The 36th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2019

Translator's note: this file contains the official reference solutions printed together with the problems in the source document ("Problems and reference solutions of the theoretical examination of the 36th CPhO finals", 26 September 2019). The equation numbers are those of the source; where the source's own numbering is inconsistent (repeated or skipped numbers) this has been kept and pointed out in a footnote. Passages that the source puts in square brackets — alternative derivations — are reproduced as "Alternative method". In problem 6 the Chinese subscripts of the original have been replaced by Latin abbreviations: S_{SE} (Sun–Earth distance), S_{SV} (Sun–Venus distance), T_{E}, T_{V} (orbital periods of the Earth and of Venus), M_{S}, R_{S}, W_{S}, I_{S} (mass, radius, radiated power and surface irradiance of the Sun).

Problem 1 (Wheel with one spoke missing)

Solution

The moment of inertia of the wheel disc about the axis through O perpendicular to the plane of the wheel is

$$J_{\mathrm{O1}} = \int_{R_1}^{R_0} \varsigma \cdot 2\pi r\,\mathrm{d}r \cdot r^2 = \frac{1}{2}\varsigma\pi\left(R_0^4 - R_1^4\right) = \frac{1}{2}M\left(R_0^2 + R_1^2\right),$$

where ς is the mass per unit area of the annular disc. The moment of inertia of the two spokes about the same O axis is

$$J_{\mathrm{O2}} = \frac{1}{3}mR_1^2 \times 2.$$

The total moment of inertia of the wheel about the O axis is

$$J_{\mathrm{O}} = J_{\mathrm{O1}} + J_{\mathrm{O2}} = \frac{1}{6}(3M + 4m)R_1^2 + \frac{1}{2}MR_0^2 = \frac{524}{75}mR_0^2. \tag{1}$$

Take the centre O of the wheel as the origin and set up the coordinate system of solution figure a. Put

$$M' = M + 2m = 10m.$$

The x coordinate of the centre of mass C of the wheel is

$$x_{\mathrm{C}} = \frac{(M' + m)\cdot 0 + (-m)\cdot(-R_1/2)}{M'} = \frac{mR_1}{2M'} = \frac{R_0}{25}. \tag{2}$$

By the parallel-axis theorem, the moment of inertia of the wheel about the axis through the centre of mass C perpendicular to the plane of the wheel is

$$J_{\mathrm{C}} = J_{\mathrm{O}} - M'x_{\mathrm{C}}^2 = \frac{2614}{375}mR_0^2. \tag{3}$$

While the spoke OB turns from its initial position to the vertical position, the moment of inertia of the wheel at any position about the instantaneous axis P at which the wheel touches the ground in pure rolling is

$$\begin{aligned} J_{\mathrm{P}} &= J_{\mathrm{C}} + M' r_{\mathrm{CP}}^2 \\ &= J_{\mathrm{C}} + M'\left[R_0^2 + x_{\mathrm{C}}^2 - 2R_0 x_{\mathrm{C}} \cos\left(\frac{\pi}{2} - \theta\right)\right] \\ &= J_{\mathrm{C}} + M'\left(R_0^2 + x_{\mathrm{C}}^2 - 2R_0 x_{\mathrm{C}} \sin\theta\right), \end{aligned} \tag{4}$$

where θ is the angle through which the centre of mass C has turned from its initial position, as shown in solution figure b.
By the law of conservation of mechanical energy,

$$\frac{1}{2} J_{\mathrm{P}} \omega^2 = M' g x_{\mathrm{C}} \sin\theta, \tag{5}$$

whence

$$\omega^2 = \frac{2M' g x_{\mathrm{C}} \sin\theta}{J_{\mathrm{P}}}.$$

Substituting (4) into (5) and differentiating both sides with respect to t gives the angular acceleration α:

$$\frac{1}{2}\omega^2\left(-M' 2R_0 x_{\mathrm{C}} \cos\theta \cdot \omega\right) + J_{\mathrm{P}} \omega \alpha = M' g x_{\mathrm{C}} \cos\theta \cdot \omega.$$

The result is

$$\alpha = \frac{M' x_{\mathrm{C}} \cos\theta \left(g + \omega^2 R_0\right)}{J_{\mathrm{P}}}. \tag{6}$$

(1) When the wheel is released from rest at the position of figure a, $\theta = 0$; from (4) the moment of inertia of the wheel about the instantaneous axis P is

$$J_{\mathrm{P},\theta=0} = J_{\mathrm{C}} + M'\left(R_0^2 + x_C^2\right) = \frac{2614}{375} m R_0^2 + 10m\left(R_0^2 + x_C^2\right) = \frac{1274}{75} m R_0^2.$$

At this instant $\omega = 0$, so (6) gives

$$\alpha_{\theta=0} = \frac{M' g x_{\mathrm{C}}}{J_{\mathrm{P},\theta=0}} = \frac{15}{637} \frac{g}{R_0}. \tag{7}$$

Alternative method. One may also use directly the law of rotation about an arbitrary instantaneous axis P,

$$\frac{\mathrm{d}\boldsymbol{L}_P}{\mathrm{d}t} = \boldsymbol{M}_P + \boldsymbol{r}_{PC} \times \left(-M' \boldsymbol{a}_p\right);$$

but at this instant the wheel has only just begun to rotate, its angular velocity is zero, and the acceleration $\boldsymbol{a}_p$ $(\omega^2 R_0)$ relative to the contact point on the ground is zero, so that

$$J_{\mathrm{P},\theta=0}\, \alpha = M' g \cdot x_{\mathrm{C}},$$

and the angular acceleration at this instant is

$$\alpha_{\theta=0} = \frac{M' g x_{\mathrm{C}}}{J_{\mathrm{P},\theta=0}} = \frac{15}{637} \frac{g}{R_0}. \tag{7}$$

(2) When the spoke OB has turned to the vertical position, $\theta = 30°$; from (4)

$$J_{\mathrm{P},\theta=30°} = J_{\mathrm{C}} + M'\left(R_0^2 + x_C^2 - 2R_0 x_{\mathrm{C}} \sin 30°\right) = \frac{1244}{75} m R_0^2.$$

From (5)

$$\omega^2_{\theta=30^\circ} = \frac{2M'gx_\mathrm{C}\sin 30^\circ}{J_{\mathrm{P},\theta=30^\circ}} = \frac{15}{622}\frac{g}{R_0}. \tag{8}$$

From (6)

$$\alpha_{\theta=30^\circ} = \frac{M'x_\mathrm{C}\cos 30^\circ\left(g+\omega^2_{\theta=30^\circ}R_0\right)}{J_{\mathrm{P},\theta=30^\circ}} = \frac{9555\sqrt{3}}{773768}\frac{g}{R_0}. \tag{9}$$

By the statement of the problem, the acceleration a_O of the wheel centre O relative to the ground and the tangential and normal accelerations a^t_CO and a^n_CO of the centre of mass C relative to the wheel centre O are

$$a_\mathrm{O} = \alpha R_0, \qquad a^t_\mathrm{CO} = \alpha x_C, \qquad a^n_\mathrm{CO} = \omega^2 x_C. \tag{10}$$

The acceleration of the centre of mass C in the ground frame of reference (see solution figure c) is

$$\begin{aligned} a_{Cx} &= a_o - a^t_{CO}\sin\theta - a^n_{CO}\cos\theta, \\ a_{Cy} &= a^n_{CO}\sin\theta - a^t_{CO}\cos\theta. \end{aligned} \tag{11}$$

From the theorem of the motion of the centre of mass the friction force F_f and the normal force F_N at this instant follow from

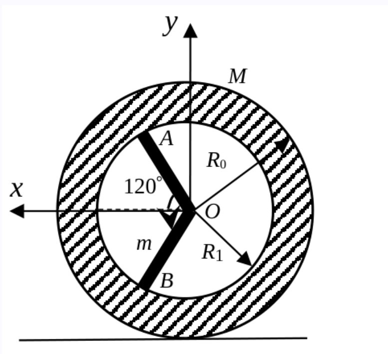
$$\begin{aligned} M'a_{Cx} &= F_f, \\ M'a_{Cy} &= F_N - M'g. \end{aligned} \tag{12}$$

Solving all the above equations simultaneously gives

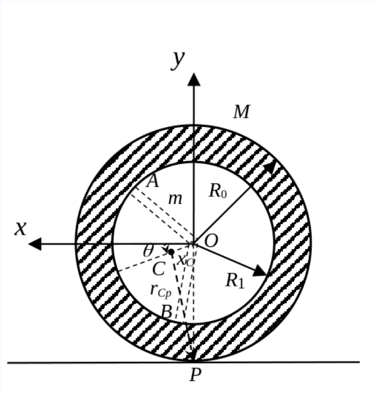
$$\begin{aligned} F_f &= \frac{89907\sqrt{3}}{773768}mg, \\ F_N &= \frac{7735679}{773768}mg. \end{aligned} \tag{13}$$

The above results may be checked with

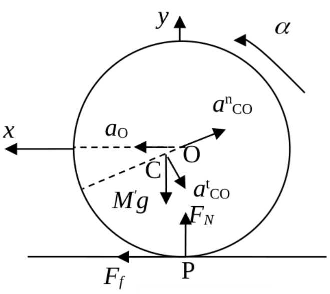
$$J_C\alpha = F_N x_C\cos\theta - F_f(R_0 - x_C\sin\theta).$$



解题图 a



解题图 b



解题图 c

解题图 = *solution figure.*

Problem 2 (Half cylinder of two lossy dielectrics)

Solution

(1) Consider a semicircle of radius r, as in solution figure a. Let the magnitudes of the electric field strength and of the electric displacement be E_1 and D_1 in the region $0 \le \varphi < \theta_0$, and E_2 and D_2 in the region $\theta_0 < \varphi \le \pi$; both are directed along the angular direction.

From the continuity of the current and Ohm's law,

$$J_1 = J_2, \qquad \sigma_1 E_1 = \sigma_2 E_2. \tag{1}$$

The sum of the voltages across the two dielectrics is

$$E_1 r\theta_0 + E_2 r(\pi - \theta_0) = V_0. \tag{2}$$

From (1) and (2),

$$E_1 = \frac{V_0}{\left[\theta_0 + \dfrac{\sigma_1}{\sigma_2}(\pi - \theta_0)\right] r}, \tag{3}$$

$$E_2 = \frac{V_0}{\left[\dfrac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)\right] r}. \tag{4}$$

Assume that the potential of the negative plate is zero; from the relation between the potential and the electric field strength:
for the region $0 \le \varphi < \theta_0$,

$$V_\varphi = V_0 - E_1 r\varphi = \left[1 - \frac{\varphi}{\theta_0 + \dfrac{\sigma_1}{\sigma_2}(\pi - \theta_0)}\right] V_0 = \frac{(\theta_0 - \varphi) + \dfrac{\sigma_1}{\sigma_2}(\pi - \theta_0)}{\theta_0 + \dfrac{\sigma_1}{\sigma_2}(\pi - \theta_0)} V_0; \tag{5}$$

for the region $\theta_0 < \varphi \le \pi$,

$$\begin{aligned} V_\varphi &= V_0 - E_1 r\theta_0 - E_2 r(\varphi - \theta_0) \\ &= \left[1 - \frac{\theta_0}{\theta_0 + \dfrac{\sigma_1}{\sigma_2}(\pi - \theta_0)} - \frac{\varphi - \theta_0}{\dfrac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)}\right] V_0 = \frac{\pi - \varphi}{\dfrac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)} V_0. \end{aligned} \tag{6}$$

(2) The total charge density on the interface $\varphi = \theta_0$ is

$$\sigma_e = \varepsilon_0(E_2 - E_1) = \frac{\varepsilon_0 V_0}{\left[\frac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)\right] r} - \frac{\varepsilon_0 V_0}{\left[\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)\right] r} = \frac{\frac{\sigma_1}{\sigma_2} - 1}{\left[\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)\right] r}\varepsilon_0 V_0. \tag{7}$$

The total charge on the interface $\varphi = \theta_0$ is

$$Q = \int_a^b \sigma_e l \, \mathrm{d}r = \frac{\varepsilon_0 \left(\frac{\sigma_1}{\sigma_2} - 1\right) l \ln \frac{b}{a}}{\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)} V_0. \tag{8}$$

(3) Compute the capacitance C_1 of the dielectric region $\varphi = 0 \sim \theta_0$. Let the free-charge density at r on the right electrode of the half cylinder be σ_{e01}; then

$$\sigma_{e01} = D_1 = \varepsilon_{r1}\varepsilon_0 E_1 = \frac{\varepsilon_{r1}\varepsilon_0 V_0}{\left[\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)\right] r}. \tag{9}$$

The total free charge on the right electrode is

$$Q_{01} = \int_a^b \sigma_{e01} l \, \mathrm{d}r = \int_a^b \frac{\varepsilon_{r1}\varepsilon_0 V_0 l}{\left[\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)\right] r} \mathrm{d}r = \frac{\varepsilon_{r1}\varepsilon_0 V_0 l}{\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)} \ln \frac{b}{a}. \tag{10}$$

The potential difference between $\varphi = 0$ and $\varphi = \theta_0$ is

$$V_{10} = E_1 \theta_0 r = \frac{\theta_0 V_0}{\theta_0 + \frac{\sigma_1}{\sigma_2}(\pi - \theta_0)}. \tag{11}$$

The capacitance C_1 of the dielectric region $\varphi = 0 \sim \theta_0$ is therefore

$$C_1 = \frac{Q_{01}}{V_{10}} = \frac{\varepsilon_{r1}\varepsilon_0 l}{\theta_0} \ln \frac{b}{a}. \tag{12}$$

Compute the capacitance C_2 of the dielectric region $\varphi = \theta_0 \sim \pi$. Let the free-charge density at r on the left electrode of the half cylinder be σ_{e02}; then

$$\sigma_{e02} = D_2 = \varepsilon_{r2}\varepsilon_0 E_2 = \frac{\varepsilon_{r2}\varepsilon_0 V_0}{\left[\frac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)\right] r}. \tag{13}$$

The free charge on the left electrode is

$$Q_{02} = \int_a^b \sigma_{e02} l \, \mathrm{d}r = \int_a^b \frac{\varepsilon_{r2}\varepsilon_0 V_0 l}{\left[\frac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)\right] r} \mathrm{d}r = \frac{\varepsilon_{r2}\varepsilon_0 V_0 l}{\frac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)} \ln \frac{b}{a}. \tag{14}$$

The potential difference between $\varphi = \theta_0$ and $\varphi = \pi$ is

$$V_{20} = E_2(\pi - \theta_0) r = \frac{(\pi - \theta_0) V_0}{\frac{\sigma_2}{\sigma_1}\theta_0 + (\pi - \theta_0)}. \tag{15}$$

Hence the capacitance C_2 of the region $\varphi = \theta_0 \sim \pi$ is

$$C_2 = \frac{Q_{02}}{V_{20}} = \frac{\varepsilon_{r2}\varepsilon_0 l}{\pi - \theta_0} \ln \frac{b}{a}. \tag{16}$$

Compute the resistance R_1 of the dielectric region $\varphi = 0 \sim \theta_0$: consider a half-cylindrical shell of thickness dr at r (see solution figure a). The conductance of the part of this shell that lies in the region $\varphi = 0 \sim \theta_0$ is

$$\mathrm{d}G_1 = \sigma_1 \frac{l\,\mathrm{d}r}{\theta_0 r},$$

so the conductance of the region $\varphi = 0 \sim \theta_0$ is

$$G_1 = \int_a^b \sigma_1 \frac{l\,\mathrm{d}r}{\theta_0 r} = \frac{\sigma_1 l}{\theta_0} \ln \frac{b}{a},$$

and the corresponding resistance is

$$R_1 = \frac{1}{G} = \frac{\theta_0}{\sigma_1 l \ln \dfrac{b}{a}}. \tag{17}$$

Compute the resistance R_2 of the dielectric region $\varphi = \theta_0 \sim \pi$. The conductance of the part of the same shell that lies in the region $\varphi = \theta_0 \sim \pi$ is

$$\mathrm{d}G_2 = \sigma_2 \frac{l\,\mathrm{d}r}{(\pi - \theta_0) r},$$

so the conductance of the region $\varphi = \theta_0 \sim \pi$ is

$$G_2 = \int_a^b \sigma_2 \frac{l\,\mathrm{d}r}{(\pi - \theta_0) r} = \frac{\sigma_2 l}{\pi - \theta_0} \ln \frac{b}{a},$$

and the corresponding resistance is

$$R_2 = \frac{1}{G_2} = \frac{\pi - \theta_0}{\sigma_2 l \ln \dfrac{b}{a}}. \tag{18}$$

(4) After the power supply has been disconnected, two mutually independent RC circuits are formed, whose equivalent circuit is shown in solution figure b; their potential differences decay from V_{10} and V_{20} to zero:

$$iR_i + \frac{q_i}{C_i} = 0,$$

where the subscript $i = 1, 2$ corresponds to the first and to the second RC circuit, and q_i is the charge on the positive plate of the capacitor C_i at the time t after the supply has been disconnected. The above equation is

$$\frac{\mathrm{d}q_i}{q_i} = -\frac{1}{R_i C_i} \,[\,\mathrm{d}t\,]$$

[a]with the solution

$$q_i = q_0 \mathrm{e}^{-\frac{1}{R_i C_i} t},$$

where q_0 is the charge on the positive plate of C_i at the moment the supply is disconnected; or

$$V_i = \frac{q_i}{C_i} = \frac{q_0}{C_i} \mathrm{e}^{-\frac{1}{R_i C_i} t} = V_{i0} \mathrm{e}^{-\frac{1}{R_i C_i} t}, \tag{19}$$

where

$$R_1C_1 = \frac{\theta_0}{\sigma_1 l \ln\dfrac{b}{a}} \times \frac{\varepsilon_{r1}\varepsilon_0 l}{\theta_0} \ln\frac{b}{a} = \frac{\varepsilon_1\varepsilon_0}{\sigma_1}, \qquad \frac{1}{R_1C_1} = \frac{\sigma_1}{\varepsilon_{r1}\varepsilon_0}, \tag{20}$$

$$R_2C_2 = \frac{\pi-\theta_0}{\sigma_2 l \ln\dfrac{b}{a}} \times \frac{\varepsilon_{r2}\varepsilon_0 l}{\pi-\theta_0} \ln\frac{b}{a} = \frac{\varepsilon_{r2}\varepsilon_0}{\sigma_2}, \qquad \frac{1}{R_2C_2} = \frac{\sigma_2}{\varepsilon_{r2}\varepsilon_0}. \tag{21}$$

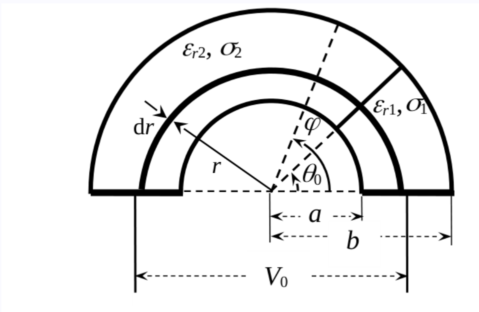
[b]Hence

$$V_1(t) = \frac{\theta_0 V_0}{\theta_0 + \dfrac{\sigma_1}{\sigma_2}(\pi-\theta_0)} \mathrm{e}^{-\frac{\sigma_1}{\varepsilon_1\varepsilon_0}t}, \tag{22}$$

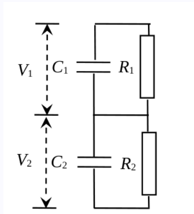
$$V_2(t) = \frac{(\pi-\theta_0)V_0}{\dfrac{\sigma_2}{\sigma_1}\theta_0 + \pi - \theta_0} \mathrm{e}^{-\frac{\sigma_2}{\varepsilon_{r2}\varepsilon_0}t}. \tag{23}$$

The voltage between the two metal films is

$$V(t) = V_1(t) + V_2(t) = \frac{\theta_0 V_0}{\theta_0 + \dfrac{\sigma_1}{\sigma_2}(\pi-\theta_0)} \mathrm{e}^{-\frac{\sigma_1}{\varepsilon_1\varepsilon_0}t} + \frac{(\pi-\theta_0)V_0}{\dfrac{\sigma_2}{\sigma_1}\theta_0 + \pi - \theta_0} \mathrm{e}^{-\frac{\sigma_2}{\varepsilon_{r2}\varepsilon_0}t}. \tag{24}$$



解题图 a



解题图 b

[a] *Translator's note:* the differential $\mathrm{d}t$ is missing on the right-hand side in the source; it has been restored in square brackets.

[b] *Translator's note:* in (20) and in (22) below the source writes ε_1 where ε_{r1} is meant.

Problem 3 (Superconducting gravimeter)

Solution

(1) As in solution figure a, the underground lake is at the depth d below the ground surface. Let the mean density of the Earth be ρ_0 and the density of water be ρ_{W}; the mass M_{E} of the Earth and the mass M_{W} of the water in the underground lake are

$$M_{\mathrm{E}} = \frac{4}{3}\pi R_{\mathrm{E}}^3\rho_0, \qquad M_{\mathrm{W}} = \frac{4}{3}\pi r_{\mathrm{W}}^3\rho_{\mathrm{W}}.$$

Let g_0 be the magnitude of the gravitational acceleration at the ground surface when there is no water in the underground lake; the law of gravitation gives

$$mg_0 = mG\frac{M_{\mathrm{E}}}{R_{\mathrm{E}}^2}. \tag{1}$$

From $M_\mathrm{E} = \frac{4}{3}\pi R_\mathrm{E}^3 \rho_0$ and (1),

$$\rho_0 = \frac{3g_0}{4\pi R_\mathrm{E} G} = 5.51 \times 10^3\ \mathrm{kg/m^3}. \tag{2}$$

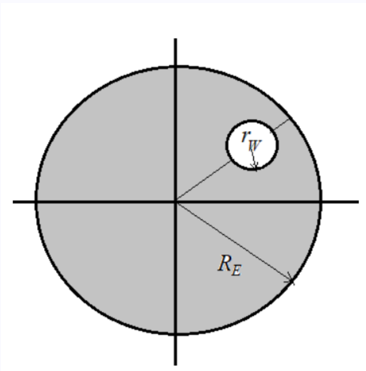
Let Δg be the change of the magnitude of the gravitational acceleration when the underground lake is filled with water; likewise

$$m(g_0 - \Delta g) = mG\left(\frac{M_\mathrm{E}}{R_\mathrm{E}^2} - \frac{m_\mathrm{E}}{d^2} + \frac{M_\mathrm{W}}{d^2}\right), \tag{3}$$

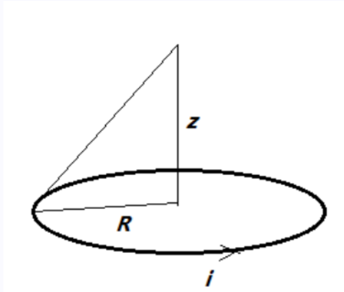
where m_E is the mass of the Earth material of mean density ρ_0 that the water has replaced. From $M_\mathrm{E} = \frac{4}{3}\pi R_\mathrm{E}^3 \rho_0$, $M_\mathrm{W} = \frac{4}{3}\pi r_\mathrm{W}^3 \rho_\mathrm{W}$ and (1)(2)(3),

$$\Delta g = G\frac{\frac{4}{3}\pi r^3\left[\rho_0 - \rho_\mathrm{W}\right]}{d^2} = 5.60 \times 10^{-3}\ \mathrm{m\cdot s^{-2}} = 5.71 \times 10^{-4} g_0. \tag{4}$$

This shows that the presence of the water of the underground lake has only a very small effect on the local gravitational acceleration.



解题图 a



解题图 b

(2) (i) The magnitude of the magnetic induction produced on the axis through the centre of a single-turn coil carrying the current i (see solution figure b) is

$$B_z = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}, \qquad \textit{(4 points)} \tag{5}$$

directed along the positive z direction (according to the direction of flow of the current i). Near the centre of the plane of the upper coil, the z component of the magnetic induction is the superposition of the fields produced by the upper and the lower coils (see figure a of the problem):

$$\begin{aligned} B_z &= B_{1z} + B_{2z} \\ &= \frac{\mu_0 \alpha N i_0 R^2}{2\left[R^2 + (z-R)^2\right]^{3/2}} + \frac{\mu_0 N i_0 R^2}{2(R^2+z^2)^{3/2}} \\ &= \frac{\mu_0 N i_0 R^2}{2}\left\{f_1(z) + f_2(z)\right\}. \end{aligned} \tag{6}$$

In the above, $z - R$ is a small quantity; keeping only the first-order terms,

$$f_1(z) = \frac{\alpha}{\left[R^2 + (z-R)^2\right]^{3/2}} \approx \frac{\alpha}{R^3}, \tag{7}$$

$$f_2(z) = \frac{1}{[R^2+z^2]^{3/2}} \approx \frac{1}{[2(z-R)R+2R^2]^{3/2}} \approx \frac{1}{2\sqrt{2}R^3}\left[1-\frac{3}{2R}(z-R)\right]. \tag{8}$$

Hence

$$B_z = \frac{\mu_0 NiR^2}{2}\left\{\frac{\alpha}{R^3}+\frac{1}{2\sqrt{2}R^3}\left[1-\frac{3}{2R}(z-R)\right]\right\} = \frac{\mu_0 Ni}{2R}\left[\left(\alpha+\frac{1}{2\sqrt{2}}\right)-\frac{3}{4\sqrt{2}R}(z-R)\right]$$

$$= \frac{\mu_0 Ni}{2R}\left[\alpha+\frac{1}{2\sqrt{2}}\right]\left[1-\frac{3}{4\sqrt{2}R\left(\alpha+\frac{1}{2\sqrt{2}}\right)}(z-R)\right] = B_0\left[1-\beta(z-R)\right], \tag{9}$$

where

$$B_0 = \frac{\mu_0 Ni_0}{2R}\left(\alpha+\frac{1}{2\sqrt{2}}\right) = \frac{\sqrt{2}\mu_0 Ni_0}{8R}\left(2\sqrt{2}\alpha+1\right), \tag{10}$$

$$\beta = \frac{3}{2R\left(2\sqrt{2}\alpha+1\right)}. \tag{11}$$

(ii) Since the resistance of the superconducting ring is zero, the magnetic flux through the ring always stays unchanged while it rises and until it is in equilibrium in the plane of the upper coil. Let the induced current in the ring at the equilibrium position be I_0; then

$$\varepsilon = -\frac{\Delta\Phi}{\Delta t} = 0, \tag{12}$$

so that

$$\Phi = (B_z)_{z=R}\,\pi\left(\frac{D}{2}\right)^2 + LI_0 = \text{const}. \tag{13}$$

From the initial conditions

$$t=0, \quad z=0, \quad I=0, \quad \Phi=0$$

and from (9) it follows that

$$LI_0 = -B_0\pi\left(\frac{D}{2}\right)^2. \tag{14}$$

From (17)[a]

$$I_0 = -\frac{1}{4L}\pi D^2 B_0 = -\frac{\mu_0 Ni_0\pi D^2}{8RL}\left[\alpha+\frac{1}{2\sqrt{2}}\right]. \tag{15}$$

The minus sign means that the current in the superconducting ring is opposite in direction to the current in the superconducting coils.

(iii) When the superconducting ring is at rest in the plane of the upper coil ($z=R$), the upward Ampère force exerted on it by the radial magnetic field exactly balances its weight, i.e.

$$mg = |I_0|\,B_r\pi D. \tag{16}$$

From the given conditions and (19)(20)[b]

$$g = \frac{\frac{1}{4L}\pi D^2 B_0\cdot\frac{1}{2}B_0\cdot\beta\cdot\frac{D}{2}\cdot\pi D}{m} = \frac{\beta\pi^2 D^4 B_0^2}{16Lm}$$

$$= \frac{\frac{3}{2R\left(2\sqrt{2}\alpha+1\right)}\pi^2 D^4\frac{\mu_0^2N^2i_0^2}{4R^2}\left(\alpha+\frac{1}{2\sqrt{2}}\right)^2}{16Lm} \tag{17}$$

$$= \frac{3\pi^2D^4\mu_0^2N^2i_0^2}{256\sqrt{2}LmR^3}\left(\alpha+\frac{1}{2\sqrt{2}}\right).$$

This shows that, once the parameters of the quasi-Helmholtz superconducting coils and of the levitated superconducting ring are known, the gravitational acceleration g_0 at the place can be measured simply by measuring i_0.

(3) When the magnitude of the gravitational acceleration at the ground is g, the magnetic levitation force balances the weight mg of the superconducting ring, and the ring is in the plane of the upper coil ($z_0 = R$). The attraction of the upper plate on the superconducting "plate" and that of the lower plate on the superconducting "plate" cancel each other, and the voltage applied to the two capacitors is V_C in both cases. Let the distance between the upper and the lower plate be d, and the distance between the superconducting "plate" and the lower plate be x; then the difference of the attractions exerted on the superconducting "plate" by the upper and the lower plate is

$$F = \frac{1}{2}\varepsilon_0 A V_\mathrm{C}^2 \left[\frac{1}{(d-x)^2} - \frac{1}{x^2}\right].$$

When $x = \dfrac{d}{2}$,

$$F_{x=d/2} = 0. \tag{18}$$

When the gravitational acceleration changes by Δg, the superconducting "plate" experiences an extra downward weight $m\Delta g$. In order that the superconducting "plate" regain mechanical equilibrium at the balance position of the AC bridge ($x = \dfrac{d}{2}$), voltages ΔV of opposite sign are applied to the two capacitors C_1 and C_2, so that the attraction of the upper plate on the superconducting "plate" becomes larger than that of the lower plate; the difference is

$$F = \frac{1}{2}\varepsilon_0 A \left[\frac{(V_C + \Delta V)^2}{(d-x)^2} - \frac{(V_C - \Delta V)^2}{x^2}\right]. \tag{19}$$

At $x = \dfrac{d}{2}$, when the superconducting "plate" has regained mechanical equilibrium,

$$F = \frac{1}{2}\frac{\varepsilon_0 A}{\left(\dfrac{d}{2}\right)^2} 4V_C\Delta V = \frac{8\varepsilon_0 A}{d^2} V_C \Delta V = m\Delta g, \tag{20}$$

so that

$$\Delta g = \frac{8\varepsilon_0 A}{md^2} V_C \Delta V. \tag{21}$$

The small change Δg of the gravitational acceleration can therefore be obtained by measuring V_C and ΔV.

[a] *Translator's note:* the source refers here to "(17)"; the relation used is in fact (14).
[b] *Translator's note:* the source refers here to "(19)(20)"; the relations used are (11) and (15).

Problem 4 (Turbocharger and intercooler)

Solution

(1) By hypothesis air may be regarded as an ideal gas; the equation of state of an ideal gas gives

$$PV = \nu RT, \tag{1}$$

where (p, V, T) is a set of state parameters of the gas, denoting its pressure, volume and temperature; m and M are the mass and the molar mass of the gas, and $\nu = \frac{m}{M}$ is the number of moles. The density of the gas in this state is

$$\rho = \frac{m}{V} = \frac{pM}{RT}. \tag{2}$$

From (2) the density of the air in the initial state (p_0, V_0', T_0) is

$$\rho_0 = \frac{p_0 M}{RT_0} = \frac{1.01 \times 10^5 \times 29.0 \times 10^{-3}}{8.31 \times 288.15} = 1.22\ \text{kg/m}^3. \tag{3}$$

After the air in the initial state (p_0, V_0', T_0) has passed through the turbocharger, its state becomes (p_1, V_1, T_1) and its density becomes ρ_1; after the isobaric cooling in the intercooler its state becomes $(p_2 = p_1,\ V_2 = V_0,\ T_2 = T_0)$. The mass of the air in the cylinder is

$$\begin{aligned} m_1 &= \rho_1 V_1 = \frac{p_1 M}{RT_1} V_1 = \frac{p_1 M}{RT_0} V_0 = \frac{p_1}{p_0} \rho_0 V_0 \\ &= \frac{1.45 \times 10^5}{1.01 \times 10^5} \times 1.22 \times 400 \times 10^{-6}\ \text{kg} = 0.703 \times 10^{-3}\ \text{kg}. \end{aligned} \tag{4}$$

In the derivation the equation of the isobaric process $\frac{V_1}{T_1} = \frac{V_0}{T_0}$ and equation (3) have been used. If the two steps with the turbocharger and the intercooler are omitted, and the outside air is pressed directly into the cylinder,

$$m_0 = \rho_0 V_0 = 1.22 \times 400 \times 10^{-6}\ \text{kg} = 0.490 \times 10^{-3}\ \text{kg}. \tag{5}$$

The factor by which the output power of the cylinder is increased is

$$R_1 = \frac{\frac{m_1}{V_0}}{\frac{m_0}{V_0}} = \frac{m_1}{m_0} = \frac{0.703 \times 10^{-3}}{0.490 \times 10^{-3}} = 1.43. \tag{6}$$

(2) Without the intercooler, the air compressed by the turbocharger is used directly. At this moment the temperature of the air has already risen to T_1 and the corresponding volume has become V_1. Using the equation of the adiabatic process

$$PV^\gamma = \text{const.} \tag{7}$$

and equation (1), the temperature and the volume of the gas in this state (p_1, V_1, T_1) can be found:

$$T_1 = T_0 \left(\frac{p_1}{p_0}\right)^{(\gamma-1)/\gamma}, \tag{8}$$

$$V_1 = \frac{nRT_1}{p_1} = \frac{nR}{p_1} T_0 \left(\frac{p_1}{p_0}\right)^{(\gamma-1)/\gamma}. \tag{9}$$

Using (2) again, the density ρ_1 of the air in this state is

$$\rho_1 = \rho_0 \left(\frac{T_0}{T_1}\right)\left(\frac{p_1}{p_0}\right) = \rho_0 \left(\frac{p_1}{p_0}\right)^{(1-\gamma)/\gamma} \left(\frac{p_1}{p_0}\right) = \rho_0 \left(\frac{p_1}{p_0}\right)^{1/\gamma}. \tag{10}$$

Correspondingly, the mass of the air fed into the cylinder is

$$m_1' = \rho_1 V_0 = 1.22 \times \left(\frac{1.45 \times 10^5}{1.01 \times 10^5}\right)^{5/7} \times 400 \times 10^{-6}\ \text{kg} = 0.632 \times 10^{-3}\ \text{kg}. \tag{11}$$

The factor by which the output power of the cylinder is increased is

$$R_2 = \frac{\rho_1}{\rho_0} = \frac{m_1'}{m_0} = \left(\frac{1.45\times 10^5}{1.01\times 10^5}\right)^{5/7} = 1.295 = 1.30. \tag{12}$$

(3) Consider separately the work done by the surroundings on the gas in the two processes in the turbocharger and in the intercooler. Turbocharging is an adiabatic compression; from (4), the number of moles of air that enters the turbocharger is

$$\nu = \frac{m_1}{M} = \frac{0.703}{29.0} = 0.02424\ \text{mol}. \tag{13}$$

By hypothesis $\gamma = \dfrac{7}{5}$, so

$$\gamma = \frac{C_p}{C_V} = \frac{C_V + R}{C_V} = \frac{7}{5}, \qquad C_V = \frac{5}{2}R, \tag{14}$$

where C_V and C_p denote the molar heat capacities of the gas at constant volume and at constant pressure. From (14) air is a mixture of diatomic molecules. For an adiabatic process the first law of thermodynamics gives

$$\Delta Q = \Delta U + A = 0, \tag{15}$$

so the work W_1 done by the surroundings in this adiabatic compression is

$$W_1 = -A = \Delta U = \nu C_V (T_1 - T_0). \tag{16}$$

From (8)(16) and the relevant data,

$$W_1 = 0.02424 \times \frac{5}{2} \times 8.31 \times 288.00 \times \left(\left(\frac{1.45\times 10^5}{1.01\times 10^5}\right)^{0.40/1.40} - 1\right)\ \text{J} = 15.78\ \text{J}. \tag{17}$$

Alternative method. Directly from the definition of work $\int P\,\mathrm{d}V$ and the adiabatic equation $pV^\gamma = \text{const.}$,

$$\begin{aligned} W_1 &= -\int_{p_0}^{p_1} p\,\mathrm{d}V = \frac{1}{\gamma}\int_{p_0}^{p_1} V\,\mathrm{d}p = \frac{p_1^{1/\gamma}V_1}{\gamma}\int_{p_0}^{p_1} p^{-1/\gamma}\,\mathrm{d}p \\ &= \frac{p_1^{1/\gamma}V_1}{\gamma}\left(\frac{\gamma}{\gamma-1}\right)\left(p_1^{\frac{\gamma-1}{\gamma}} - p_0^{\frac{\gamma-1}{\gamma}}\right) = \frac{p_1^{\frac{1-\gamma}{\gamma}}nRT_1}{\gamma}\left(\frac{\gamma}{\gamma-1}\right)\left(p_1^{\frac{\gamma-1}{\gamma}} - p_0^{\frac{\gamma-1}{\gamma}}\right). \end{aligned}$$

Substituting T_1 from (8) into the above,

$$W_1 = \left(\frac{\nu RT_0}{\gamma-1}\right)\left(\frac{1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\left(p_1^{\frac{\gamma-1}{\gamma}} - p_0^{\frac{\gamma-1}{\gamma}}\right) = \left(\frac{\nu RT_0}{\gamma-1}\right)\left[\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]. \tag{16}$$

Substituting the relevant data,

$$W_1 = \frac{0.02424\times 8.31\times 288.0}{0.4}\left(\left(\frac{1.45\times 10^5}{1.01\times 10^5}\right)^{0.40/1.40} - 1\right)\ \text{J} = 15.78\ \text{J}. \tag{17}$$

The process of the gas through the intercooler is an isobaric cooling; the work done by the surroundings on the gas is

$$W_2 = -p_1(V_0 - V_1), \tag{18}$$

where V_1 is determined by (9):

$$V_1 = \frac{\nu R}{p_1} T_0 \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{0.02424 \times 8.31 \times 288.0}{1.45 \times 10^5} \times 1.1088\ \mathrm{m}^3 = 4.436 \times 10^{-4}\ \mathrm{m}^3. \tag{19}$$

Substituting (19) into (18) and using the relevant data,

$$W_2 = -1.45 \times 10^5 \times \left(4.000 \times 10^{-4} - 4.436 \times 10^{-4}\right)\ \mathrm{J} = 6.32\ \mathrm{J}. \tag{20}$$

Hence, for the two processes through the turbocharger (adiabatic compression) and through the intercooler (isobaric cooling), the total work done by the surroundings on the gas is

$$W = W_1 + W_2 = 15.78\ \mathrm{J} + 6.32\ \mathrm{J} = 22.1\ \mathrm{J}. \tag{21}$$

The entropy change of the gas passing through the turbocharger (adiabatic compression) is

$$\Delta S_1 = \int_{\text{adiabatic}} \frac{\mathrm{d}Q}{T} = 0. \tag{22}$$

The entropy change of the gas passing through the intercooler (isobaric cooling) is

$$\begin{aligned}\Delta S_2 &= \int_{\text{cooling}} \frac{\mathrm{d}Q}{T} = \int_{T_1}^{T_0} \frac{\nu C_p \mathrm{d}T}{T} = \nu C_p \ln \frac{T_0}{T_1} \\ &= -\frac{7\nu R}{2}\frac{\gamma-1}{\gamma} \ln \frac{p_1}{p_0} = -\nu R \ln \frac{p_1}{p_0} = -0.07284\ \mathrm{J/K}.\end{aligned} \tag{23}$$

Alternative calculation.

$$\begin{aligned}\Delta S_2 &= \int_{\text{adiabatic+cooling}} \frac{\mathrm{d}Q}{T} = \int_{\text{isothermal}} \frac{p\,\mathrm{d}V}{T} = \int_{V_0'}^{V_0} \frac{\nu R\,\mathrm{d}V}{V} \\ &= -\nu R \ln \frac{V_0'}{V_0} = -\nu R \ln \frac{p_1}{p_0} = -0.07284\ \mathrm{J/K}.\end{aligned}$$

In the derivation the equation of the isothermal process $p_0 V_0' = p_2 V_2 = p_1 V_0$ has been used.

Problem 5 (Optical tweezers)

Solution

(1) The wavelength of the light emitted by the He–Ne laser is $\lambda = 632.8$ nm, and the focal length of the focusing objective is $f = 4.00$ mm. When the cross-sectional diameter of the emitted laser beam is $D_1 = 1.50$ mm, the size of the focused spot is limited by circular-aperture diffraction, and the size of the spot is

$$D_{o1} = \frac{2 \times 1.22\lambda}{R_1} f = \frac{2 \times 1.22 \times 0.6328\ \mu\mathrm{m}}{1.50\ \mathrm{mm}} \times 4.00\ \mathrm{mm} = 4.12\ \mu\mathrm{m}. \tag{1}$$

a

(2) The beam expander consists of a concave lens of focal length $f_1 = -6.3$ mm and a convex lens of focal length $f_2 = 25.2$ mm; the distance between the two lenses is

$$l = f_1 + f_2 = -6.3\ \mathrm{mm} + 25.2\ \mathrm{mm} = 18.9\ \mathrm{mm}. \tag{2}$$

After the expansion the diameter of the laser beam is

$$R_2 = \frac{f_2}{-f_1} R_1 = \frac{25.2}{6.3} \times 1.5 \text{ mm} = 6.0 \text{ mm}. \tag{3}$$

(3) In the paraxial approximation the position of the focal point in the sample cell can be found from the refraction imaging formula applied twice. When the expanded laser beam passes through the focusing objective and reaches the glass surface at the bottom of the sample cell, it is first imaged at the outer surface of the bottom of the cell; the image position i_1 satisfies

$$\frac{n_\mathrm{a}}{o_1} + \frac{n_\mathrm{g}}{i_1} = \frac{n_\mathrm{g} - n_\mathrm{a}}{r_1}, \tag{4}$$

where o_1 is the object distance. Since the outer surface of the bottom is at the distance $x = 2$ mm from the focusing objective and the focal length of the objective is $f = 4.0$ mm, we have $o_1 = -2$ mm and $r_1 = \infty$, hence

$$i_1 = -\frac{n_\mathrm{g}}{n_\mathrm{a}} o_1 = 1.46 \times 2 \text{ mm} = 2.92 \text{ mm}. \tag{5}$$

The position i_2 of the second refraction image, at the inner surface of the bottom of the sample cell, satisfies

$$\frac{n_\mathrm{g}}{o_2} + \frac{n_\mathrm{w}}{i_2} = \frac{n_\mathrm{w} - n_\mathrm{g}}{r_2}, \tag{6}$$

where o_2 is the object distance; here, measured from the inner surface of the bottom, $o_2 = o_1 - h = 2.92 - 1 = 1.92$ mm[b], and since the object is virtual here, o_2 takes the negative value, i.e. $o_2 = -1.92$ mm, $r_2 = \infty$, hence

$$i_1 = -\frac{n_\mathrm{w}}{n_\mathrm{g}} o_2 = \frac{1.33}{1.46} \times 1.92 \text{ mm} = 1.75 \text{ mm} \tag{7}$$

[c]The distance from the position of the focal point in the sample cell to the inner surface of the bottom of the cell is therefore 1.75 mm.
The diameter of the expanded laser beam is $D_2 = 6.0$ mm; when it is focused by the focusing lens into the distilled water in the sample cell, the geometry shows that the beam diameter at the outer surface of the bottom of the cell equals $D_2/2$, and that at the inner surface the beam diameter can be written as

$$D_\mathrm{W} = \frac{D_2}{2} \frac{2n_\mathrm{g} - 1}{2n_\mathrm{g}} = 1.97 \text{ mm}. \tag{8}$$

The diameter of the light spot at the focus of the laser in the liquid is

$$D_{\mathrm{o}2} = \frac{2 \times 1.22\lambda}{n_\mathrm{w} D_\mathrm{W}} i_2 = \frac{2 \times 1.22 \times 0.6328\ \mu\text{m}}{1.33 \times 1.97 \text{ mm}} \times 1.75 \text{ mm} = 1.03\ \mu\text{m}. \tag{9}$$

Alternative form.

$$D_{o2} = \frac{2 \times 1.22\lambda}{n_\mathrm{a} D_2} f = \frac{2 \times 1.22 \times 0.6328\ \mu\text{m}}{1.0 \times 6.0 \text{ mm}} \times 4.0 \text{ mm} = 1.03\ \mu\text{m}. \tag{9}$$

Indeed,

$$D_{o2} = \frac{2 \times 1.22\lambda}{D_\mathrm{W}} \frac{i_2}{n_\mathrm{w}} = \frac{2 \times 1.22\lambda}{\left(\frac{-o_1}{f} D_2\right) \frac{n_\mathrm{a} i_2}{n_\mathrm{w}(-o_1)}} \frac{i_2}{n_\mathrm{w}} = \frac{2 \times 1.22\lambda}{n_\mathrm{a} D_2} f.$$

Considering separately the influence on the size of the spot of the refraction at the outer and at the inner surface of the bottom of the sample cell, the magnifications of the light at the outer and at the inner surface are, respectively,

$$V_1 = -\frac{n_a i_1}{n_g o_1} = \frac{2.92}{1.46 \times 2} = 1, \tag{9}$$

$$V_2 = -\frac{n_g i_2}{n_w o_2} = \frac{1.46 \times 1.75}{1.33 \times 1.92} = 1. \tag{10}$$

[d]The influence of the refraction at the outer and inner surfaces of the bottom of the sample cell on the size of the focused spot may therefore in fact be neglected.

(4) In distilled water, the momentum carried in a time Δt by a beam of power P can be written as

$$p = N\frac{h}{\lambda_{\rm w}} = \frac{P\Delta t}{h\nu}\frac{n_{\rm w}h}{\lambda} = \frac{Pn_{\rm w}\Delta t}{c}, \tag{11}$$

where N is the number of photons that pass through the cross section of the beam in the time Δt. The initial momentum of the beam L_1 is

$$\boldsymbol{p}_{1i} = \frac{p_1 n_{\rm W}\Delta t}{c}\boldsymbol{i}, \tag{12}$$

where $\boldsymbol{i}$ is the unit vector along the x axis. The initial momentum of the beam L_2 is

$$\boldsymbol{p}_{2i} = \frac{p_2 n_{\rm W}\Delta t}{c}\boldsymbol{i}. \tag{13}$$

After two refractions in the sphere, the momentum of the beam L_1 is

$$\boldsymbol{p}_{1f} = \frac{p_1 n_{\rm W}\Delta t}{c}\cos 15^\circ\boldsymbol{i} + \frac{p_1 n_{\rm W}\Delta t}{c}\sin 15^\circ\boldsymbol{j}. \tag{14}$$

After two refractions in the sphere, the momentum of the beam L_2 is

$$\boldsymbol{p}_{2f} = \frac{p_2 n_{\rm W}\Delta t}{c}\cos 15^\circ\boldsymbol{i} - \frac{p_2 n_{\rm W}\Delta t}{c}\sin 15^\circ\boldsymbol{j}. \tag{15}$$

After the refraction by the sphere, the change of the momentum of the beams is

$$\begin{aligned}\Delta\boldsymbol{p}_l &= (\boldsymbol{p}_{1f}+\boldsymbol{p}_{2f}) - (\boldsymbol{p}_{1i}+\boldsymbol{p}_{2i})\\ &= \frac{(p_1+p_2)(\cos 15^\circ - 1)n_{\rm W}\Delta t}{c}\boldsymbol{i} + \frac{(p_1-p_2)\sin 15^\circ n_{\rm W}\Delta t}{c}\boldsymbol{j}.\end{aligned} \tag{16}$$

The change of the momentum of the sphere is

$$\Delta\boldsymbol{p}_{\rm b} = -\Delta\boldsymbol{p}_l = \frac{(p_1+p_2)(1-\cos 15^\circ)n_{\rm W}\Delta t}{c}\boldsymbol{i} + \frac{(p_2-p_1)\sin 15^\circ n_{\rm W}\Delta t}{c}\boldsymbol{j}. \tag{17}$$

The optical force (light-intensity gradient force) acting on the sphere is

$$\boldsymbol{F}_{\rm b} = \frac{\Delta\boldsymbol{p}_{\rm b}}{\Delta t} = \frac{(p_1+p_2)(1-\cos 15^\circ)n_{\rm W}}{c}\boldsymbol{i} + \frac{(p_2-p_1)\sin 15^\circ n_{\rm W}}{c}\boldsymbol{j}, \tag{18}$$

$$\boldsymbol{F}_{\rm b} = (0.45\boldsymbol{i} + 1.15\boldsymbol{j}) \times 10^{-12}\ \mathrm{N}. \tag{19}$$

Under the action of the light-intensity gradient force the sphere therefore experiences one force along the positive x direction and one along the positive y direction.

The magnitude of the weight of the sphere is

$$mg = \frac{4}{3}\pi\left(\frac{D_\mathrm{b}}{2}\right)^3 \rho_\mathrm{b} = \frac{4}{3} \times 3.14 \times 1.0 \times 10^{-18} \times 10^3 \times 9.8 = 4.11 \times 10^{-14}\ \mathrm{N}. \tag{20}$$

The ratios of the optical forces on the sphere in the x and in the y direction to its weight are

$$\eta_x = \frac{0.45 \times 10^{-12}}{4.11 \times 10^{-14}} = 10.9, \tag{21}$$

$$\eta_y = \frac{1.15 \times 10^{-12}}{4.11 \times 10^{-14}} = 28.0. \tag{22}$$

(5) In water at ordinary temperature (300 K), the mean kinetic energy of a sphere is

$$E_k = \frac{3}{2}kT = 1.5 \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21}\ \mathrm{J}. \tag{23}$$

The root-mean-square speed of a sphere is

$$v_p = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{4.19 \times 10^{-15}}} = \sqrt{\frac{12.42 \times 10^{-6}}{4.19}} = 1.72 \times 10^{-3}\ \mathrm{m/s}. \tag{24}$$

(6) Suppose that only spheres whose speed does not exceed v_m can be trapped by the optical potential well; conservation of energy gives

$$\frac{1}{2}mv_m^2 + U_\mathrm{min} = 0, \tag{25}$$

where

$$U_\mathrm{min} = -U_0\left(1 - \frac{y^2}{4}\right)_{y=0} = -U_0. \tag{27}$$

[e]From (25)(26) the maximum trapping speed is

$$v_m = \sqrt{\frac{2U_0}{m}} = \sqrt{\frac{2 \times 78 \times 10^{-3} \times 1.6 \times 10^{-19}}{4.19 \times 10^{-15}}} = \sqrt{\frac{24.96 \times 10^{-6}}{4.19}} = 2.44 \times 10^{-3}\ \mathrm{m/s}. \tag{28}$$

(7) The probability that a polystyrene sphere can be trapped may be written as

$$P = \int_0^{v_m} f(v)\,\mathrm{d}v = \int_0^{v_m} 4\pi\left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} v^2 \mathrm{e}^{-\frac{mv^2}{2k_\mathrm{B}T}}\,\mathrm{d}v. \tag{29}$$

Put

$$v_\mathrm{p} = \sqrt{\frac{2k_\mathrm{B}T}{m}}; \tag{30}$$

then

$$P = \int_0^{v_m} \frac{4}{\sqrt{\pi}}\left(\frac{v}{v_\mathrm{p}}\right)^2 \mathrm{e}^{-\frac{v^2}{v_\mathrm{p}^2}}\,\mathrm{d}\left(\frac{v}{v_\mathrm{p}}\right). \tag{31}$$

Put

$$x = \frac{v}{v_\mathrm{p}};$$

the probability that a polystyrene sphere is trapped by the optical potential well can then be written as

$$P = \int_0^{\sqrt{U_0/kT}} \frac{4}{\sqrt{\pi}} x^2 \mathrm{e}^{-x^2}\,\mathrm{d}x. \tag{32}$$

From (31) the probability P depends on U_0 and on the temperature T, but not on m. Therefore the probability P_m for a polystyrene sphere of mass m and the probability P_{2m} for a polystyrene sphere of mass $2m$ to be trapped by the optical potential well are equal:

$$P_m = P_{2m}. \tag{33}$$

[a] *Translator's note:* in (1) and (3) the source writes R_1 and R_2 for the beam diameters D_1 and D_2.
[b] *Translator's note:* printed exactly so in the source; the quantity used is $i_1 - d = 2.92 - 1.00$ mm.
[c] *Translator's note:* the left-hand side of (7) is printed as i_1; it is i_2.
[d] *Translator's note:* the source uses the number (9) three times in this part.
[e] *Translator's note:* the source jumps from (25) to (27) and then refers to "(25)(26)".

Problem 6 (Building a model of the Sun)

Solution

(1)(i)

$$\Phi = \frac{D}{S_{\mathrm{SE}}} = \frac{d}{L}. \tag{1}$$

(ii) The main error comes from whether the solar disc exactly fills the inner tube of the bamboo pole; one can put in front of the round hole of the pole a shading disc of adjustable radius — the principle is similar to that of a coronagraph.

(iii)

$$\Phi = \frac{1}{80} \qquad \left(\text{the currently accepted value is } \frac{1}{109}\right). \tag{2}$$

(2)(i) From the law of gravitation,

$$G\frac{Mm}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r \tag{3}$$

(or directly from Kepler's third law), whence

$$r_{\mathrm{VE}} = \frac{S_{\mathrm{SV}}}{S_{\mathrm{SE}}} = \left(\frac{T_{\mathrm{V}}}{T_{\mathrm{E}}}\right)^{2/3} = 0.72. \tag{4}$$

(ii) At the beginning of the transit, i.e. at the ingress point, the line joining the Earth and Venus is tangent to the solar disc; after the transit time t_{P}, i.e. at the egress point, the line joining the Earth and Venus is tangent to the solar disc on the other side, as shown in solution figure a. Neglecting the very small inclination between the orbital planes of the Earth and of Venus, and using the small-angle approximation,

$$\frac{D_{\mathrm{P}}}{S_{\mathrm{SV}}} = \frac{D_{\mathrm{P}}}{S_{\mathrm{SE}}} + 2\pi\left(\frac{t_{\mathrm{P}}}{T_{\mathrm{V}}} - \frac{t_{\mathrm{P}}}{T_{\mathrm{E}}}\right), \tag{5}$$

whence

$$\frac{D_{\mathrm{P}}}{S_{\mathrm{SE}}} = \frac{r_{\mathrm{VE}}}{1 - r_{\mathrm{VE}}} 2\pi t_{\mathrm{P}} \left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right). \tag{6}$$

From figure a one reads off

$$t_{\mathrm{P}} = 6\ 21\ 47 = 6.36\ \mathrm{h} = 0.265\ \mathrm{d}.$$

Substituting (4) and the above data into (6),

$$\frac{D_{\mathrm{P}}}{S_{\mathrm{SE}}} = \frac{1.44}{0.28}\pi\left(1.18\times10^{-3} - 0.72\times10^{-3}\right) = 7.4\times10^{-3} = \frac{1}{135}.$$

Let the distance of the chord from the centre of the disc be h_{P}; the geometry gives

$$h_{\mathrm{P}} = \frac{1}{2}\sqrt{D^2 - D_{\mathrm{P}}^2}. \tag{7}$$

Since $h_{\mathrm{P}} = 5D/16$, (7) gives

$$D = 1.28 D_{\mathrm{P}},$$

and the angular size of the Sun is

$$\Phi = \frac{D}{S_{\mathrm{SE}}} = \frac{1.28}{135} = \frac{1}{105}. \tag{8}$$

(iii) From (6), the chord D_{P} swept during the transit is proportional to the transit time t_{P}:

$$\frac{D_{\mathrm{P}}}{S_{\mathrm{SE}}} = \frac{r_{\mathrm{VE}}}{1-r_{\mathrm{VE}}}2\pi\left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right)t_{\mathrm{P}} = 2.57\ \mathrm{s}^{-1}\times t_{\mathrm{P}}. \tag{9}$$

[a]From (7), the angle subtended at Venus by D_{P} and $D_{\mathrm{P'}}$ is

$$\begin{aligned}\Theta = \frac{h_{\mathrm{P}} - h_{\mathrm{P'}}}{S_{\mathrm{SV}}} &= \frac{1}{2r_{\mathrm{VE}}}\left(\sqrt{\frac{D^2}{S_{\mathrm{SE}}^2} - \frac{D_{\mathrm{P}}^2}{S_{\mathrm{SE}}^2}} - \sqrt{\frac{D^2}{S_{\mathrm{SE}}^2} - \frac{D_{\mathrm{P'}}^2}{S_{\mathrm{SE}}^2}}\right)\\
&= \frac{1}{2r_{\mathrm{VE}}}\left(\sqrt{\Phi^2 - \left(\frac{2\pi r_{\mathrm{VE}}t_{\mathrm{P}}}{1-r_{\mathrm{VE}}}\right)^2\left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right)^2} - \sqrt{\Phi^2 - \left(\frac{2\pi r_{\mathrm{VE}}t_{\mathrm{P'}}}{1-r_{\mathrm{VE}}}\right)^2\left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right)^2}\right).\end{aligned}$$

Using the small-angle approximation (see solution figure b),

$$\Theta = \frac{H}{S_{\mathrm{SE}} - S_{\mathrm{SV}}} = \frac{H}{S_{\mathrm{SE}}(1-r_{\mathrm{VE}})},$$

where H is the straight-line distance between the two observers P and P′ on the same meridian (in the projection perpendicular to the direction of observation). Equating the right-hand sides of the two expressions gives

$$\begin{aligned}S_{\mathrm{SE}} = \frac{2r_{\mathrm{VE}}H}{1-r_{\mathrm{VE}}}\Bigg[&\sqrt{\Phi^2 - \left(\frac{2\pi r_{\mathrm{VE}}t_{\mathrm{P}}}{1-r_{\mathrm{VE}}}\right)^2\left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right)^2}\\
&- \sqrt{\Phi^2 - \left(\frac{2\pi r_{\mathrm{VE}}t_{\mathrm{P'}}}{1-r_{\mathrm{VE}}}\right)^2\left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right)^2}\ \Bigg]^{-1}.\end{aligned} \tag{10}$$

(iv) The latitude of Beijing is 39.5° and that of Hong Kong is 22.5°; the shortest distance on the ground between the two corresponding parallels of latitude is

$$H = \frac{17^\circ}{180^\circ}\times\pi\times6370\ \mathrm{km} = 1900\ \mathrm{km}. \tag{11}$$

Beijing: first (exterior) contact at ingress 6:10:00, third (interior) contact at egress 12:31:57, hence

$$t_{\mathrm{P}} = 6\ 21\ 57.$$

Hong Kong: first (exterior) contact at ingress 6:11:48, third (interior) contact at egress 12:31:19, hence

$$t_{\mathrm{P}'} = 6\ 19\ 31.$$

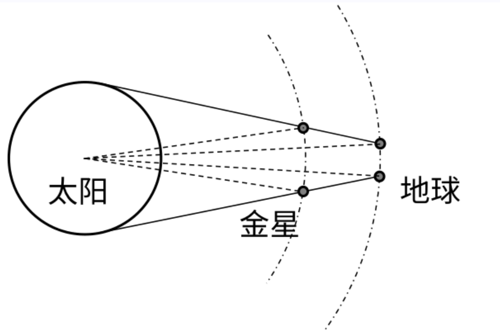
Substituting the relevant data into (10),

$$S_{\mathrm{SE}} = 1.6 \times 10^8\ \mathrm{km}. \tag{12}$$

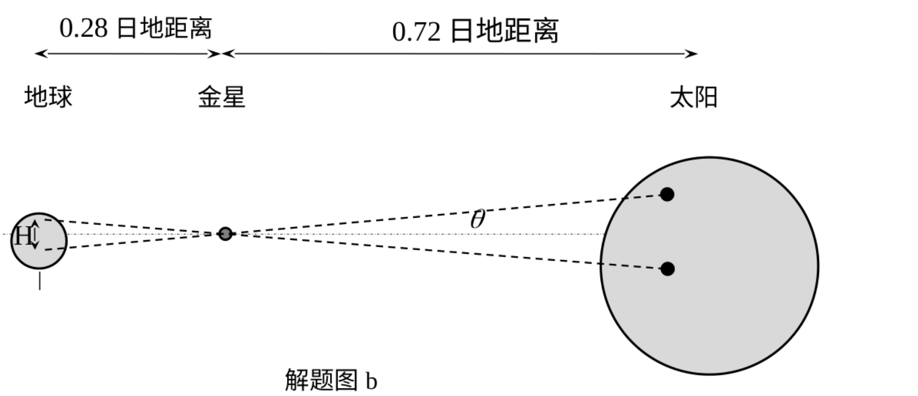
The diameter of the Sun is

$$D = \frac{D}{S_{\mathrm{SE}}} S_{\mathrm{SE}} = \frac{1}{105} \times 1.5 \times 10^6\ \mathrm{km} = 1.5 \times 10^6\ \mathrm{km}$$

b



解题图 a



解题图 b

Labels in the solution figures: 太阳 *= Sun;* 金星 *= Venus;* 地球 *= Earth;* 日地距离 *= Sun–Earth distance.*

(3)(i) The power of the solar radiation received by the Earth is

$$W_{\mathrm{tot}} = \pi R_{\mathrm{E}}^2 I = 3.14 \times 6.4^2 \times 10^{12} \times 1.37 \times 10^3\ \mathrm{W} = 1.76 \times 10^{17}\ \mathrm{W}. \tag{13}$$

(ii) The total energy consumed by the world per year is

$$E_{\mathrm{cons}} = 70 \times 10^8 \times 3 \times 10^3 \times 4 \times 3.6 \times 10^6\ \mathrm{J} = 3 \times 10^{20}\ \mathrm{J},$$

and the solar radiation energy received by the Earth in one year is

$$E_{\mathrm{rad}} = 365 \times 24 \times 3600 \times W_{\mathrm{tot}} = 5.5 \times 10^{24}\ \mathrm{J}.$$

The fraction of the energy consumed is therefore

$$E_{\mathrm{cons}} : E_{\mathrm{rad}} = 1 : 1.6 \times 10^4, \tag{14}$$

less than one part in ten thousand, which shows that the potential for us to use solar energy is still very large.
(iii) The total energy radiated by the Sun per unit time is

$$W_{\mathrm{S}} = 4\pi S_{\mathrm{SE}}^2 I = 3.87 \times 10^{26}\ \mathrm{W}. \tag{15}$$

(iv) The energy radiated by the Sun per unit time per unit area is

$$I_{\mathrm{S}} = \frac{W_{\mathrm{S}}}{4\pi R_{\mathrm{S}}^2} = \frac{S_{\mathrm{SE}}^2}{R_{\mathrm{S}}^2} I = 6.5 \times 10^7\ \mathrm{W/m^2},$$

so the surface temperature of the Sun is

$$T_{\mathrm{surf}} = \left(\frac{I_{\mathrm{S}}}{\sigma}\right)^{1/4} = 5.8 \times 10^3\ \mathrm{K}. \tag{16}$$

(4)(i) From the law of gravitation,

$$G\frac{M_{\mathrm{S}} m}{S_{\mathrm{SE}}^2} = m\frac{4\pi^2}{T_{\mathrm{E}}^2} S_{\mathrm{SE}},$$

so that the mass of the Sun is

$$\begin{aligned} M_{\mathrm{S}} &= \frac{4\pi^2}{G T_{\mathrm{E}}^2} S_{\mathrm{SE}}^3 \\ &= \frac{4\pi^2}{6.67 \times 10^{-11} \times (365 \times 24 \times 3600)^2} \times (1.5 \times 10^{11})^3\ \mathrm{kg} \\ &= 2.0 \times 10^{30}\ \mathrm{kg}. \end{aligned} \tag{17}$$

The mean density of the Sun is

$$\rho = \frac{M_{\mathrm{S}}}{\frac{4}{3}\pi R_{\mathrm{S}}^3} = 1.4 \times 10^3\ \mathrm{kg/m^3}, \tag{18}$$

i.e. 1.4 times the density of water.
(ii) If the energy source of the Sun came from a chemical reaction, assumed to be the burning of coal, then the mass of coal that must be burnt per second is

$$\frac{\Delta m}{\Delta t} = \frac{W_{\mathrm{S}}}{4 \times 3.6 \times 10^6} = 2.7 \times 10^{19}\ \mathrm{kg/s},$$

and the time for which the burning could be sustained is

$$t_{\mathrm{tot}} = \frac{M}{\frac{\Delta m}{\Delta t}} = 7.4 \times 10^{10}\ \mathrm{s} = 2400\ \mathrm{y}. \tag{19}$$

In his day Darwin argued from the point of view of biological evolution that the Earth must have existed for at least one hundred million years; if the Sun has existed as a star for 5 billion years or more, then the energy-production efficiency of the Sun must be at least

2 million times the efficiency of burning coal. Hence the energy source of the Sun cannot be chemical energy; it can only be nuclear energy.

(5)(i) The whole fusion process amounts to the consumption of 4 protons and the production of one helium nucleus, with 2 positrons annihilating with two electrons of the plasma into photons. Neglecting the energy carried away by the neutrinos, the energy released is

$$1\ \text{MeV} = 1.60217733(49) \times 10^{-13}\ \text{J},$$

$$\begin{aligned}\Delta E &= \left(4M_\text{p} - M_{^4\text{He}} - 2m_\text{e} + 4m_\text{e}\right)c^2 \\ &= 938.3 \times 4 - 3727.5 + 0.51 \times 2 = 26.7\ \text{MeV} = 4.27 \times 10^{-12}\ \text{J}.\end{aligned} \tag{20}$$

The number of protons that must be consumed per unit time is

$$\frac{\Delta N_\text{P}}{\Delta t} = \frac{3.87 \times 10^{26}}{4.27 \times 10^{-12}/4} = 3.62 \times 10^{38}\ \text{s}^{-1}. \tag{21}$$

(ii) The total number of protons in the Sun is

$$N_\text{P} = \frac{0.71 \times 2.0 \times 10^{30}\ \text{kg}}{1.67 \times 10^{-27}\ \text{kg}} = 8.5 \times 10^{56}.$$

Assuming that in the end all the protons take part in the fusion reactions (in fact the outer layers of the Sun are too cold for fusion), the time for which the fusion can be maintained is

$$\Delta t = \frac{8.5 \times 10^{56}}{3.62 \times 10^{38}}\ \text{s} = 2.3 \times 10^{18}\ \text{s} = 7.3 \times 10^{10}\ \text{y}. \tag{22}$$

The reaction probability per unit time is

$$P_\text{PP} = \frac{3.62 \times 10^{38}}{8.5 \times 10^{56}}\ \text{s}^{-1} = 4.2 \times 10^{-19}\ \text{s}^{-1}. \tag{23}$$

(iii) Two protons reacting produce one neutrino, so the number of neutrinos produced per second is

$$\frac{\Delta N_{\nu_\text{e}}}{\Delta t} = 1.81 \times 10^{38}\ \text{s}^{-1},$$

and the number of neutrinos that can be received on the Earth per unit time per unit area is

$$\frac{\Delta N_{\nu_\text{e}}}{\Delta t}\frac{1}{4\pi S_\text{SE}^2} = 6.4 \times 10^{14}\ \text{s}^{-1}\text{m}^{-2}. \tag{24}$$

(iv)

$$\frac{\Delta M_\text{S}}{\Delta t} = \frac{W_\text{S}}{c^2} = \frac{3.87 \times 10^{26}\ \text{kg}}{9 \times 10^{16}\ \text{s}} = 4.3 \times 10^{9}\ \text{kg/s} = 1.3 \times 10^{17}\ \text{kg/y}. \tag{25}$$

(6) The mean density $\bar{\rho}_\text{core}$ of the core region is higher than the mean density $\bar{\rho}$, but smaller than $64\bar{\rho}$:

$$\bar{\rho} = \frac{M_\text{S}}{\frac{4}{3}\pi R^3} < \bar{\rho}_\text{core} = \frac{M_\text{S,core}}{\frac{4}{3}\pi\left(\frac{R}{4}\right)^3} = 64\frac{M_\text{S,core}}{\frac{4}{3}\pi R^3} < 64\frac{M_\text{S}}{\frac{4}{3}\pi R^3} = 64\bar{\rho},$$

i.e. it is smaller than

$$64\bar{\rho} = 9 \times 10^4\ \text{kg}\cdot\text{m}^{-3}.$$

The proton number density therefore satisfies

$$8.4 \times 10^{23}\ \mathrm{cm}^{-3} = \frac{9 \times 10^4\ \mathrm{kg \cdot m^{-3}}}{64 \times 1.67 \times 10^{-27}\ \mathrm{kg}} < n_\mathrm{p} < \frac{9 \times 10^4\ \mathrm{kg \cdot m^{-3}}}{1.67 \times 10^{-27}\ \mathrm{kg}} = 5.4 \times 10^{25}\ \mathrm{cm}^{-3},$$

so that

$$\begin{aligned} 5.4 \times 10^{-45}\ \mathrm{cm^3/s} = \frac{2.9 \times 10^{-19}\ \mathrm{s}^{-1}}{5.4 \times 10^{25}\ \mathrm{cm}^{-3}} < R(T) = \frac{P_\mathrm{PP}}{n_\mathrm{p}} \\ < \frac{64 \times 2.9 \times 10^{-19}\ \mathrm{s}^{-1}}{5.4 \times 10^{25}\ \mathrm{cm}^{-3}} = 3.5 \times 10^{-43}\ \mathrm{cm^3/s}. \end{aligned} \tag{26}$$

[c]Since the proton–proton reaction rate is far lower than those of the other two reactions, the overall reaction probability is determined by the proton–proton probability. Drawing a horizontal line at the above reaction rate and intersecting it with the PP reaction-rate curve gives the reaction temperature

$$T \approx 1.0 \times 10^7\ \mathrm{K}. \tag{27}$$

(7) The mass defect of the fusion process (a) is 0.49 MeV, much larger than the kinetic energy of a proton (1.5 keV); the protons may therefore be regarded as at rest, the total momentum after the reaction is zero, and the total kinetic energy equals the mass defect times the square of the speed of light.
For this three-body problem, the computation of the energy and the momentum of each particle from the conservation of energy and momentum is not unique, but the maximum kinetic energy of the neutrino corresponds to the case in which the deuteron and the positron have no relative motion (i.e. have the same velocity). Since the kinetic energy of the deuteron is much smaller than its rest mass times the square of the speed of light, the non-relativistic expression may be used for it, whereas the rest mass of the neutrino is approximately zero and the relativistic expression must be used. Hence

$$(m_\mathrm{D} + m_\mathrm{e})v - p_\nu = 0, \tag{28}$$

$$\frac{p_\mathrm{D}^2}{2m_\mathrm{D}} + \frac{p_\mathrm{e}^2}{2m_\mathrm{e}} + p_\nu c = 0.49\ \mathrm{MeV}. \tag{29}$$

Also

$$p_\mathrm{e} = \frac{m_\mathrm{e}}{m_\mathrm{D}} p_\mathrm{D},$$

so that

$$\frac{p_\mathrm{e}^2}{2m_\mathrm{e}} = \frac{m_\mathrm{e} p_\mathrm{D}^2}{2m_\mathrm{D}^2} \ll \frac{p_\mathrm{D}^2}{2m_\mathrm{D}}.$$

From (28)(29) again,

$$\frac{p_\nu^2 c^2}{2m_\mathrm{D} c^2} + p_\nu c = 0.49\ \mathrm{MeV};$$

dropping the small quantity $\frac{p_\nu^2 c^2}{2m_\mathrm{D} c^2}$ gives

$$E_{\nu\,\max} = 0.49\ \mathrm{MeV}. \tag{30}$$

The ratio of the total neutrino energy to the energy released in the fusion is smaller than

$$\frac{0.49 \times 2}{26.7} = 3.7\%. \tag{31}$$

(8)(i) The decrease ΔM of the mass caused by the radiation in one hundred million years is

$$\Delta M = \frac{\Delta M_{\mathrm{S}}}{\Delta t} \times \Delta t = \left(1.3 \times 10^{17}\ \mathrm{kg/y}\right)\left(1.0 \times 10^{8}\ \mathrm{y}\right) = 1.3 \times 10^{25}\ \mathrm{kg}. \tag{32}$$

(ii) Gravity provides the centripetal force, so

$$G\frac{Mm}{r_1^2} = m\left(\frac{2\pi}{T_1}\right)^2 r_1, \qquad G\frac{(M-\Delta M)m}{r_2^2} = m\left(\frac{2\pi}{T_2}\right)^2 r_2,$$

hence

$$\frac{MT_1^2}{r_1^3} = \frac{(M-\Delta M)T_2^2}{r_2^3}. \tag{33}$$

Since the mass decrease is caused by the radial radiation of the Sun, the angular momentum of the Earth with respect to the Sun is conserved:

$$mv_1r_1 = mv_2r_2,$$

or

$$m\frac{2\pi r_1}{T_1}r_1 = m\frac{2\pi r_2}{T_2}r_2,$$

i.e.

$$\frac{r_1^2}{T_1} = \frac{r_2^2}{T_2}. \tag{34}$$

From (32)(33)(34),

$$\frac{T_2}{T_1} = \left(1 - \frac{\Delta M}{M}\right)^{-2} = 1 + 2\frac{\Delta M}{M}, \tag{35}$$

so the change of the period of revolution of the Earth is

$$T_2 - T_1 = 1.3 \times 10^{-5} T_1. \tag{36}$$

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[a] *Translator's note:* printed exactly so; the numerical factor $\frac{r_{\mathrm{VE}}}{1-r_{\mathrm{VE}}}2\pi\left(\frac{1}{T_{\mathrm{V}}} - \frac{1}{T_{\mathrm{E}}}\right)$ actually equals 2.77×10^{-2} per day, and the unit s^{-1} is a misprint.

[b] *Translator's note:* printed exactly so; the factor should be 1.6×10^{8} km, which does give 1.5×10^{6} km.

[c] *Translator's note:* the value $2.9 \times 10^{-19}\ \mathrm{s}^{-1}$ used here differs from the value $P_{\mathrm{PP}} = 4.2 \times 10^{-19}\ \mathrm{s}^{-1}$ obtained in (23); both are printed as they stand in the source.