The 37th Chinese Physics Olympiad

Final — Theoretical Examination (Problems)

2020

Problem 1

China's large scientific facility, the spallation neutron source, was completed and put into service in 2018; it has wide applications in many fields. Historically, Chadwick first confirmed the existence of the neutron and measured its mass in 1932. In 1939 Hahn and others found that bombarding uranium nuclei with neutrons makes them fission and release neutrons at the same time, which triggers a chain reaction. To make the chain reaction proceed in a sustained and controllable way, the neutrons released by the uranium nuclei can be moderated by elastic collisions. Neglect relativistic effects.

(1) Chadwick bombarded stationary target nuclei of mass m_1 (a hydrogen nucleus H or a nitrogen nucleus ^{14}N, of mass m_{H} or $14\,m_{\mathrm{H}}$) with neutrons of mass m, and observed their motion. Let α be the angle between the outgoing momentum of the target nucleus and the initial momentum of the incident neutron. Derive the relations that connect the outgoing speed v_1 of the target nucleus and the final speed v of the neutron with the initial speed v_0 of the neutron. The experiment found that the maximum outgoing speed of the hydrogen nuclei is 3.30×10^7 m/s and the maximum outgoing speed of the nitrogen nuclei is 4.70×10^6 m/s. Find m and v_0.

(2) In the above experiment a nitrogen nucleus may also be struck repeatedly by a beam of neutrons. Suppose the nitrogen nucleus is initially at rest, the speed of every neutron that hits it is v_0, and every collision gives the nitrogen nucleus the largest possible increase of speed. Calculate after how many collisions the kinetic energy of the nitrogen nucleus becomes approximately equal to the initial kinetic energy of a neutron.

(3) Suppose the neutrons that have been slowed down by many collisions are in thermal equilibrium, so that their speeds obey the Maxwell distribution

$$f(v) = 4\pi \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} v^2 \mathrm{e}^{-\frac{mv^2}{2k_{\mathrm{B}}T}},$$

where k_{B} is Boltzmann's constant and T is the absolute temperature of the system. Calculate, at thermal equilibrium, the kinetic energy that corresponds to the most probable speed of the neutrons, and the most probable kinetic energy.

Problem 2

When two conducting spheres in an electric field are brought close together, the field between them is markedly enhanced. This problem explores that phenomenon. The permittivity of vacuum is ε_0.

(1) An isolated conducting sphere of radius R_0 has its centre a distance a ($a > R_0$) from a stationary point charge Q. Find the magnitude and the position of the image charge.

(2) The conducting sphere is placed in a uniform external field of magnitude E_0. This applied field may be regarded as the field produced, at the midpoint of their line of centres, by two equal and opposite point charges $\pm Q$ that are very far apart. Show that the effect of the charge induced on the conducting sphere is equivalent to an electric dipole formed by two image charges, and find the relation between the moment of that dipole and the external field E_0.

(3) Two equal and opposite point charges $\pm q$ outside the conducting sphere are separated by Δl. The dipole $q\,\Delta l$ they form is placed radially near the conducting sphere of radius R_0; the centre of the dipole is a distance r ($r \gg \Delta l$) from the centre of the sphere. Find the magnitude of the image dipole moment of this dipole inside the conducting sphere.

(4) In a uniform external field of magnitude E_0, two conducting spheres of radius R_0 are placed along the direction of the external field, the distance between their centres being r ($r > 2R_0$). Find the potential at an arbitrary point $\mathrm{P}'(x, y)$ of the plane through the two centres, outside the spheres, and the field distribution along the line of centres (a recursion relation may be used). Take the midpoint of the line of centres as the origin and the line of centres as the X axis.

(5) Show that, if spark discharge is not taken into account, when the two conductors above are slowly brought infinitely close together the field at the midpoint of their line of centres tends to infinity because of electrostatic induction.

Problem 3

A quantum heat engine is a heat engine that uses quantum matter as its working substance. The working principle of a quantum heat engine is described below, taking a two-level atomic system as an example. The mean energy of a two-level atom is defined as

$$\langle E\rangle = p_0 \cdot E_0 + p_1 \cdot E_1,$$

where E_0, p_0 and E_1, p_1 denote the energy and the probability of the atom being in the ground state and in the excited state respectively. For simplicity assume $E_0 = 0$; during the cycle the energy difference between the excited state and the ground state is an adjustable parameter. The probability that the atom is in a state of energy E obeys the Boltzmann distribution

$$p \propto \mathrm{e}^{-E/(k_\mathrm{B}T)},$$

where T is the thermodynamic temperature and k_B is Boltzmann's constant. In a quasi-static process the change of the mean energy is

$$\mathrm{d}\langle E\rangle = p_1\,\mathrm{d}E_1 + E_1\,\mathrm{d}p_1,$$

where $p_1\,\mathrm{d}E_1$ is the energy change caused by the change of the energy level, and corresponds to the work $\mathrm{d}W$ done by the surroundings on the two-level atomic system; $E_1\,\mathrm{d}p_1$ is the energy change caused by the change of the probability, and corresponds to the heat $\mathrm{d}Q$ transferred by the surroundings to the two-level atomic system.

(1) Bring the two-level atomic system into contact with a heat reservoir at temperature T. Find the probability p_0 that the two-level atomic system is in the ground state and the probability p_1 that it is in the excited state at thermal equilibrium.

(2) The P–V diagram of the classical Otto cycle is shown in Fig. 2a, where A→B and C→D are isochoric processes and B→C and D→A are adiabatic processes. Draw a schematic diagram of the relation between the two-level energy difference E_1 and the excited-state probability p_1 during the quantum Otto cycle, and calculate the heat transferred, the increment of internal energy and the work done on the surroundings in the four processes of the quantum Otto cycle.

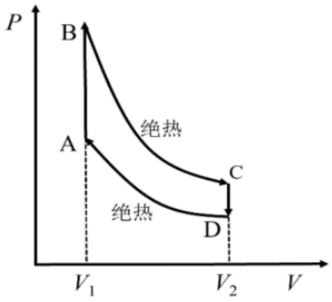

图 2a

Labels: 绝热 = *adiabatic;* 图 2a = *Fig. 2a.*

(3) Similarly, calculate the heat transferred, the increment of internal energy and the work done on the surroundings in each process of the quantum Carnot cycle. Assume that the temperature T_h of the hot reservoir and the temperature T_l of the cold reservoir of the quantum Carnot cycle are equal to the highest and the lowest temperature of the quantum Otto cycle respectively, and compare the efficiencies of the quantum Otto engine and of the quantum Carnot engine.

Problem 4

Since Yeh and Cummins observed in 1964 the Doppler shift of the light scattered by particles in a water flow, laser Doppler velocimetry has found wide application. In order to obtain a sufficient scattered-light intensity, tracer scattering particles of suitable size and concentration are usually spread in the fluid; when laser light falls on the moving particles it is scattered, and this gives information about the velocity of the particles (and of the fluid). Let the angle between the particle velocity v and the vertical direction be β.

(1) As shown in Fig. 4a, a beam of light of frequency f is scattered by a moving particle in the fluid. The speed of light in the fluid is c/n (n is the refractive index of the fluid), and the particle moves with speed v ($v \ll c/n$). The angles between the propagation directions of the incident light and of the observed scattered light and the particle velocity are θ_1 and θ_2 respectively. Find the relation between the frequency shift Δf of the scattered light relative to the incident light and the direction of the scattered light. If the speed of the particle is $v = 1\,\mathrm{m/s}$ and the frequency of the light is $f = 10^{14}\,\mathrm{Hz}$, can Δf be detected directly with a spectrometer of resolution $5\,\mathrm{MHz}$?

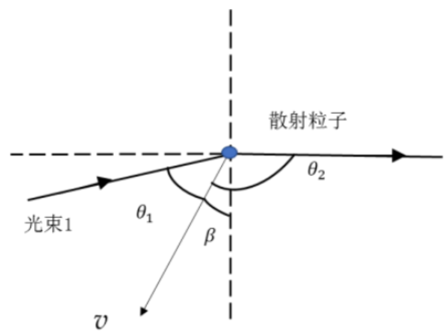

图 4a

Labels: 散射粒子 = *scattering particle;* 光束1 = *beam 1;* 图 4a = *Fig. 4a.*

(2) As shown in Fig. 4b, two coherent parallel beams of frequency f (their propagation directions lie in the same vertical plane) illuminate the same particle in the fluid; each beam makes an angle $\alpha/2$ with the horizontal plane (α is rather small). The phases with which the light scattered from the two beams reaches the photodetector are ϕ_1 and ϕ_2 respectively; the electric-field vectors of the two scattered beams are approximately in the same direction and both have amplitude E_0. The current delivered by the photodetector is proportional to the light intensity it receives, with proportionality constant k. Assume the frequency-response range of the photodetector is $10^2 \sim 10^7$ Hz, and find the expression for the output current of the photodetector. For simplicity, assume that the particle velocity lies in the plane of the two incident beams that illuminate the particle.

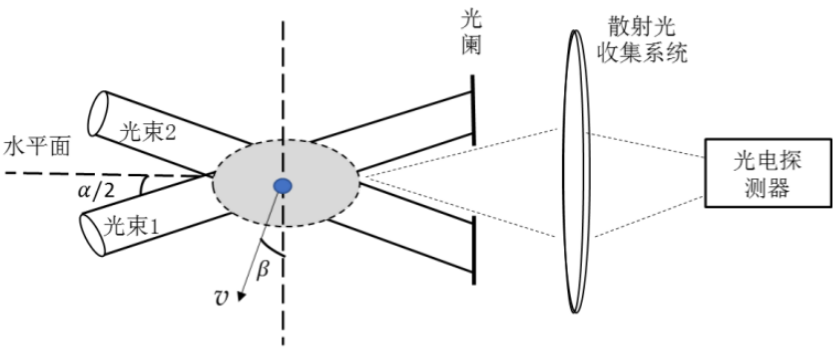

图 4b

Labels: 水平面 = *horizontal plane;* 光束1, 光束2 = *beam 1, beam 2;* 光阑 = *aperture stop;* 散射光收集系统 = *scattered-light collecting system;* 光电探测器 = *photodetector;* 图 4b = *Fig. 4b.*

(3) Two coherent parallel beams in the XOZ plane are incident symmetrically about the X axis and reach the crossing region (see Fig. 4c). Find the spacing of the interference fringes. A particle (whose velocity lies in the XOZ plane) passing through the region of alternating bright and dark interference fringes scatters light pulses; find the frequency of these light pulses.

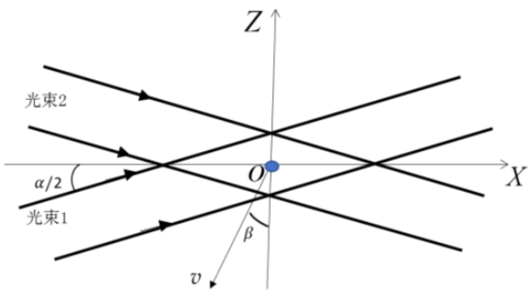

图 4c

Labels: 光束1, 光束2 = *beam 1, beam 2;* 图 4c = *Fig. 4c.*

Problem 5

In the vast cosmic space between the stars there is an interstellar medium. Under the action of cosmic rays the molecules of the interstellar medium lose part of their electrons and become positive ions, while the electrons wander freely outside, so that the medium is in a plasma state. A pulsar is a rapidly rotating neutron star with an extremely strong magnetic field; the electromagnetic radiation from it that is observed on the Earth is a pulse signal. Pulsar signals can be used for interstellar navigation and high-precision timekeeping; for this one must obtain the precise time of arrival of the pulse signal at the Earth and study the phase change and the dispersion of its electromagnetic radiation.

The distance from a pulsar to the Earth is d; the mean number density of electrons in the interstellar medium is n_e (of order $10^4\,\mathrm{m}^{-3}$). Assume there is a uniform static magnetic field B in the medium (of order $10^{-10}\,\mathrm{T}$) whose direction is parallel to the propagation direction of the electromagnetic wave. The electron mass is m_e and its charge is $-e$ ($e > 0$); the permittivity of vacuum is ε_0 and the speed of light in vacuum is c.

(1) Take the direction of the electromagnetic radiation from this pulsar to the Earth as the $+z$ direction. For an electromagnetic wave of frequency f the electric field is

$$E_x = E_0\cos(kz - 2\pi f t), \qquad E_y = E_0\cos\left(kz - 2\pi f t \pm \frac{\pi}{2}\right),$$

where E_0 and k are the amplitude and the wave number of the wave. For simplicity take E_0 to be constant and neglect the action of the magnetic field of the wave itself on the electron. Find the cyclotron radius R_e of the motion of the electron.

(2) A pulsar signal is dispersed as it propagates in the interstellar medium; its propagation speed (the magnitude v_g of the group velocity) depends on the rate of change of the frequency of the electromagnetic wave with respect to the wave number: $v_\mathrm{g} = 2\pi\dfrac{\mathrm{d}f}{\mathrm{d}k}$. Find the lowest frequency f_c of the electromagnetic waves of the pulsar signal that can reach the Earth through the medium.

(3) Assume that along the propagation path of the pulsar signal the mean electron number density n_e of the interstellar medium stays constant. By how much is the time of arrival at the Earth of an electromagnetic wave of frequency f in the pulsar signal delayed compared with the time it would take propagating in vacuum?

(4) Observation shows that electromagnetic waves of frequency f that leave the pulsar at the same moment display a phase difference when they arrive at the Earth. Find this phase difference. (The approximations $\dfrac{e}{2\pi}\sqrt{\dfrac{n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}}} \ll f$ and $\dfrac{eB}{2\pi m_\mathrm{e}} \ll f$ may be used.)

(5) If the electron number density has a fluctuation Δn_e over a section of length a along the propagation path, find the phase shift caused by this electron-density fluctuation for the electromagnetic wave of frequency f of the pulsar signal after it has passed through this section.

Problem 6

Antiparticles were first predicted by Dirac's theory. In 1932 Anderson, while studying cosmic rays, discovered the antiparticle of the electron — the positron. Since then antiprotons and other antiparticles have been discovered one after another.

Bombarding a stationary proton target with a high-energy proton beam of single-particle energy 6.8 GeV can produce antiprotons; the reaction is

$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{p} + \mathrm{p} + \underbrace{\mathrm{p} + \bar{\mathrm{p}}}_{\longrightarrow\ \pi^- + \cdots}$$

The annihilation of protons and antiprotons can produce π^- mesons, so the antiproton beam is accompanied by a large number of π^- mesons as well. Figure 6a is a schematic diagram of the experimental apparatus used to detect antiprotons. The antiproton $\bar{\mathrm{p}}$ and π^- meson beams pass in turn through the scintillation counters S_1, S_2 and S_3; S_1 and S_2 are a distance $l = 12\,\mathrm{m}$ apart. Between S_2 and S_3 are placed the Cherenkov counters C_1 and C_2. A Cherenkov counter determines the speed of a charged particle by detecting Cherenkov radiation — the electromagnetic radiation excited when a charged particle moves in a medium faster than the speed of light in that medium. C_1 registers only the faster π^- mesons and C_2 only the slower antiprotons. The role of S_3 in the experiment is to check whether the preceding counting result is genuine.

(1) In the reaction above, assume that the final-state protons and the antiproton have approximately the same speed. Find the time $t_{\bar{\mathrm{p}}}$ that the antiproton needs to travel from S_1 to S_2. If a π^- meson has the same kinetic energy as the antiproton, find the time t_{π^-} that the π^- meson needs to travel from S_1 to S_2.

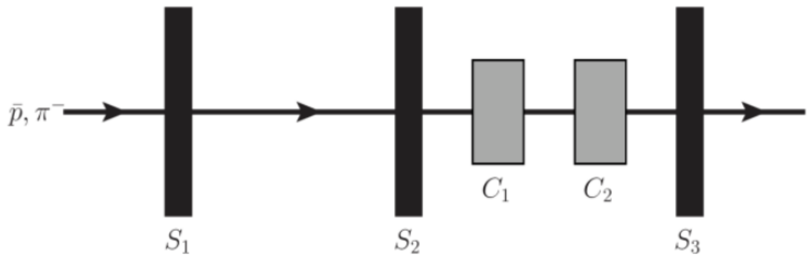

图 6a

Labels: 图 6a = *Fig. 6a.*

(2) A charged particle moving with speed v ($v > c/n$) passes through a medium of refractive index n. Find the angle θ between the propagation direction of the Cherenkov radiation produced and the direction of motion of the charged particle.

(3) A differential Cherenkov counter can register the number of particles whose speed lies in a certain interval. Figure 6b is a schematic section of it, taken so that the picture is rotationally symmetric about the axis (the dashed line in the figure). The light-collecting system consists of a spherical mirror of radius R and an annular aperture of adjustable radius. When the angle θ between the propagation direction of the Cherenkov radiation and the direction of motion of the charged particle is very small, the spherical mirror focuses the Cherenkov radiation excited by the charged particle onto its focal plane, forming a radiation light ring of radius r that is projected on the counter. Find the radius of the light ring for a charged particle of speed v.

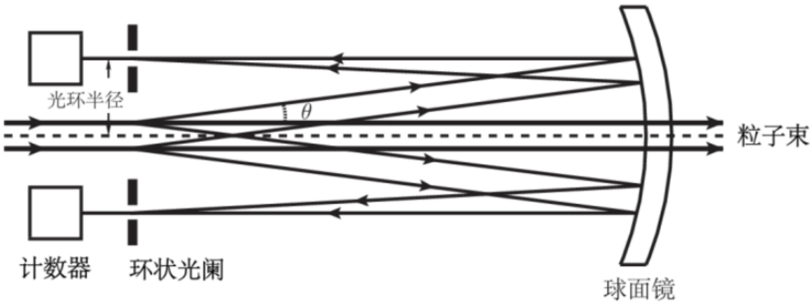

图 6b

Labels: 光环半径 = *radius of the light ring;* 计数器 = *counter;* 环状光阑 = *annular aperture;* 粒子束 = *particle beam;* 球面镜 = *spherical mirror;* 图 6b = *Fig. 6b.*

(4) When an antiproton meets a proton the annihilation reaction $\mathrm{p} + \bar{\mathrm{p}} \to 3\pi^0$ takes place. The difference between the total kinetic energy of the final-state particles and that of the initial-state particles is the reaction energy Q. For simplicity assume the kinetic energies of the proton and the antiproton are negligible. The total kinetic energy of the three final π^0 mesons is a constant, and the way this total kinetic energy is shared among the three π^0 mesons can be represented by a Dalitz plot. As shown in Fig. 6c, the Dalitz plot is an equilateral triangle of height 1; the distances from the point P to the three sides are equal to the ratios of the kinetic energies of the three π^0 mesons to the reaction energy, that is $d_i = E_{\mathrm{k},i}/Q$, where $E_{\mathrm{k},i}$ is the kinetic energy of the i-th π^0 meson, $i = 1, 2, 3$. Set up coordinates with the base as the X axis, the midpoint of the base as the origin, and the perpendicular bisector of the base as the Y axis. Find the expression for the boundary of the possible region of the point P, expressed in terms of the reaction energy Q, the π^0 meson mass m_{π^0} and the speed of light c in vacuum. Also discuss the expression for the boundary of the region of P when $m_{\pi^0} = 0$.

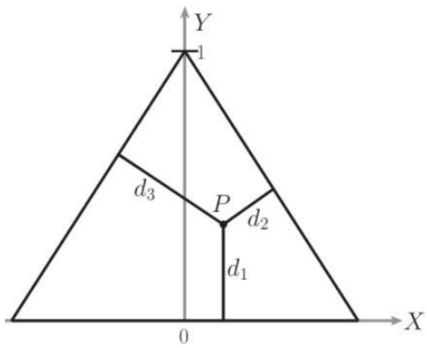

图 6c

Labels: 图 6c = *Fig. 6c.*

Given: the speed of light in vacuum $c = 2.998\times10^8$ m/s; the proton mass $M_\mathrm{p} = 938.2720\,\mathrm{MeV}/c^2$; the neutral π meson mass $m_{\pi^0} = 134.9766\,\mathrm{MeV}/c^2$; the charged π^+ meson mass $m_{\pi^+} = 139.5702\,\mathrm{MeV}/c^2$.

Problem 7

According to the “You Zuo” chapter of the *Xunzi*, Confucius visited the temple of Duke Huan of Lu, where there was a *qi* (欹, read qī) vessel. Of the *qi* vessel it is said: “empty, it tilts; half full, it stands upright; brimful, it topples." A real suspended *qi* vessel is shown in Fig. 7a: when the vessel is empty the body tilts; when the water poured in is moderate the body stands upright; when too much water is poured in the body overturns. Figure 7b is a front sectional view of the suspended *qi* vessel. The whole outer profile of the *qi* vessel is a surface of revolution about the Z axis; the Z' axis is the symmetry axis of the inner bore. The radii of the hemispherical shells of the inner wall and of the outer wall of the vessel are $R_1 = R$ and $R_2 = \sqrt{3}\,R$, and the centres O and O' of the two hemispheres both lie on the X axis. The Z' axis and the Z axis are a distance $t = 3R/10$ apart, and the length of the cylindrical part of the *qi* vessel is $l = 2R$. The straight line containing the pair of suspension points of the *qi* vessel meets the XOZ plane at the point Q, whose coordinates are $(x_Q = -R/10,\ z_Q = 2\sqrt{3}R/11)$. The material of the *qi* vessel is uniform and its density ρ_1 is 3 times the density ρ_2 of water. Neglect friction.

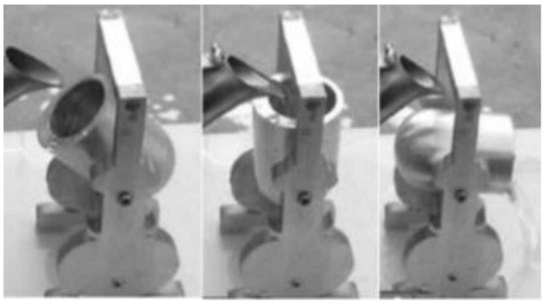

Fig. 7a

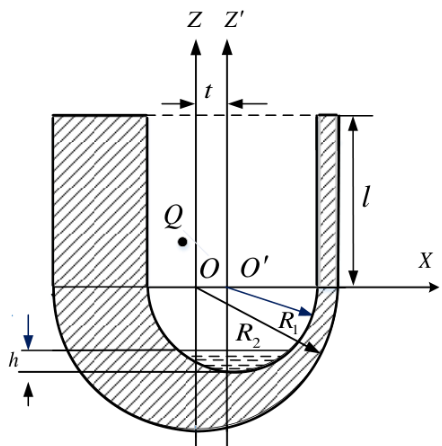

图 7b

Labels: 图 7b = *Fig. 7b.*

(1) Find the angle between the Z axis and the vertical when the empty *qi* vessel hangs freely in equilibrium.

(2) Find the moment of inertia of the empty *qi* vessel about the axis containing the pair of suspension points, and the angular frequency of small oscillations about its equilibrium position. (Given: the moment of inertia of a uniform sphere of density ρ and radius R about an axis through its centre is $I_1 = \frac{8}{15}\pi\rho R^5$; the moment of inertia of a uniform cylinder of radius R and length L about an axis through its centre parallel to the base of the cylinder is $I_2 = \dfrac{\pi\rho R^2 L(3R^2 + L^2)}{12}$.)

(3) Let the empty *qi* vessel hang freely and begin to pour water into it slowly. At what distance h of the water surface inside the vessel from the bottom does the body stand upright?

(4) State briefly the critical condition for the *qi* vessel to "topple when brimful".