:31$. Using these data and the result of part (iii), compute the Sun–Earth distance S_{SE}.

(3) The results of the transit observations of 1882 gave a mean Sun–Earth distance $S_{\mathrm{SE}} = 1.5 \times 10^8$ km and a solar diameter $D = 1.4 \times 10^6$ km. The solar irradiance received by the Earth on a surface perpendicular to the direction of the solar radiation is $I = 1.37\ \mathrm{kW{\cdot}m^{-2}}$. Find

(i) the total energy received by the Earth from the Sun per unit time;

(ii) the ratio of the energy consumed by the 7 billion people of the world to the solar energy received by the Earth (take an average annual consumption of 3 tonnes of standard coal per person, and 1 kg of standard coal as producing 4 kW·h of electricity);

(iii) the total energy radiated by the Sun per unit time;

(iv) an estimate of the surface temperature of the Sun.

(4) (i) Compute the total mass and the mean density of the Sun;

(ii) if the energy source of the Sun came from a chemical reaction, and if the energy released per unit mass were the same as for coal, how long could it last, computed with the energy the Sun radiates per unit time at present? The Sun has existed as a star for about 5 billion years or more; how many times the efficiency of a chemical reaction must its energy-production efficiency then be?

(5) Spectral analysis shows that the main constituent of the Sun is hydrogen; the basic nuclear-fusion reactions involving hydrogen are as follows:

$$\mathrm{P} + \mathrm{P} \to \mathrm{D} + \mathrm{e}^+ + \nu_\mathrm{e} \qquad \text{(a)}$$

$$\mathrm{D} + \mathrm{P} \to {}^3\mathrm{He} + \gamma \qquad \text{(b)}$$

$${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + \mathrm{P} + \mathrm{P} \qquad \text{(c)}$$

The positrons produced in these fusion reactions annihilate, and the neutrinos escape. The nuclei are all positively charged and repel each other; only when the temperature is very high can their kinetic energy overcome the Coulomb repulsion, so that the fusion reactions take place. When the temperature is high enough, all the atoms are ionized into nuclei and electrons, in the plasma state. (Here the energy carried away by the neutrinos is not taken into account, and neither is the unfused hydrogen carried away by the solar wind.)

(i) How many protons (hydrogen nuclei) must be consumed per unit time in order to maintain the radiation of the Sun?

(ii) at present 71% of the mass of the Sun is hydrogen; at the above consumption rate, for how many more years can this at most be maintained? Compute also the proton–proton reaction probability per unit time (the ratio of the number of particles that react per unit time to the total number of particles), P_{PP};

(iii) the number of escaping neutrinos does not decrease during their propagation; how many neutrinos arrive at the Earth per unit time on unit area perpendicular to the direction of the solar radiation? (The number of electron neutrinos that actually arrive at the Earth is only 35% of the theoretical value, but if the numbers of the other two kinds of neutrinos are added, the result agrees with the theoretical value; this shows that some of the electron neutrinos are converted into other kinds of neutrinos during the propagation. This is related to neutrino oscillations; the corresponding results won the Nobel Prizes in Physics of 2002 and 2015.)

(iv) by how much does the mass of the Sun decrease per unit time because of the radiation?

(6) The probability P of a fusion reaction depends on the temperature T of the plasma, and the higher the number density n (the number of particles per unit volume) the higher the reaction probability; the relation is $P = nR(T)$, where $R(T)$ is called the reaction rate. Figure c gives the reaction rates of three fusion reactions as functions of the temperature. Estimate a lower limit for the temperature of the core region of the Sun (whose radius is about one quarter of the solar radius) in which nuclear fusion can occur.

(7) Estimate the maximum energy carried away by a single neutrino, and estimate the maximum relative error, in the radiated energy computed in part (5), that is caused by neglecting the energy carried away by the neutrinos.

(8) From the preceding calculations the mass of the Sun decreases gradually, so the period and the mean orbital radius of the revolution of the Earth around the Sun will also change. Compute

(i) the decrease ΔM of the mass due to the radiation in one hundred million years;

(ii) the resulting change of the period of revolution of the Earth.

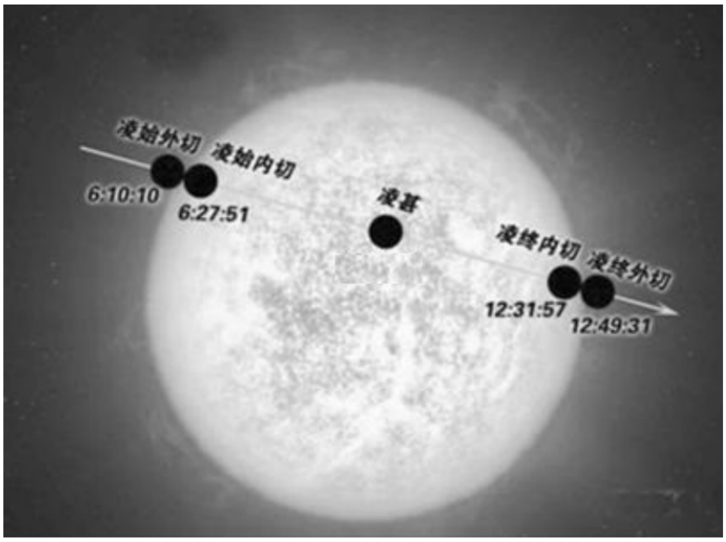

figure a. Sketch of the transit of Venus of 6 June 2012

Labels in figure a: 凌始外切 *= first (exterior) contact at ingress;* 凌始内切 *= second (interior) contact at ingress;* 凌甚 *= greatest transit;* 凌终内切 *= third (interior) contact at egress;* 凌终外切 *= fourth (exterior) contact at egress.*

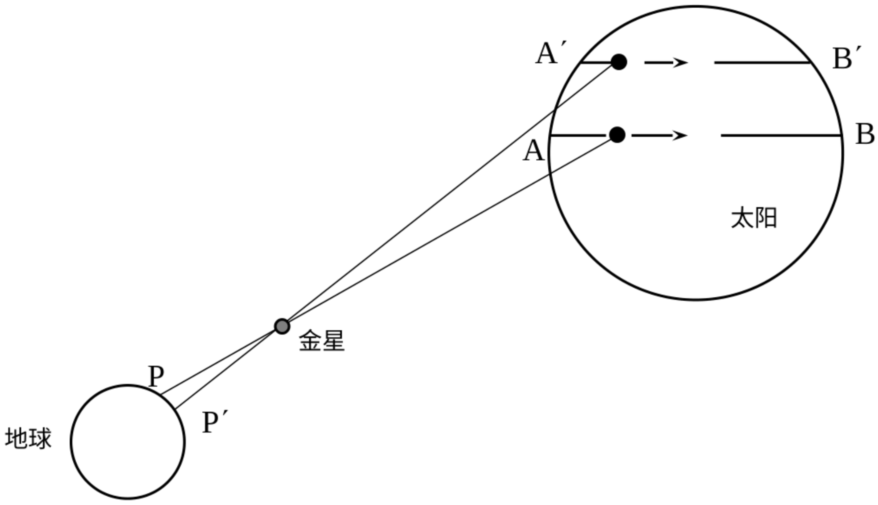

figure b. Sketch of the transit of Venus

Labels in figure b: 地球 *= Earth;* 金星 *= Venus;* 太阳 *= Sun.*

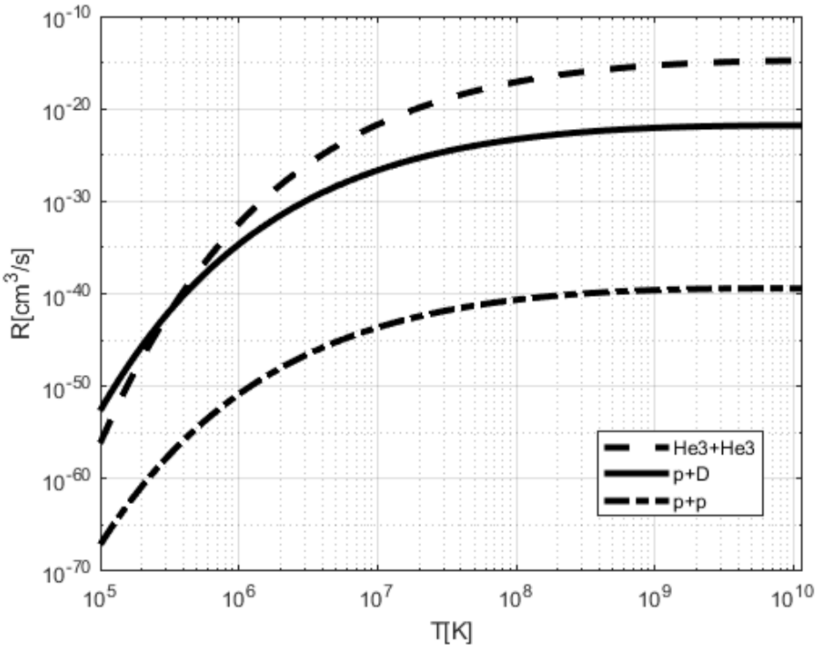

figure c. Reaction rate as a function of temperature