The 37th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2020

Note. The official answer sheet prints no per-step mark values, so none are shown. Equation numbers reproduce the circled numbers of the source and restart within each problem. Passages that the original encloses in square brackets are alternative methods (解法（二） = “Method 2”); they are kept inside brackets here as well.

Problem 1 (Neutron collisions and moderation)

Solution

(1) A neutron with some initial velocity collides with a target nucleus at rest; just before the collision the angle between the direction of the initial velocity and the line joining the centres of the two spheres is α. In the plane determined by the line of centres and the initial velocity of the neutron, let the x axis lie along the line of centres and let the y axis be the direction perpendicular to the line of centres in that plane. Let the speed of the neutron after the collision be v, with velocity component v_x along x and v_y along y, and let the speed of the target nucleus after the collision be v_1. Momentum is conserved separately along x and along y:

$$\begin{cases} mv_0\cos\alpha = mv_x + m_1v_1, \\ mv_0\sin\alpha = mv_y, \end{cases} \tag{1}$$

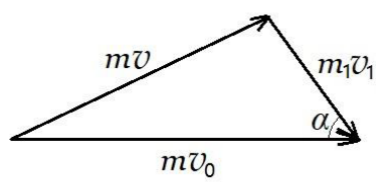
where

$$v_x^2 + v_y^2 = v^2.$$

[**Method 2.** From momentum conservation before and after the collision, the momentum $m\boldsymbol{v}_0$ of the neutron before the collision, the momentum $m\boldsymbol{v}$ of the neutron after the collision and the momentum $m_1\boldsymbol{v}_1$ of the nucleus after the collision form a closed vector triangle, as shown in solution Fig. 1a. By the law of cosines,

$$m^2v_0^2 + m_1^2v_1^2 - 2mm_1v_0v_1\cos\alpha = m^2v^2 \tag{1}$$

where α is the angle between the momentum of the neutron before the collision and the momentum of the nucleus after the collision.]



解题图 1a

Labels: 解题图 1a = *solution Fig. 1a.*

Energy conservation gives

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}m_1v_1^2 \tag{2}$$

From (1) and (2),

$$m_1v_1^2 + \frac{m_1^2}{m}v_1^2 - 2m_1v_0v_1\cos\alpha = 0,$$

whence

$$v_1 = \frac{2m\cos\alpha}{m+m_1}\,v_0, \tag{3}$$

and substituting (3) into (2),

$$v = \frac{\sqrt{(m+m_1)^2 - 4mm_1\cos^2\alpha}}{m+m_1}\,v_0 = \frac{\sqrt{m^2+m_1^2-2mm_1\cos 2\alpha}}{m+m_1}\,v_0. \tag{4}$$

From (3), v_1 reaches its maximum when $\alpha = 0$:

$$v_{1\,\mathrm{max}} = \frac{2m}{m+m_1}\,v_0,$$

so the maximum speed of the hydrogen nucleus is

$$v_\mathrm{H} = \frac{2m}{m+m_\mathrm{H}}\,v_0,$$

and the maximum speed of the nitrogen nucleus is

$$v_\mathrm{N} = \frac{2m}{m+14\,m_\mathrm{H}}\,v_0.$$

From these two relations,

$$m = \frac{14v_\mathrm{N} - v_\mathrm{H}}{v_\mathrm{H} - v_\mathrm{N}}\,m_\mathrm{H} = \frac{14\times 4.7\times 10^6 - 3.3\times 10^7}{3.3\times 10^7 - 4.7\times 10^6}\,m_\mathrm{H} = 1.16\,m_\mathrm{H}, \tag{5}$$

$$v_0 = \frac{m+m_\mathrm{H}}{2m}\,v_\mathrm{H} = 3.07\times 10^7\ \mathrm{m/s}. \tag{6}$$

(2) A nitrogen-14 nucleus of velocity V_i goes on to collide with the i-th neutron, of speed v_0. Under the condition that every collision gives the largest possible increase of speed, the velocity of the nitrogen-14 nucleus becomes V_{i+1} and the final velocity of the neutron is v_0'. Momentum and energy conservation give

$$\begin{cases} mv_0 + m_\mathrm{N}V_i = mv_0' + m_\mathrm{N}V_{i+1}, \\ \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}m_\mathrm{N}V_i^2 = \tfrac{1}{2}mv_0'^2 + \tfrac{1}{2}m_\mathrm{N}V_{i+1}^2 \end{cases} \tag{7}$$

From (7),

$$\frac{V_i}{v_0} = 1 - a + a\frac{V_{i-1}}{v_0} \tag{8}$$

where

$$a = \frac{m_\mathrm{N} - m}{m_\mathrm{N} + m} = 0.847. \tag{9}$$

Iterating (8) step by step,

$$\begin{aligned} \frac{V_n}{v_0} &= 1 - a + a\frac{V_{n-1}}{v_0} = 1 - a + a\left(1 - a + a\frac{V_{n-2}}{v_0}\right) = 1 - a^2 + a^2\frac{V_{n-2}}{v_0} \\ &= 1 - a^3 + a^3\frac{V_{n-3}}{v_0} = \cdots = 1 - a^n + a^n\frac{V_0}{v_0} = 1 - a^n. \end{aligned} \tag{10}$$

Here the condition $V_0 = 0$ given in the problem has been used.
The required number n satisfies

$$\tfrac{1}{2} m_{\mathrm{N}} V_n^2 = \tfrac{1}{2} \times 1.16\, m_{\mathrm{H}} v_0^2,$$

that is

$$\tfrac{1}{2}\, 14\, m_{\mathrm{H}} (1 - a^n)^2 v_0^2 = \tfrac{1}{2} \times 1.16\, m_{\mathrm{H}} v_0^2. \tag{11}$$

From (9) and (11), the closest value satisfying the equation is

$$n = 2. \tag{12}$$

(3) From the Maxwell speed distribution[a]

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \mathrm{e}^{-\frac{mv^2}{2k_{\mathrm{B}}T}} v^2$$

the condition for a maximum of the speed distribution is

$$\frac{\mathrm{d}f(v)}{\mathrm{d}v} = 0,$$

so the most probable speed is

$$v_{\mathrm{p}} = \sqrt{\frac{2k_{\mathrm{B}}T}{m}} \tag{13}$$

and the kinetic energy corresponding to the most probable speed is

$$E_{\mathrm{p}} = \tfrac{1}{2} m v_{\mathrm{P}}^2 = k_{\mathrm{B}}T. \tag{14}$$

Let the total number of particles be N. By the definition of the kinetic-energy distribution function,

$$f(E_{\mathrm{k}}) = \frac{\mathrm{d}N}{N\,\mathrm{d}E_{\mathrm{k}}} \tag{15}$$

By the definition of the speed distribution function,

$$f(v) = \frac{\mathrm{d}N}{N\,\mathrm{d}v}, \qquad \text{while} \qquad \mathrm{d}v = \frac{\mathrm{d}E_{\mathrm{k}}}{\sqrt{2mE_{\mathrm{k}}}},$$

so

$$f(E_{\mathrm{k}}) = \frac{\mathrm{d}N}{N\,\mathrm{d}E_{\mathrm{k}}} = \frac{1}{\sqrt{2mE_{\mathrm{k}}}}\, f\left(\sqrt{\frac{2E_{\mathrm{k}}}{m}}\right).$$

Combining the above,

$$f(E_{\mathrm{k}}) = \frac{2\pi}{(\pi k_{\mathrm{B}}T)^{3/2}}\, \mathrm{e}^{-\frac{E_{\mathrm{k}}}{k_{\mathrm{B}}T}} E_{\mathrm{k}}^{1/2} \tag{16}$$

The condition for the kinetic-energy distribution to be a maximum is $\dfrac{\mathrm{d}f(E_{\mathrm{k}})}{\mathrm{d}E_{\mathrm{k}}} = 0$, from which the most probable kinetic energy is

$$E_{\mathrm{kp}} = \tfrac{1}{2} k_{\mathrm{B}}T = \tfrac{1}{2} E_{\mathrm{p}}. \tag{17}$$

[a] *Translator's note:* The source prints kT inside the bracket and $k_{\mathrm{B}}T$ in the exponent; both denote $k_{\mathrm{B}}T$.

Problem 2 (Two conducting spheres and the image series)

Solution

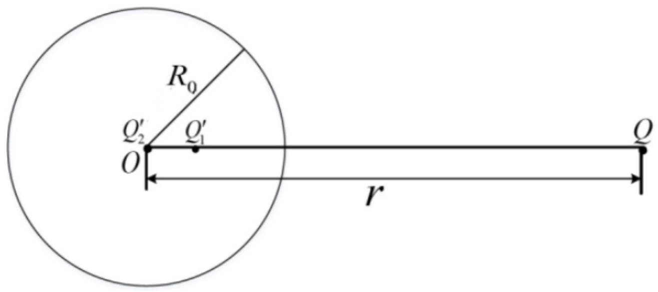
(1) As shown in solution Fig. 2a, take the centre of the sphere as the origin and the line through the centre of the sphere and the position of the external point charge as the X axis (the same below). The point charge Q outside the conducting sphere is at $x = r$ $(r > R_0)$; the charge outside the conducting sphere and its image charge lie on the X axis (the same below). From the spherical image formula for a grounded conducting sphere, the magnitude and the position of the image charge Q_1' of the point charge Q are

$$Q_1' = -\frac{R_0}{r}Q, \qquad x' = \frac{R_0^2}{r} \tag{1}$$

Because the charge on the ungrounded conducting sphere is conserved and the outer surface of the conducting sphere is an equipotential, there is a further image charge Q_2' at the centre of the sphere, of magnitude and position

$$Q_2' = -Q_1' = \frac{R_0}{r}Q, \qquad x_2' = 0 \tag{2}$$

The magnitudes and the position coordinates of the image charges inside the conducting sphere and of the charges outside it (image dipole moments included) are distinguished by a prime and by the absence of a prime respectively (the same below, unless stated otherwise).



解题图 2a

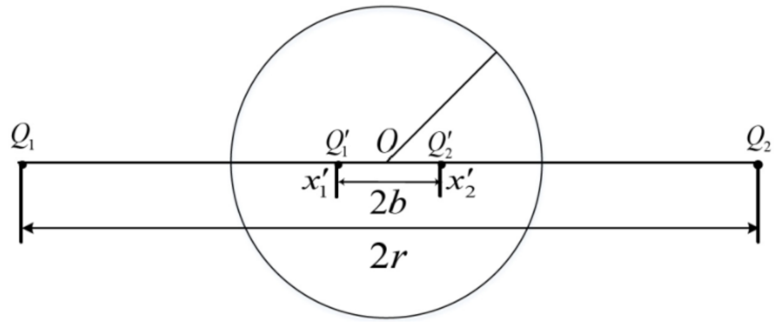
Labels: 解题图 2a = *solution Fig. 2a.*

(2) Suppose that on the X axis there are equal and opposite point charges $Q_1 = Q$ $(Q > 0)$ and $Q_2 = -Q$, each at distance r $(r > R_0)$ from the centre O of the metal (conducting) sphere, as shown in solution Fig. 2b. From the spherical image formula, the magnitudes and the positions of the image charges Q_1' and Q_2' of Q_1 and Q_2 are

$$\begin{cases} Q_1' = -\dfrac{R_0}{r}Q_1 = -\dfrac{R_0}{r}Q, & x_1' = -b \equiv -\dfrac{R_0^2}{r}; \\ Q_2' = -\dfrac{R_0}{r}Q_2 = \dfrac{R_0}{r}Q, & x_2' = b \equiv \dfrac{R_0^2}{r}. \end{cases}$$

The sum of the image charges Q_1' and Q_2' is zero, which keeps the conducting sphere electrically neutral. When $r \to \infty$, $b \to 0$; the pair of image charges Q_1' and Q_2' may be regarded as a point dipole located at the centre O of the sphere, whose dipole moment p' has magnitude[a]

$$p' = Q_2'(x_2' - x_1') = 2Q_1'b = 2\frac{R_0^3}{r^2}Q \tag{2}$$



解题图 2b

Labels: 解题图 2b = *solution Fig. 2b.*

The uniform external field E_0 may be regarded as the field produced at the midpoint of the line joining equal and opposite point charges $\pm Q$, so

$$E_0 = 2\,\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \tag{3}$$

and hence

$$p' = 4\pi\varepsilon_0 E_0 R_0^3 \tag{4}$$

(3) By the statement of the problem, the point charges outside the sphere, $q_1 = -q$ $(q > 0)$ and $q_2 = q$, are at $x_1 = r - \frac{\Delta l}{2}$ and $x_2 = r + \frac{\Delta l}{2}$ respectively, as shown in solution Fig. 2c. From (1), the magnitudes and the positions of the image charges q_1' and q_2' of q_1 and q_2 are

$$q_1' = \frac{qR_0}{r - \dfrac{\Delta l}{2}} > 0, \qquad x_1' = \frac{R_0^2}{r - \dfrac{\Delta l}{2}} \tag{5}$$

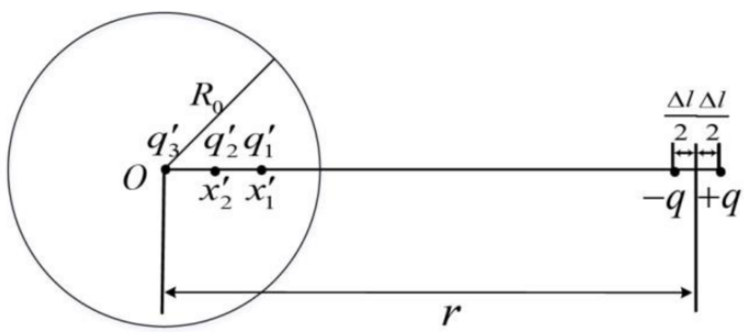
$$q_2' = -\frac{qR_0}{r + \dfrac{\Delta l}{2}} < 0, \qquad x_2' = \frac{R_0^2}{r + \dfrac{\Delta l}{2}} \tag{6}$$

Because the charge on the conducting sphere is conserved and its outer surface is an equipotential, there is a further image charge Q' at the centre of the sphere, whose magnitude and position are[b]

$$p_2 = \frac{R_0^3}{r_1^3}p_1, \quad b_2 = \frac{R_0^2}{r_1}, \quad r_2 = r - b_2; \quad Q_1 = -q_1^{(2)} < 0, \quad q_1^{(2)} = \frac{R_0}{r_1^2}p_1 > 0$$

$$Q' = -(q_1' + q_2') = -\left(\frac{qR_0}{r - \dfrac{\Delta l}{2}} + \frac{-qR_0}{r + \dfrac{\Delta l}{2}}\right) = -\frac{R_0 q\,\Delta l}{r^2 - \left(\dfrac{\Delta l}{2}\right)^2} = -\frac{R_0}{r^2}q\,\Delta l < 0, \qquad x_3' = 0 \tag{7}$$

Here the condition $\Delta l \ll r$ has been used (the same below).



解题图 2c

Labels: 解题图 2c *= solution Fig. 2c.*

The system of image charges made up of q_1' and q_2' can be regarded as an image point charge located at

$$x_0' = b \equiv \frac{R_0^2}{r}$$

of charge

$$-Q' = q_1' + q_2' = \frac{qR_0}{r - \dfrac{\Delta l}{2}} + \frac{-qR_0}{r + \dfrac{\Delta l}{2}} = \frac{R_0}{r^2} q\,\Delta l = -q_3' > 0$$

together with an image point dipole, whose dipole moment has magnitude

$$\begin{aligned} p' &= q_1'(x_1' - x_0') + q_2'(x_2' - x_0') \\ &= \frac{R_0^3}{r^2} \frac{q}{1 - \dfrac{\Delta l}{2r}} \left[\left(\frac{1}{1 - \dfrac{\Delta l}{2r}} - 1 \right) - \left(\frac{1}{1 + \dfrac{\Delta l}{2r}} - 1 \right) \right] = \frac{R_0^3}{r^3} q\,\Delta l \end{aligned}$$

directed along the $+X$ direction, that is, the same as the direction of the dipole moment outside the sphere.

In addition, the system of the pair of point charges $Q' < 0$ and $-Q' > 0$ has a non-vanishing point dipole moment (and higher moments as well), of magnitude

$$p'' = (-Q')(x_0' - 0) = \frac{R_0}{r^2} q\,\Delta l\, \frac{R_0^2}{r} = \frac{R_0^3}{r^3} q\,\Delta l$$

directed along the $+X$ direction, that is, the same as the direction of the dipole moment outside the sphere.

Taken together, the magnitude of the image dipole moment of the dipole outside the sphere is

$$p_{\text{tot}}' = p' + p'' = 2\frac{R_0^3}{r^3} q\,\Delta l \tag{8}$$

directed along the $+X$ direction, that is, the same as the direction of the dipole moment outside the sphere.

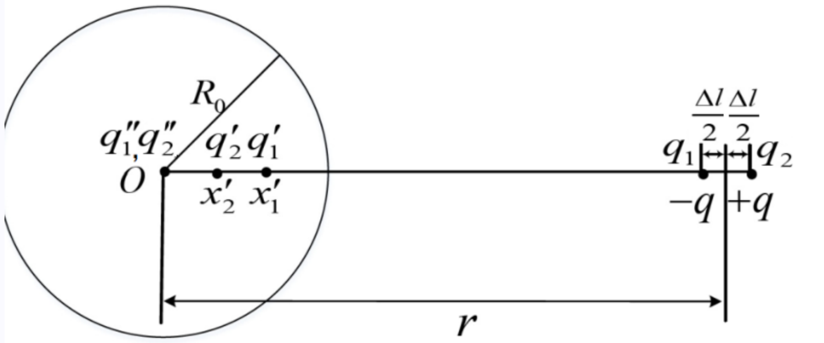
[**Method 2.** Let the dipole outside the sphere consist of the point charge $q_1 = -q$ $(q > 0)$ at $x_1 = r - \frac{\Delta l}{2}$ and the point charge $q_2 = q$ at $x_2 = r + \frac{\Delta l}{2}$. Then q_1 has **two** image charges, namely

$$q_1' = \frac{qR_0}{r - \dfrac{\Delta l}{2}}\ (> 0), \quad x_1' = \frac{R_0^2}{r - \dfrac{\Delta l}{2}}; \qquad q_1'' = -\frac{qR_0}{r - \dfrac{\Delta l}{2}}\ (< 0), \quad x_1'' = 0. \tag{5}$$

Similarly q_2 also has **two** image charges, namely

$$q_2' = -\frac{qR_0}{r+\dfrac{\Delta l}{2}}\ (<0),\quad x_2' = \frac{R_0^2}{r+\dfrac{\Delta l}{2}};\qquad q_2'' = \frac{qR_0}{r+\dfrac{\Delta l}{2}}\ (>0),\quad x_2''=0. \tag{6}$$

The dipole outside the sphere and its image charges are shown in solution Fig. 2d.



解题图 2d

Labels: 解题图 2d = *solution Fig. 2d.*

The total dipole moment of the charge system formed by the four image charges q_1', q_1'', q_2', q_2'' is

$$p_{\text{tot}}' = q_1'x_1' + q_1''x_1'' + q_2'x_2' + q_2''x_2'' \tag{7}$$

that is

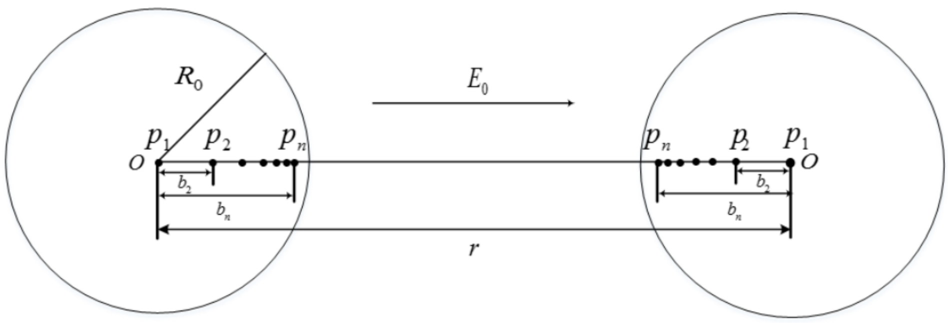
$$p_{\text{tot}}' = \frac{qR_0^3}{[\,r-(\Delta l/2)\,]^2} - \frac{qR_0^3}{[\,r+(\Delta l/2)\,]^2} = 2\frac{R_0^3}{r^3}q\,\Delta l \tag{8}$$

and its direction is the same as the direction of the dipole moment outside the sphere.]

(4) From here on the prime that marks the magnitudes and the position coordinates of the image charges inside the conducting sphere (image dipole moments included) is dropped. As the figure shows, when there are two identical conducting spheres along the direction of the field in space, the constant external field produces at the positions of the two sphere centres a pair of point dipoles

$$p_1 = 4\pi\varepsilon_0 E_0 R_0^3 \tag{9}$$

This pair of image dipoles in turn produces secondary dipoles p_2, together with a pair of equal positive and negative point charges located at the position of the dipole p_2 and at the sphere centre; these then produce p_3, p_4, ... together with the image charges (isolated point charges) of the separated equal positive and negative point charges, and so on, an infinite chain of image mappings (see solution Fig. 2e, in which the isolated image point charges are not drawn). From the analysis of part (3), the dipole moment of each pair of point dipoles, the distance of the position of that dipole from the two sphere centres, and the magnitudes of the isolated point charges are, in turn,



解题图 2e

Labels: 解题图 2e = *solution Fig. 2e.*

$$p_1 = 4\pi\varepsilon_0 E_0 R_0^3, \quad b_1 = 0, \quad r_1 = r - b_1 = r \tag{10}$$

$$p_2 = \frac{R_0^3}{r_1^3} p_1, \quad b_2 = \frac{R_0^2}{r_1}, \quad r_2 = r - b_2; \quad Q_1 = -q_1^{(2)} < 0, \quad q_1^{(2)} = \frac{R_0}{r_1^2} p_1 > 0 \tag{11}$$

$$\begin{gathered} p_3 = \frac{R_0^3}{r_2^3} p_2, \quad b_3 = \frac{R_0^2}{r_2}, \quad r_3 = r - b_3; \\ Q_2 = -(q_2^{(2)} + q_2^{(3)}), \quad q_2^{(2)} = \frac{R_0}{r_2} Q_1 < 0, \quad q_2^{(3)} = \frac{R_0}{r_2}\left(\frac{p_2}{r_2} + q_1^{(2)}\right) > 0 \end{gathered} \tag{12}$$

$$p_4 = \frac{R_0^3}{r_3^3} p_3, \quad b_4 = \frac{R_0^2}{r_3}, \quad r_4 = r - b_4;$$

$$Q_3 = -(q_3^{(2)} + q_3^{(3)} + q_3^{(4)}), \quad q_3^{(2)} = \frac{R_0}{r_2} Q_2, \quad q_3^{(3)} = \frac{R_0}{r_3} q_2^{(2)}, \quad q_3^{(4)} = \frac{R_0}{r_3}\left(\frac{p_3}{r_3} + q_2^{(3)}\right) > 0$$

$$\vdots$$

$$\begin{gathered} p_n = \frac{R_0^3}{r_{n-1}^3} p_{n-1}, \quad b_n = \frac{R_0^2}{r_{n-1}}, \quad r_n = r - b_n; \\ Q_{n-1} = -\sum_{i=2}^{n} q_{n-1}^{(i)}, \quad q_{n-1}^{(2)} = \frac{R_0}{r_2} Q_{n-2} < 0, \quad q_{n-1}^{(3)} = \frac{R_0}{r_3} q_{n-2}^{(2)} < 0, \ \cdots, \\ q_{n-1}^{(n-1)} = \frac{R_0}{r_{n-2}} q_{n-2}^{(n-2)} < 0, \quad q_{n-1}^{(n)} = \frac{R_0}{r_{n-1}}\left(\frac{p_{n-1}}{r_{n-1}} + q_{n-2}^{(n-1)}\right) > 0 \end{gathered} \tag{13}$$

$$\vdots$$

Here $q_n^{(i)}$ is the amount of image point charge at the position b_i (that is, at that distance from the sphere centre) inside the left-hand sphere after the n-th image mapping, and Q_i is the compensating charge at the sphere centre (which guarantees the electrical neutrality of the conducting sphere and the equipotentiality of its outer surface).
The potential of a dipole of dipole moment $\boldsymbol{p}$ located at $\boldsymbol{b}_0$ is

$$\varphi(\boldsymbol{r}) = \frac{\boldsymbol{p} \cdot (\boldsymbol{r} - \boldsymbol{b}_0)}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{b}_0|^3}$$

Take the midpoint of the line joining the two spheres as the origin, the x axis horizontally to the right, and the origin as the zero of potential. Then the potential at an arbitrary point $P(x,y)$ of the space outside the spheres is

$$\begin{aligned}\varphi(x,y) =& -E_0x+\sum_{n=1}^{\infty}\frac{\left(x+\frac{r}{2}-b_n\right)p_n}{4\pi\varepsilon_0\left[\left(x+\frac{r}{2}-b_n\right)^2+y^2\right]^{3/2}}+\sum_{n=1}^{\infty}\frac{\left(x-\frac{r}{2}+b_n\right)p_n}{4\pi\varepsilon_0\left[\left(x-\frac{r}{2}+b_n\right)^2+y^2\right]^{3/2}}\\ &+\sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{4\pi\varepsilon_0\left[\left(x+\frac{r}{2}-b_i\right)^2+y^2\right]^{1/2}}+\sum_{n=2}^{\infty}\frac{Q_{n-1}}{4\pi\varepsilon_0\left[\left(x+\frac{r}{2}\right)^2+y^2\right]^{1/2}}\\ &+\sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{4\pi\varepsilon_0\left[\left(x-\frac{r}{2}+b_i\right)^2+y^2\right]^{1/2}}+\sum_{n=2}^{\infty}\frac{Q_{n-1}}{4\pi\varepsilon_0\left[\left(x-\frac{r}{2}\right)^2+y^2\right]^{1/2}}\end{aligned} \tag{14}$$

When $y=0$, the potential distribution on the line of centres (outside the spheres) is

$$\begin{aligned}\varphi(x,y=0) =& -E_0x+\sum_{n=1}^{\infty}\frac{\left(x+\frac{r}{2}-b_n\right)p_n}{4\pi\varepsilon_0\left|x+\frac{r}{2}-b_n\right|^3}+\sum_{n=1}^{\infty}\frac{\left(x-\frac{r}{2}+b_n\right)p_n}{4\pi\varepsilon_0\left|x-\frac{r}{2}+b_n\right|^3}\\ &+\sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{4\pi\varepsilon_0\left|x+\frac{r}{2}-b_i\right|}+\sum_{n=2}^{\infty}\frac{Q_{n-1}}{4\pi\varepsilon_0\left|x-\frac{r}{2}\right|}\\ &+\sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{4\pi\varepsilon_0\left|x+\frac{r}{2}-b_i\right|}+\sum_{n=2}^{\infty}\frac{Q_{n-1}}{4\pi\varepsilon_0\left|x-\frac{r}{2}\right|}\end{aligned} \tag{15}$$

and the field distribution on the line of centres (outside the spheres) is

$$\begin{aligned}E(x,y=0) =& -\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x,y=0)=E_0+\sum_{n=1}^{\infty}\frac{p_n}{2\pi\varepsilon_0\left|x+\frac{r}{2}-b_n\right|^3}+\sum_{n=1}^{\infty}\frac{p_n}{2\pi\varepsilon_0\left|x-\frac{r}{2}+b_n\right|^3}\\ &+\sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{4\pi\varepsilon_0\left|x+\frac{r}{2}-b_i\right|^2}+\sum_{n=2}^{\infty}\frac{Q_{n-1}}{4\pi\varepsilon_0\left|x-\frac{r}{2}\right|^2}\\ &+\sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{4\pi\varepsilon_0\left|x+\frac{r}{2}-b_i\right|^2}+\sum_{n=2}^{\infty}\frac{Q_{n-1}}{4\pi\varepsilon_0\left|x-\frac{r}{2}\right|^2}\end{aligned} \tag{16}$$

Here the direction of the dipoles formed by the image charges is the same as the direction of the uniform field applied outside the conducting spheres.

(5) Let $r\to 2R_0$, the two spheres remaining insulated from each other. From (10), (11) and (12),

$$p_2=\left(\frac{1}{2}\right)^3p_1,\qquad b_2=\frac{1}{2}R_0,\qquad r_2=\frac{3}{2}R_0 \tag{17}$$

$$p_3=\left(\frac{2}{3}\right)^3\left(\frac{1}{2}\right)^3p_1=\left(\frac{1}{3}\right)^3p_1,\qquad b_3=\frac{2}{3}R_0,\qquad r_3=\frac{4}{3}R_0 \tag{18}$$

$$\vdots$$

$$p_n = \left(\frac{1}{n}\right)^3 p_1, \qquad b_n = \frac{n-1}{n}R_0, \qquad r_n = \frac{n+1}{n}R_0; \tag{19}$$

$$\vdots$$

From (16),

$$\begin{aligned}
E(x=0,y=0) &= E_0 + \sum_{n=1}^{\infty}\frac{p_n}{\pi\varepsilon_0\left|\frac{r}{2}-b_n\right|^3} + \sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{2\pi\varepsilon_0\left|\frac{r}{2}-b_i\right|^2} + \sum_{n=2}^{\infty}\frac{Q_n}{2\pi\varepsilon_0\left|\frac{r}{2}\right|^2}\\
&= E_0 + \sum_{n=1}^{\infty}\frac{p_n}{\pi\varepsilon_0\left|\frac{r}{2}-b_n\right|^3} + \sum_{n=2}^{\infty}\sum_{i=2}^{n}\frac{q_{n-1}^{(i)}}{2\pi\varepsilon_0\left|\frac{r}{2}-b_i\right|^2} + \sum_{n=2}^{\infty}\frac{-\sum_{i=2}^{n}q_{n-1}^{(i)}}{2\pi\varepsilon_0\left|\frac{r}{2}\right|^2}\\
&> E_0 + \sum_{n=1}^{\infty}\frac{p_n}{\pi\varepsilon_0\left|\frac{r}{2}-b_n\right|^3}\\
&= E_0 + E_0\sum_{n=1}^{\infty}\frac{4}{n^3\left|1-\frac{n-1}{n}\right|^3} = E_0 + 4E_0\sum_{n=1}^{\infty}1 \longrightarrow \infty
\end{aligned} \tag{20}$$

Here the inequality

$$-Q_{n-1} \equiv \sum_{i=2}^{n} q_{n-1}^{(i)} > 0$$

has been used. It can be proved by a direct calculation as follows:

$$\begin{aligned}
-Q_{n-1} \equiv \sum_{i=2}^{n} q_{n-1}^{(i)} &= \frac{R_0}{r_2}Q_{n-2} + \frac{R_0}{r_3}q_{n-2}^{(2)} + \cdots + \frac{R_0}{r_{n-2}}q_{n-2}^{(n-1)} + \frac{R_0}{r_{n-1}}q_{n-2}^{(n)}\\
&= \frac{R_0}{r_2}Q_{n-2} + \frac{R_0}{r_3}q_{n-2}^{(2)} + \cdots + \frac{R_0}{r_{n-2}}q_{n-2}^{(n-1)} + \left[\frac{R_0}{r_{n-1}}\left(\frac{p_{n-1}}{r_{n-1}} + q_{n-2}^{(n-1)}\right)\right]\\
&= \frac{R_0}{r_2}Q_{n-2} + \frac{R_0}{r_3}q_{n-2}^{(2)} + \cdots + \frac{R_0}{r_{n-2}}q_{n-2}^{(n-2)} + \left[\frac{R_0}{r_{n-1}^2}p_{n-1} + \frac{R_0}{r_{n-1}}\left(\frac{p_{n-2}}{r_{n-2}} + q_{n-3}^{(n-2)}\right)\right]\\
&= \frac{R_0}{r_2}Q_{n-2} + \frac{R_0}{r_3}q_{n-2}^{(2)} + \cdots + \frac{R_0}{r_{n-2}}q_{n-2}^{(n-2)} + \frac{R_0}{r_{n-1}}q_{n-2}^{(n-1)} + \frac{R_0}{r_{n-1}^2}p_{n-1}\\
&> \frac{R_0}{r_{n-1}}\left[Q_{n-2} + \sum_{i=2}^{n-1}q_{n-2}^{(i)}\right] + \frac{R_0}{r_{n-1}^2}p_{n-1} = \frac{R_0}{r_{n-1}^2}p_{n-1} > 0
\end{aligned}$$

[a] *Translator's note:* Equation (2) is printed as shown. Its middle member should read $2Q_2'b$; with $Q_2' = (R_0/r)Q$ and $x_2' - x_1' = 2b = 2R_0^2/r$ the final result $2R_0^3Q/r^2$ is correct. Note also that the source labels two different equations of this problem with the number (2).

[b] *Translator's note:* The display that follows is stray text in the source: it duplicates equation (11) of part (4) and has nothing to do with part (3). It is reproduced here as printed; the sentence continues with equation (7) below.

Problem 3 (Quantum Otto and quantum Carnot engines)

Solution

(1) The probability that the two-level system has energy E_1 obeys the Boltzmann distribution $p \propto e^{-E/k_\mathrm{B}T}$, so

$$p_1 = p_0\, e^{-\frac{E_1}{k_\mathrm{B}T}} \tag{1}$$

The total probability of the atom being in the different levels is 1, that is

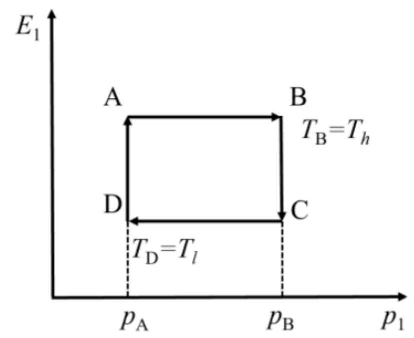
$$p_1 + p_0 = 1 \tag{2}$$

From (1) and (2),

$$p_0 = \frac{1}{1 + e^{-\frac{E_1}{k_\mathrm{B}T}}} \tag{3}$$

$$p_1 = \frac{e^{-\frac{E_1}{k_\mathrm{B}T}}}{1 + e^{-\frac{E_1}{k_\mathrm{B}T}}} \tag{4}$$

(2) The schematic diagram of the quantum Otto cycle is shown in solution Fig. 3a. Below we compute the increments of the various physical quantities in each process of the quantum Otto engine cycle.



解题图 3a

Labels: 解题图 3a = *solution Fig. 3a.*

A→B (quantum isochoric) process: no work is done, i.e. $E_\mathrm{A} = E_\mathrm{B}$; the heat absorbed goes entirely into increasing the internal energy, so the heat absorbed is

$$\Delta_1\langle E\rangle = Q_1 = E_\mathrm{B}\,(p_\mathrm{B} - p_\mathrm{A}) \tag{5}$$

B→C (quantum adiabatic) process: no heat is absorbed or released, so $p_\mathrm{B} = p_\mathrm{C}$, and the increment of internal energy is

$$\Delta_2\langle E\rangle = (E_\mathrm{C} - E_\mathrm{B})\,p_\mathrm{B} \tag{6}$$

The work done on the surroundings is

$$W_2 = -\Delta_2\langle E\rangle = -(E_\mathrm{C} - E_\mathrm{B})p_\mathrm{B} = (E_\mathrm{B} - E_\mathrm{C})p_\mathrm{B} \tag{7}$$

C→D (quantum isochoric) process: no work is done, i.e. $E_\mathrm{C} = E_\mathrm{D}$; the heat released comes from the decrease of the internal energy, and the increment of internal energy is

$$\Delta_3\langle E\rangle = E_\mathrm{D}\,(p_\mathrm{A} - p_\mathrm{B}) \tag{8}$$

The heat released is

$$Q_2 = -\Delta_3\langle E\rangle = E_\mathrm{C}\,(p_\mathrm{B} - p_\mathrm{A}) \tag{9}$$

D→A (quantum adiabatic) process: no heat is absorbed or released; the increment of internal energy is

$$\Delta_4\langle E\rangle = (E_\mathrm{A} - E_\mathrm{D})\, p_\mathrm{A} \tag{10}$$

The work done on the surroundings is

$$W_4 = -\Delta_4\langle E\rangle = -(E_\mathrm{A} - E_\mathrm{D})p_\mathrm{A} = (E_\mathrm{D} - E_\mathrm{A})p_\mathrm{A} \tag{11}$$

(3) For the quantum Otto engine, let the heat absorbed in the cycle be Q_1 and the heat released be Q_2; the efficiency is

$$\eta_\mathrm{Otto} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} \tag{12}$$

Substituting (5) and (9) into (12),

$$\eta_\mathrm{Otto} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{E_\mathrm{C}}{E_\mathrm{B}} \tag{13}$$

Since $p_\mathrm{B} = p_\mathrm{C}$ in the B→C process,

$$\frac{e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_\mathrm{B}}}}{1 + e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_\mathrm{B}}}} = \frac{e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_\mathrm{C}}}}{1 + e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_\mathrm{C}}}} \tag{14}$$

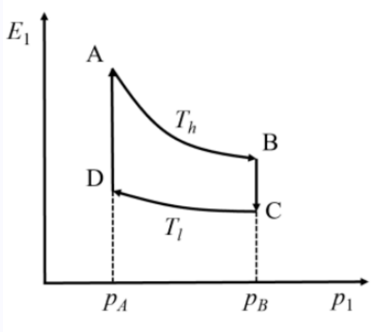
After simplification, and using $T_\mathrm{B} = T_h$,

$$\frac{E_\mathrm{B}}{T_h} = \frac{E_\mathrm{C}}{T_\mathrm{C}} \tag{15}$$

From (13) and (15),

$$\eta_\mathrm{Otto} = 1 - \frac{E_l}{E_h} = 1 - \frac{T_\mathrm{C}}{T_h} \tag{16}$$

The schematic diagram of the quantum Carnot cycle is shown in solution Fig. 3b. Below we compute the increments of the various physical quantities in each process of the quantum Carnot engine cycle.



解题图 3b

Labels: 解题图 3b = *solution Fig. 3b.*

A→B process:

$$\mathrm{d}Q_1 = E_1 \mathrm{d}p_1 = E_1\,\mathrm{d}\left(\frac{e^{-\frac{E_1}{k_\mathrm{B}T_h}}}{1 + e^{-\frac{E_1}{k_\mathrm{B}T_h}}}\right)$$

$$= E_1\,\frac{-\dfrac{1}{k_\mathrm{B}T_h}e^{-\frac{E_1}{k_\mathrm{B}T_h}}\left(1 + e^{-\frac{E_1}{k_\mathrm{B}T_h}}\right) - e^{-\frac{E_1}{k_\mathrm{B}T_h}}\left(-\dfrac{1}{k_\mathrm{B}T_h}e^{-\frac{E_1}{k_\mathrm{B}T_h}}\right)}{\left(1 + e^{-\frac{E_1}{k_\mathrm{B}T_h}}\right)^2}\,\mathrm{d}E_1 = \frac{-\dfrac{E_1}{k_\mathrm{B}T_h}e^{-\frac{E_1}{k_\mathrm{B}T_h}}}{\left(1 + e^{-\frac{E_1}{k_\mathrm{B}T_h}}\right)^2}\,\mathrm{d}E_1$$

The heat absorbed in this process is

$$
\begin{aligned}
Q_1 &= \int_{E_\mathrm{A}}^{E_\mathrm{B}} \frac{-\frac{E_1}{k_\mathrm{B}T_h} e^{-\frac{E_1}{k_\mathrm{B}T_h}}}{\left(1+e^{-\frac{E_1}{k_\mathrm{B}T_h}}\right)^2} \mathrm{d}E_1 = \int_{E_\mathrm{A}}^{E_\mathrm{B}} \frac{E_1}{\left(1+e^{-\frac{E_1}{k_\mathrm{B}T_h}}\right)^2} \mathrm{d}e^{-\frac{E_1}{k_\mathrm{B}T_h}} = \int_{E_\mathrm{A}}^{E_\mathrm{B}} E_1 \,\mathrm{d}\left(-\frac{1}{1+e^{-\frac{E_1}{k_\mathrm{B}T_h}}}\right) \\
&= \frac{E_\mathrm{A}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} - \frac{E_\mathrm{B}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} + \int_{E_\mathrm{A}}^{E_\mathrm{B}} \frac{1}{1+e^{-\frac{E_1}{k_\mathrm{B}T_h}}} \mathrm{d}E_1 \\
&= \frac{E_\mathrm{A}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} - \frac{E_\mathrm{B}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} + \int_{E_\mathrm{A}}^{E_\mathrm{B}} \frac{k_\mathrm{B}T_h \,\mathrm{d}e^{\frac{E_1}{k_\mathrm{B}T_h}}}{1+e^{\frac{E_1}{k_\mathrm{B}T_h}}} \\
&= \frac{E_\mathrm{A}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} - \frac{E_\mathrm{B}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} + k_\mathrm{B}T_h \ln \frac{1+e^{\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}}
\end{aligned} \tag{17}
$$

The increment of internal energy is

$$
\Delta E_1 = \frac{e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} E_\mathrm{B} - \frac{e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} E_\mathrm{A} \tag{18}
$$

The work done on the surroundings is

$$
W_1 = Q_1 - \Delta E = E_\mathrm{A} - E_\mathrm{B} + k_\mathrm{B}T_h \ln \frac{1+e^{\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} \tag{19}
$$

B→C process: in this process the probability distribution stays unchanged, $p_\mathrm{B} = p_\mathrm{C}$, and no heat is transferred, so

$$
\frac{e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} = \frac{e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_l}}}{1+e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_l}}}
$$

which after simplification gives

$$
E_\mathrm{C} = \frac{T_l}{T_h} E_\mathrm{B} \tag{20}
$$

Similarly $p_\mathrm{D} = p_\mathrm{A}$, so no heat is absorbed or released in the D→A process:

$$
\frac{e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} = \frac{e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}}{1+e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}}
$$

which after simplification gives

$$
E_\mathrm{D} = \frac{T_l}{T_h} E_\mathrm{A} \tag{21}
$$

The heat absorbed is

$$
Q_2 = 0 \tag{22}
$$

and the work done on the surroundings equals the decrease of internal energy,

$$
W_2 = -\Delta E_2 = \frac{e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} E_\mathrm{B} - \frac{e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_\mathrm{C}}}}{1+e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_h}}} E_\mathrm{C} \tag{23}
$$

C→D process: the process C→D is similar to A→B, both being isothermal; the heat released can be computed by analogy with (17), and after substituting (20) and (21) the heat released is

$$\begin{aligned} Q_3 &= \frac{E_\mathrm{D}}{1+e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}} - \frac{E_\mathrm{C}}{1+e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_l}}} + k_\mathrm{B}T_l \ln \frac{1+e^{\frac{E_\mathrm{C}}{k_\mathrm{B}T_l}}}{1+e^{\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}} \\ &= \frac{\frac{T_l}{T_h}E_\mathrm{A}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} - \frac{\frac{T_l}{T_h}E_\mathrm{B}}{1+e^{-\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}} + k_\mathrm{B}T_l \ln \frac{1+e^{\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} \end{aligned} \tag{24}$$

The increment of internal energy is

$$\Delta E_3 = \frac{e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}}{1+e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}}E_\mathrm{D} - \frac{e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_l}}}{1+e^{-\frac{E_\mathrm{C}}{k_\mathrm{B}T_l}}}E_\mathrm{C} \tag{25}$$

and the work done on the surroundings is

$$W_3 = Q_3 - \Delta E_3 = E_\mathrm{C} - E_\mathrm{D} + k_\mathrm{B}T_l \ln \frac{1+e^{\frac{E_\mathrm{B}}{k_\mathrm{B}T_h}}}{1+e^{\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}} \tag{26}$$

D→A process: the heat absorbed is

$$Q_4 = 0 \tag{27}$$

and the work done on the surroundings equals the decrease of internal energy,

$$W_4 = -\Delta E_4 = \frac{e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}}{1+e^{-\frac{E_\mathrm{D}}{k_\mathrm{B}T_l}}}E_\mathrm{D} - \frac{e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}}{1+e^{-\frac{E_\mathrm{A}}{k_\mathrm{B}T_h}}}E_\mathrm{A} \tag{28}$$

For the quantum Carnot engine, the heat absorbed in the cycle is Q_1 and the heat released is Q_3; substituting (17) and (24) into the efficiency formula,[a]

$$\eta_\mathrm{Carnot} \equiv \frac{Q_1 - Q_4}{Q_1} = 1 - \frac{Q_4}{Q_1} = 1 - \frac{T_l}{T_h} \tag{29}$$

This agrees with the efficiency of the classical Carnot engine. However $T_l < T_\mathrm{C} < T_h$, so from (16) and (29)

$$\eta_\mathrm{Otto} < \eta_\mathrm{Carnot} \tag{30}$$

Therefore the efficiency of the quantum Otto engine is lower than that of the quantum Carnot engine.

[a] *Translator's note:* Equation (29) is printed as shown; the subscript should be 3, not 4, since the ratio actually evaluated is $Q_3/Q_1 = T_l/T_h$ while $Q_4 = 0$ by (27).

Problem 4 (Laser Doppler velocimetry)

Solution

(1) A light wave of frequency f is scattered by a moving particle in the fluid. The magnitude of the propagation speed of the light wave in the fluid is c, and the speed of the particle is v ($v \ll c$).[a] Because of the Doppler effect the frequency of the light received by the particle is

$$f' = f\left(1 + \frac{vn}{c}\cos\theta_1\right) \tag{1}$$

where θ_1 is the angle between the direction of incidence of the light wave and the velocity of the scattering particle. Again because of the Doppler effect, the frequency of the scattered light from the particle detected in the scattering direction is

$$f'' = \frac{f'}{1 - \dfrac{vn}{c}\cos\theta_2} \tag{2}$$

where θ_2 is the angle between the propagation direction of the scattered light wave and the velocity of the scattering particle.
The frequency difference between the scattered light and the incident light is

$$\Delta f = f'' - f = \frac{fvn}{c}(\cos\theta_1 + \cos\theta_2) \tag{3}$$

Given the particle speed $v = 1\,\mathrm{m/s}$ and a light frequency of order $10^{14}\,\mathrm{Hz}$,

$$\Delta f = \frac{fvn}{c}(\cos\theta_1 + \cos\theta_2) < \frac{2fvn}{c} \approx 0.7\,\mathrm{MHz} < 5\,\mathrm{MHz} \tag{4}$$

Therefore a spectrometer of resolution $5\,\mathrm{MHz}$ cannot detect particles moving at this speed.
(2) The electric fields of the two scattered beams 1 and 2 that reach the detector are

$$\begin{aligned} E_1 &= E_0\cos[2\pi(f+\Delta f)t + \phi_1] \\ E_2 &= E_0\cos[2\pi(f+\Delta f')t + \phi_2] \end{aligned} \tag{5}$$

where Δf and $\Delta f'$ are respectively the frequency differences of scattered beams 1 and 2 from the frequency of the original light. The combined intensity of the two scattered beams is

$$\begin{aligned} I(t) &\propto |E_1 + E_2|^2 \\ &= E_0^2 + \frac{E_0^2}{2}\cos[4\pi(f+\Delta f)t + 2\phi_1] + \frac{E_0^2}{2}\cos[4\pi(f+\Delta f')t + 2\phi_2] \\ &\quad + E_0^2\cos[4\pi(2f+\Delta f'+\Delta f)t + (\phi_1+\phi_2)] + E_0^2\cos[2\pi(\Delta f - \Delta f')t + \phi_1 - \phi_2] \end{aligned} \tag{6}$$

In this expression the first term is a constant; the middle three terms vary at optical frequencies and their sum frequency, which the frequency response of the detector cannot follow, so that in practice they show up as their time averages, which are also constants; the variation frequency of the fifth term is the difference frequency of the optical frequencies. Therefore the photocurrent delivered by the detector is

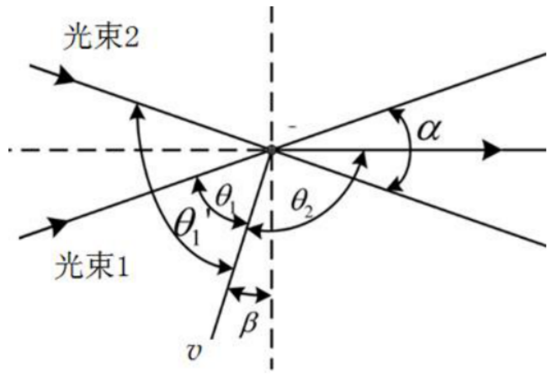
$$i(t) = kE_0^2\cos(2\pi\Delta f_{\mathrm{D}}t + \phi_1 - \phi_2) \tag{7}$$

Here

$$\Delta f_{\mathrm{D}} = \Delta f - \Delta f'$$

is the difference between the scattered-light frequency shifts of beam 1 and beam 2. Beam 1 and beam 2, and the light they scatter from a particle of velocity v, are shown in solution Fig. 4a. By the statement of the problem, (3) is the scattered-light frequency shift of beam 1; similarly, the frequency shift obtained for beam 2 is

$$\Delta f' = \frac{fvn}{c}(\cos\theta_1' + \cos\theta_2) \tag{8}$$



解题图 4a

Labels: 光束1, 光束2 = *beam 1, beam 2;* 解题图 4a = *solution Fig. 4a.*

Therefore the difference between the scattered-light frequency shifts of beam 1 and beam 2 is

$$\begin{aligned}\Delta f_{\mathrm{D}} &= \Delta f' - \Delta f \\ &= \frac{fvn}{c}(\cos\theta_1' - \cos\theta_1) \\ &= \frac{2fvn}{c}\sin\frac{\alpha}{2}\cos\beta\end{aligned} \tag{9}$$

where

$$\alpha = \theta_1' - \theta_1, \qquad \beta = \tfrac{1}{2}(\theta_1' + \theta_1 - \pi)$$

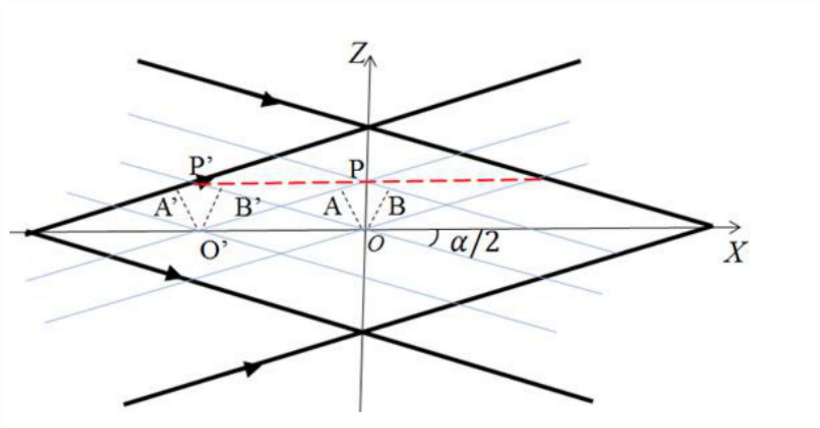
Because the photocurrent signal depends only on the difference of the scattered-light frequency shifts and has nothing to do with θ_2, the velocity measurement by this method is independent of the direction of the scattered light.

(3) As shown in solution Fig. 4b, the rhombus enclosed by the thick black lines is the region where the two symmetrically incident coherent parallel beams cross; the thin blue lines represent the rays inside that region. For every point on the z axis $\Delta\phi = 0$, so all the points of the z axis inside the crossing region interfere constructively (for example the point O and the point O$'$), and a constructive bright fringe appears there. Draw from the point O the perpendicular OA to AP and the perpendicular OB to BP; the phase difference of the two coherent beams arriving at the point P, relative to the point O, is then

$$\Delta\phi = \frac{2\pi}{\lambda}(\mathrm{AP} + \mathrm{BP}) = \frac{4\pi}{\lambda}d\sin\frac{\alpha}{2} \tag{10}$$

where d is the distance from the point P to the X axis. Similarly, draw from the point O$'$ the perpendicular O$'$A$'$ to A$'$P$'$ and the perpendicular O$'$B$'$ to B$'$P$'$; the phase difference of the two coherent beams arriving at the point P$'$ is

$$\Delta\phi' = \frac{2\pi}{\lambda}(\mathrm{A'P'} + \mathrm{B'P'}) = \frac{4\pi}{\lambda}d\sin\frac{\alpha}{2} \tag{11}$$



解题图 4b

Labels: 解题图 4b = *solution Fig. 4b.*

Therefore the interference is the same at all points of P′P inside the crossing region; from the geometry of the figure, P′P is parallel to O′O. If the phase difference at each point is an even multiple of π, the interference is constructive:

$$\frac{4\pi}{\lambda} d \sin\frac{\alpha}{2} = 2j\pi, \qquad (j = 0, 1, 2, \cdots) \tag{12}$$

So the zeroth-order bright fringe lies on the X axis inside the crossing region, and the remaining bright fringes are parallel to the X axis and equally spaced. The phase difference between two adjacent bright fringes is 2π, so

$$2d\sin(\alpha/2) = \lambda \tag{13}$$

and the spacing of the interference fringes is

$$d = \frac{\lambda}{2\sin(\alpha/2)} \tag{14}$$

Because the intensity alternates between bright and dark, the particle scatters a light pulse each time it moves into a bright field, so the frequency of the light pulses is related to the vertical component of the particle velocity. The time for the particle to cross two adjacent bright fringes is

$$\tau = \frac{d}{v\cos\beta} \tag{15}$$

and hence the frequency of the light pulses is

$$f_{\mathrm{D}} = \frac{2fvn}{c}\sin\frac{\alpha}{2}\cos\beta \tag{16}$$

[**Method 2.** The directions of the two incident beams are

$$\begin{aligned} \boldsymbol{e}_1 &= \cos\frac{\alpha}{2}\,\boldsymbol{i} + \sin\frac{\alpha}{2}\,\boldsymbol{j} \\ \boldsymbol{e}_2 &= \cos\frac{\alpha}{2}\,\boldsymbol{i} - \sin\frac{\alpha}{2}\,\boldsymbol{j} \end{aligned} \tag{10}$$

At the position vector $\boldsymbol{r}$ the phase difference of the two beams is

$$\Delta\phi = \frac{2\pi}{\lambda}(\boldsymbol{r}\cdot\boldsymbol{e}_1 - \boldsymbol{r}\cdot\boldsymbol{e}_2) \tag{11}$$

The equation satisfied by all the field points with phase difference $\Delta\phi$ is

$$\Delta\phi = 2y\sin\frac{\alpha}{2}\cdot\frac{2\pi}{\lambda} \tag{12}$$

so the interference fringes are a family of straight lines parallel to the X axis. The fringe spacing d satisfies

$$2d\sin\frac{\alpha}{2}\cdot\frac{2\pi}{\lambda} = 2\pi \tag{13}$$

whence

$$d = \frac{\lambda}{2\sin(\alpha/2)} \tag{14}$$

The time for the particle to cross two adjacent bright fringes is

$$\tau = \frac{d}{v\cos\beta} \tag{15}$$

and hence the frequency of the light pulses is

$$f_{\mathrm{D}} = \frac{2fvn}{c}\sin\frac{\alpha}{2}\cos\beta \tag{16}$$

]

[a] *Translator's note:* As printed. The speed in the medium is of course c/n; the formulae that follow use the factor vn/c accordingly.

Problem 5 (Pulsar signals in a magnetized interstellar plasma)

Solution

(1) The interstellar medium is in a plasma state. Because the mass of a positive ion is far larger than the mass of a negative ion (an electron), it is the electrons that move under the action of the electric field, while the ions may be regarded as fixed. The ion number density of the interstellar medium is very small, so collisions between ions may be neglected. The motion of the electrons is non-relativistic, and the action of the magnetic field of the electromagnetic wave on the electrons may also be neglected. Let the velocity of an electron be $\boldsymbol{v}$; the Lorentz force on the electron is

$$\boldsymbol{F} = -e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{1}$$

Since $\boldsymbol{E}$ is perpendicular to the propagation direction of the wave, $\boldsymbol{B}$ is parallel to the propagation direction and $\boldsymbol{v}\times\boldsymbol{B}$ is perpendicular to the propagation direction, $\boldsymbol{F}$ is perpendicular to the propagation direction, so the electron performs circular motion in the plane perpendicular to the propagation direction of the signal. By Newton's second law,

$$eE_0 \pm evB = \frac{m_{\mathrm{e}}v^2}{R_{\mathrm{e}}} \tag{2}$$

where

$$v = 2\pi f R_{\mathrm{e}} \tag{3}$$

The sign $\pm$ corresponds to the two different senses of rotation of the electron. Solving (2) and (3) together,

$$R_{\mathrm{e}\mp} = \frac{eE_0}{2\pi m_\mathrm{e} f\left(2\pi f \mp \dfrac{eB}{m_\mathrm{e}}\right)} \tag{4}$$

[**Method 2.** The equation of motion of the electron under the Lorentz force is

$$m_\mathrm{e}\ddot{\boldsymbol{x}}(t) = -e[\boldsymbol{E}(t) + \dot{\boldsymbol{x}}(t) \times \boldsymbol{B}] \tag{1}$$

where $\boldsymbol{E}(t)$ is the electric field of the electromagnetic wave at that point,

$$\boldsymbol{E}(t) \equiv E_x \pm iE_y = E_0 e^{\pm i(\omega t - kz)}$$

Here $\omega = 2\pi f$, and $\theta(t)$ is the argument of the electric field in the plane perpendicular to the propagation direction of the wave (that plane being treated as a complex plane). In this complex plane

$$\boldsymbol{x}(t) = R_\mathrm{e} e^{\pm i\omega t} \tag{2}$$

where ω can be understood as the angular velocity of the gyration of the electron. From (2) and the fact that $\boldsymbol{x}(t)$ is orthogonal to $\boldsymbol{B}$,

$$\dot{\boldsymbol{x}}(t) \times \boldsymbol{B} = \pm i[\boldsymbol{\omega} \times \boldsymbol{x}(t)] \times \boldsymbol{B} = \pm i\boldsymbol{x}(t)(\boldsymbol{\omega} \cdot \boldsymbol{B}) = \pm i\omega B\boldsymbol{x}(t) \tag{3}$$

From (1), (2) and (3),

$$-m_\mathrm{e}\omega^2 R_\mathrm{e} = -e[E_0 \pm \omega R_\mathrm{e} B]$$

and solving,

$$R_{\mathrm{e}\mp} = \frac{eE_0}{2\pi m_\mathrm{e} f\left(2\pi f \mp \dfrac{eB}{m_\mathrm{e}}\right)} \tag{4}$$

]

(2) The number of electrons per unit volume in the interstellar medium is n_e, so the polarization of the medium is

$$P_\mp = -n_\mathrm{e} e R_{\mathrm{e}\mp} \tag{5}$$

and the electric displacement vector is

$$D_\mp = \varepsilon_0 E + P_\mp = \left(\varepsilon_0 - \frac{n_\mathrm{e} e^2}{4\pi^2 m_\mathrm{e} f^2 \mp 2\pi f eB}\right) E \tag{6}$$

Hence the permittivity of the medium is

$$\varepsilon_\mp = \varepsilon_0 - \frac{n_\mathrm{e} e^2}{4\pi^2 m_\mathrm{e} f^2 \mp 2\pi f eB} \tag{7}$$

and the refractive index n is

$$n_\mp^2 = \frac{\varepsilon_\mp}{\varepsilon_0} = 1 - \frac{f_\mathrm{p}^2}{f\,(f \mp f_B)} \tag{8}$$

where

$$f_\mathrm{p} = \frac{e}{2\pi}\sqrt{\frac{n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}}}, \qquad f_B = \frac{eB}{2\pi m_\mathrm{e}}$$

are respectively the characteristic frequency of the plasma and the synchronous cyclotron frequency of the electron in the magnetic field.

From the relation $k = \frac{\omega}{c} n$ ($\omega = 2\pi f$) between wave number, frequency and refractive index, together with (8), the group velocity is

$$v_{\mathrm{g}\mp} = \frac{\mathrm{d}\omega}{\mathrm{d}k_\mp} = \frac{1}{\frac{\mathrm{d}k_\mp}{\mathrm{d}\omega}} = \frac{c}{n_\mp + \omega \frac{\mathrm{d}n_\mp}{\mathrm{d}\omega}} = \frac{n_\mp c}{1 \pm \frac{f_\mathrm{p}^2 f_B}{2f(f \mp f_B)^2}} \tag{9}$$

An electromagnetic wave whose group velocity in the medium is zero cannot propagate in that medium:

$$v_{\mathrm{g}\mp} = 0 \tag{10}$$

that is

$$f^2 \mp f_B f - f_\mathrm{p}^2 = 0$$

Discarding the negative root of this equation, the lowest frequency of an electromagnetic wave that can pass through the interstellar medium is

$$f_{\mathrm{c}\mp} = \frac{-(\mp f_B) + \sqrt{f_B^2 + 4f_\mathrm{p}^2}}{2} = \frac{e}{4\pi}\left(\sqrt{\frac{B^2}{m_\mathrm{e}^2} + \frac{4n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}}} \pm \frac{B}{m_\mathrm{e}}\right) \tag{11}$$

When $B = 0$, $f_{\mathrm{c}\mp}(B = 0) = f_\mathrm{p}$, which is exactly the reason why f_p is called the characteristic frequency of the plasma.

(3) The delay of the arrival time on the Earth of an electromagnetic signal of frequency f that has travelled through this medium, relative to propagation in vacuum, is

$$\Delta t_\mp = \int_0^d \frac{\mathrm{d}l}{v_{\mathrm{g}\mp}} - \frac{d}{c} \tag{12}$$

Substituting (9) into (12), and noting $f_\mathrm{p} \ll f$, $f_B \ll f$,

$$\begin{aligned}\Delta t_\mp &= \int_0^d \frac{\mathrm{d}l}{v_{\mathrm{g}\mp}} - \frac{d}{c} \approx \int_0^d \frac{\left(1 \pm \frac{f_\mathrm{p}^2 f_B}{2f^3}\right)\mathrm{d}l}{c\sqrt{1 - \frac{f_\mathrm{p}^2}{f^2} \mp \frac{f_\mathrm{p}^2 f_B}{f^3}}} - \frac{d}{c} \approx \frac{1}{c}\int_0^d \left[1 + \frac{f_\mathrm{p}^2}{2f^2} \pm \frac{f_\mathrm{p}^2 f_B}{f^3}\right]\mathrm{d}l - \frac{d}{c} \\ &= \frac{d}{c}\frac{f_\mathrm{p}^2}{2f^2}\left(1 \pm \frac{2f_B}{f}\right) = \frac{1}{2f^2 c}\frac{e^2 n_\mathrm{e} d}{4\pi^2 \varepsilon_0 m_\mathrm{e}}\left(1 \pm \frac{eB}{\pi m_\mathrm{e} f}\right)\end{aligned} \tag{13}$$

(4) From the relation $k = \frac{\omega}{c} n$ between wave number and refractive index together with (8) (in the approximation $f_\mathrm{p} \ll f$, $f_B \ll f$),

$$k_\mp(f) = \frac{2\pi f}{c} n_\mp = \frac{2\pi f}{c}\sqrt{1 - \frac{f_\mathrm{p}^2}{f^2} \mp \frac{f_\mathrm{p}^2 f_\mathrm{B}}{f^3}} \tag{14}$$

The phase difference of the two different electromagnetic signals formed after an electromagnetic wave of frequency f emitted by the pulsar has crossed the interstellar-medium section d is

$$\begin{aligned}\Delta\phi &\equiv \phi_+ - \phi_- = -\omega(t_+ - t_-) + \int_0^d (k_+ - k_-)\,\mathrm{d}l \\ &= -2\pi f \frac{d}{c}\frac{f_\mathrm{p}^2}{2f^2}\left[\left(1 - \frac{2f_B}{f}\right) - \left(1 + \frac{2f_B}{f}\right)\right] + \frac{2\pi f d}{c}\left(\sqrt{1 - \frac{f_\mathrm{p}^2}{f^2} + \frac{f_\mathrm{p}^2 f_\mathrm{B}}{f^3}} - \sqrt{1 - \frac{f_\mathrm{p}^2}{f^2} - \frac{f_\mathrm{p}^2 f_\mathrm{B}}{f^3}}\right) \\ &= 2\pi f \frac{d}{c}\frac{f_\mathrm{p}^2}{2f^2}\frac{4f_B}{f} + \frac{2\pi f d}{c}\frac{f_\mathrm{p}^2 f_\mathrm{B}}{f^3} = \frac{6\pi d}{c}\frac{f_\mathrm{p}^2 f_\mathrm{B}}{f^2} = \frac{3e^3 n_\mathrm{e} d B}{4\pi^2 \varepsilon_0 m_\mathrm{e}^2 c f^2}\end{aligned} \tag{15}$$

(5) From (15), a fluctuation of the electron density causes a change of the refractive index. From (8),

$$\Delta n_{\mp} = -\frac{f_{\mathrm{p}}^2}{2nf\,(f \mp f_B)}\frac{\Delta n_{\mathrm{e}}}{n_{\mathrm{e}}} \tag{16}$$

From (16) and $k = \frac{\omega}{c}n$, the change of the wave number caused by the electron-density fluctuation is

$$\Delta k_{\mp} = \frac{2\pi f}{c}\Delta n_{\mp} = -\frac{\pi f_{\mathrm{p}}^2}{nc(f \mp f_B)}\frac{\Delta n_{\mathrm{e}}}{n_{\mathrm{e}}} \tag{17}$$

Therefore the phase shift produced by the electron-density fluctuation for the two different electromagnetic signals formed by an electromagnetic wave of frequency f crossing the interstellar medium is

$$\Delta\phi_{\mp} = -a\,\Delta k_{\mp} = \frac{\pi f_{\mathrm{p}}^2}{nc(f \mp f_B)}\frac{\Delta n_{\mathrm{e}}}{n_{\mathrm{e}}}\,a$$

Problem 6 (Antiprotons, Cherenkov counters and the Dalitz plot)

Solution

(1) The momentum of the incident high-energy proton is

$$P = \sqrt{\frac{E_{\mathrm{p}}^2}{c^2} - M_{\mathrm{p}}^2c^2} = 6.735\ \mathrm{GeV}/c \tag{1}$$

and the momentum of the antiproton produced is approximately

$$P_{\bar{\mathrm{p}}} = \frac{P}{4} = 1.684\ \mathrm{GeV}/c \tag{2}$$

[**Method 2.**

$$E_{\mathrm{p}} + M_{\mathrm{p}}c^2 = 4E_{\bar{\mathrm{p}}} \tag{1}$$

$$E_{\bar{\mathrm{p}}} = \sqrt{P_{\bar{\mathrm{p}}}^2c^2 + M_{\mathrm{p}}^2c^4} \tag{2}$$

]

The speed of the antiproton is

$$v_{\bar{\mathrm{p}}} = \frac{P_{\bar{\mathrm{p}}}c^2}{E_{\bar{\mathrm{p}}}} = \frac{P_{\bar{\mathrm{p}}}c^2}{\sqrt{P_{\bar{\mathrm{p}}}^2c^2 + M_{\mathrm{p}}^2c^4}} = 0.873c \tag{3}$$

and the time for the antiproton to travel from S_1 to S_2 is

$$t_{\bar{\mathrm{p}}} = \frac{l}{v_{\bar{\mathrm{p}}}} = 45.8\ \mathrm{ns}. \tag{4}$$

The energy of the π meson is

$$E_\pi = E_{k,\pi} + m_\pi c^2 = E_{\bar{\mathrm{p}}} - M_{\mathrm{p}}c^2 + m_\pi c^2. \tag{5}$$

The speed of the π meson is

$$v_\pi = \frac{P_\pi c^2}{E_\pi} = \frac{c\sqrt{E_\pi^2 - m_\pi^2c^4}}{E_\pi} = 0.992c \tag{6}$$

and the time for the π meson to travel from S_1 to S_2 is

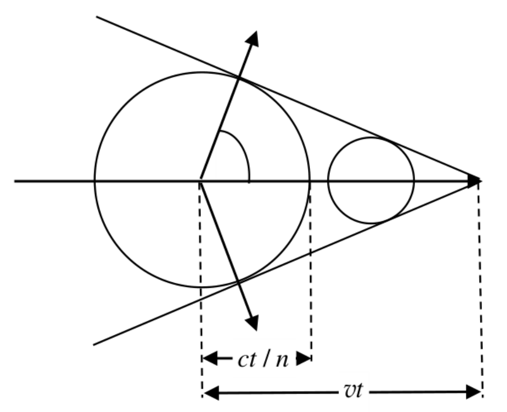
$$t_\pi = \frac{l}{v_\pi} = 40 \text{ ns.} \tag{7}$$

(2) During the process in which a charged particle moves uniformly with speed v ($v > c/n$) in a medium of refractive index n from the position O to the position P (the elapsed time interval being t), the electromagnetic fields excited in the medium at the points along its path form a conical envelope, as shown in solution Fig. 6a. The direction of the Cherenkov radiation is along the normal of the conical envelope, and the angle it makes with the direction of motion of the charged particle satisfies

$$\cos\theta = \frac{ct/n}{vt} = \frac{c}{nv}.$$

Therefore

$$\theta = \arccos\frac{c}{nv}. \tag{8}$$



解题图 6a

Labels: 解题图 6a = *solution Fig. 6a.*

(3) The focal length of the spherical mirror is

$$f = R/2. \tag{9}$$

The radius of the light ring on the focal plane is

$$r = f\tan\theta = \frac{R}{2}\sqrt{\left(\frac{nv}{c}\right)^2 - 1}. \tag{10}$$

(4) Momentum conservation gives

$$\gamma_1 m_\pi \boldsymbol{v}_1 + \gamma_2 m_\pi \boldsymbol{v}_2 + \gamma_3 m_\pi \boldsymbol{v}_3 = 0, \tag{11}$$

and energy conservation gives

$$m_\pi c^2(\gamma_1 - 1) + m_\pi c^2(\gamma_2 - 1) + m_\pi c^2(\gamma_3 - 1) = Q, \tag{12}$$

where

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}. \tag{13}$$

[**Method 2.** Momentum conservation gives

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = 0 \tag{11}$$

Energy conservation gives

$$E_{k,1} + E_{k,2} + E_{k,3} = Q \tag{12}$$

together with

$$E_{k,i} = \sqrt{p_i^2 c^2 + m_\pi^2 c^4} - m_\pi c^2 \tag{13}$$

]

It follows that

$$p_i = \frac{Q}{c}\sqrt{d_i^2 + 2d_i K},$$

where $K = m_\pi c^2/Q$ and $d_i = K(\gamma_i - 1)$; hence the momentum magnitudes p_1, p_2, p_3 of the three π mesons satisfy $p_i + p_j > p_k$ with $\{i, j, k\} = \{1, 2, 3\}$.

The possible distribution range of d_1, d_2, d_3 is

$$\begin{aligned}
\sqrt{d_1^2 + 2d_1 K} &\leq \sqrt{d_2^2 + 2d_2 K} + \sqrt{d_3^2 + 2d_3 K} \\
\sqrt{d_2^2 + 2d_2 K} &\leq \sqrt{d_1^2 + 2d_1 K} + \sqrt{d_3^2 + 2d_3 K} \\
\sqrt{d_3^2 + 2d_3 K} &\leq \sqrt{d_1^2 + 2d_1 K} + \sqrt{d_2^2 + 2d_2 K}
\end{aligned} \tag{14}$$

In the coordinate system given in the problem,

$$d_1 = y, \qquad d_2 = -(\sqrt{3}x + y - 1)/2, \qquad d_3 = (\sqrt{3}x - y + 1)/2 \tag{15}$$

Substituting (15) into (14),

$$2(3K+1)y^3 - 6(3K+1)x^2 y + (12K^2 - 1)y^2 + 3(2K+1)^2 x^2 - 2(4K+1)Ky \leq 0. \tag{16}$$

If $m_\pi = 0$, then $K = 0$, so

$$(y - 1/2)(y - \sqrt{3}x)(y + \sqrt{3}x) \leq 0 \tag{17}$$

which describes the triangular region formed by joining the midpoints of the three sides.

Problem 7 (The *qi* vessel: "empty it tilts, full it topples")

Solution

(1) Suppose the whole *qi* vessel were solid; then it splits into the long cylindrical part (I) and the hemispherical part (II). In the coordinate system XOZ the position of the centre of mass of the cylindrical part (I) is

$$x_{\mathrm{CI}} = 0, \qquad z_{\mathrm{CI}} = \frac{l}{2} = R$$

and the position of the centre of mass of the hemispherical part (II) is

$$x_{\mathrm{CII}} = 0, \qquad z_{\mathrm{CII}} = \frac{\int_{-R_2}^{0} \rho_1 \pi (R_2^2 - z^2) z \,\mathrm{d}z}{\int_{-R_2}^{0} \rho_1 \pi (R_2^2 - z^2) \,\mathrm{d}z} = -\frac{3}{8} R_2 = -\frac{3\sqrt{3}}{8} R = -0.6495R \tag{1}$$

Let the centre-of-mass coordinates of the whole solid body be ($x_{\mathrm{C,sol}} = 0,\ z_{\mathrm{C,sol}}$); then

$$\left(\rho \pi R_2^2 l + \rho \frac{2}{3} \pi R_2^3\right) z_{\mathrm{C,sol}} = \rho \pi R_2^2 l \, z_{\mathrm{CI}} + \rho \frac{2}{3} \pi R_2^3 z_{\mathrm{CII}} = \rho \pi R_2^2 l \, \frac{l}{2} - \rho \frac{2}{3} \pi R_2^3 \frac{3R_2}{8}$$

whence

$$x_{\mathrm{C,sol}} = 0, \qquad z_{\mathrm{C,sol}} = \frac{3(2l^2 - R_2^2)}{4(3l + 2R_2)} = \frac{5(3 - \sqrt{3})}{16} R = 0.3962R \tag{2}$$

Correspondingly the centre-of-mass coordinates of the hollow part of the whole *qi* vessel are

$$x_{\mathrm{C,hol}} = t = 0.3R, \qquad z_{\mathrm{C,hol}} = \frac{3(2l^2 - R_1^2)}{4(3l + 2R_1)} = \frac{21}{32} R = 0.6563R \tag{3}$$

Let m_{sol} and m_{hol} be respectively the mass of the *qi* vessel if it were entirely solid and the mass of the material that would exactly fill the hollow part of the whole vessel; then

$$m_{\mathrm{sol}} = \rho_1 \pi R_2^2 \left(\frac{2R_2}{3} + l\right), \qquad m_{\mathrm{hol}} = \rho_1 \pi R_1^2 \left(\frac{2R_1}{3} + l\right)$$

The centre-of-mass coordinates of the whole vessel are

$$x_{\mathrm{C}} = \frac{m_{\mathrm{sol}} x_{\mathrm{C,sol}} + (-m_{\mathrm{hol}}) x_{\mathrm{C,hol}}}{m_{\mathrm{sol}} + (-m_{\mathrm{hol}})} = -\frac{R_1^2 t (3l + 2R_1)}{3l(R_2^2 - R_1^2) + 2(R_2^3 - R_1^3)} = \frac{3(5 - 3\sqrt{3})}{5} R = -0.1177R \tag{4}$$

$$z_{\mathrm{C}} = \frac{m_{\mathrm{sol}} z_{\mathrm{C,sol}} + (-m_{\mathrm{hol}}) z_{\mathrm{C,hol}}}{m_{\mathrm{sol}} + (-m_{\mathrm{hol}})} = \frac{3(R_2 + R_1)(2l^2 - R_2^2 - R_1^2)}{4[3l(R_2 + R_1) + 2(R_2^2 + R_2 R_1 + R_1^2)]} = \frac{3}{2}(3\sqrt{3} - 5)R = 0.2942R \tag{5}$$

The position of the suspension point Q is $\left(-\frac{1}{10}R,\ \frac{2\sqrt{3}}{11}R\right)$, and the angle by which the *qi* vessel tilts when it hangs with no water inside is

$$\theta_0 = \operatorname{arccot} \frac{z_{\mathrm{C}} - z_Q}{x_{\mathrm{C}} - x_Q} = \operatorname{arccot} \frac{25(5\sqrt{3} - 3)}{121} = 0.707439 \times \frac{180^\circ}{\pi} = 40.5333^\circ \tag{6}$$

(2) From the moment of inertia of a sphere about a diameter, the moment of inertia of a hemisphere about the diameter of its base circle is

$$I_{11} = \frac{1}{2} I_1 = \frac{4}{15} \pi \rho R^5$$

By the parallel-axis theorem, the moment of inertia of the hemisphere about an axis through its centre of mass parallel to its base is

$$I_{1c} = I_{11} - \frac{2}{3} \pi \rho R^3 \left(\frac{3}{8} R\right)^2 = \frac{83}{480} \pi \rho R^5 = 0.1729 \, \pi \rho R^5 \tag{7}$$

The moment of inertia of the solid *qi* vessel about the horizontal axis through the suspension point is

$$\begin{aligned} I_{\text{sol}} &= I_{1c}(R=R_2) + \frac{2}{3}\rho_1\pi R_2^3\left[x_Q^2 + \left(-\frac{3}{8}R_2 - z_Q\right)^2\right] \\ &\quad + I_2(R=R_2, L=l) + \rho_1\pi R_2^2 l\left[x_Q^2 + \left(\frac{l}{2} - z_Q\right)^2\right] \\ &= \frac{79588 + 7591\sqrt{3}}{6050}\rho_1\pi R^5 = 15.3283\,\rho_1\pi R^5 \end{aligned} \tag{8}$$

The moment of inertia of the (filled-in) hollow part about the horizontal axis through the suspension point is

$$\begin{aligned} I_{\text{hol}} &= I_{1c}(R=R_1) + \frac{2}{3}\rho_1\pi R_1^3\left[(x_Q+t)^2 + \left(-\frac{3}{8}R_1 - z_Q\right)^2\right] \\ &\quad + I_2(R=R_1, L=l) + \rho_1\pi R_1^2 l\left[(x_Q+t)^2 + \left(\frac{l}{2} - z_Q\right)^2\right] \\ &= \frac{(24953 - 3850\sqrt{3})}{6050}\rho_1\pi R^5 = 3.0223\,\rho_1\pi R^5 \end{aligned} \tag{9}$$

so the moment of inertia of the *qi* vessel about the horizontal axis through the suspension point is

$$I = I_{\text{sol}} - I_{\text{hol}} = \frac{(54635 + 11441\sqrt{3})}{6050}\rho_1\pi R^5 = 12.306\,\rho_1\pi R^5 \tag{10}$$

The distance from the centre of mass C to the horizontal axis through the suspension point Q is

$$\begin{aligned} d &= \sqrt{(x_{\text{C}} - x_Q)^2 + (z_{\text{C}} - z_Q)^2} \\ &= \sqrt{\left(-\frac{R_1^2 t(3l + 2R_1)}{3l(R_2^2 - R_1^2) + 2(R_2^3 - R_1^3)} - x_Q\right)^2 + \left(\frac{3(R_2+R_1)(2l^2 - R_2^2 - R_1^2)}{4[3l(R_2+R_1) + 2(R_2^2 + R_2R_1 + R_1^2)]} - z_Q\right)^2} \\ &= \frac{(3\sqrt{3} - 5)\sqrt{2}}{220}\sqrt{30848 - 17541\sqrt{3}}\,R = 0.02722R \end{aligned}$$

The mass of the *qi* vessel is

$$m_{\text{qi}} = m_{\text{sol}} - m_{\text{hol}} = \rho_1\pi\left[R_2^2\left(\frac{2R_2}{3} + l\right) - R_1^2\left(\frac{2R_1}{3} + l\right)\right] = \frac{2}{3}(5 + 3\sqrt{3})\rho_1\pi R^3$$

and the angular frequency with which the *qi* vessel swings about the line of suspension points is

$$\omega = \sqrt{\frac{m_{\text{qi}}\,g\,d}{I}} = \sqrt{\frac{110\sqrt{2(30848 - 17541\sqrt{3})}}{163905 + 34323\sqrt{3}}}\sqrt{\frac{g}{R}} = 0.122624\sqrt{\frac{g}{R}} \tag{11}$$

(3) Let the mass of the water inside the *qi* vessel when it stands upright be m_{S}. The x coordinates x_{CS}, x_C and x_{Ct} of the centres of mass of the water, of the *qi* vessel and of the whole system are then respectively

$$x_{\text{CS}} = t \tag{12}$$

$$x_C = \frac{3(5 - 3\sqrt{3})}{5}R = -0.1177R$$

$$x_{Ct} = x_Q \tag{13}$$

so that

$$x_{Ct}(m_\mathrm{S} + m_\mathrm{qi}) = m_\mathrm{S} x_\mathrm{CS} + m_\mathrm{qi} x_C \tag{14}$$

Solving,

$$m_\mathrm{S} = \frac{m_\mathrm{qi}(x_C - x_{Ct})}{x_{Ct} - x_{CS}} = \frac{m_\mathrm{qi}(x_C - x_Q)}{x_Q - t} = 0.3006\,\rho_1 \pi R^3 = 0.9018\,\rho_2 \pi R^3 \tag{15}$$

Since

$$m_\mathrm{S} = 0.9018\,\rho_2 \pi R^3 > \frac{2}{3}\rho_2 \pi R^3 \tag{16}$$

the volume of the water is larger than that of the hemispherical container at the bottom, i.e. $h > R$. Therefore

$$m_\mathrm{S} = \frac{2}{3}\rho_2 \pi R^3 + \rho_2 \pi R^2 (h - R) = 0.9018\,\rho_2 \pi R^3 \tag{17}$$

and solving,

$$h = \frac{(23 - 9\sqrt{3})}{6} R = 1.23526R \tag{18}$$

The z coordinate of the centre of mass of the whole system is then

$$\begin{aligned} z_{Ct} &= \frac{m_\mathrm{qi} z_C + m_\mathrm{S} z_{Cs}}{m_\mathrm{qi} + m_\mathrm{S}} = \frac{3\{[\rho_1(R_2^2 - R_1^2)(2l^2 - R_1^2 - R_2^2) + \rho_2 R_1^2(2h^2 - 4hR_1 + R_1^2)]\}}{4\{\rho_1[3l(R_2^2 - R_1^2) + 2(R_1^3 - R_1^3)] + \rho_2 R_1^2(3h - R_1)\}} \\ &= \frac{3(2h^2 - 4hR + 25R^2)}{4[3h + (29 + 18\sqrt{3})R]} = 0.271326R < z_Q \end{aligned} \tag{19}$$

so the qi vessel stands upright when $h = 1.23526R$.

[**Method 2.** Suppose the qi vessel stands upright when $h \le R_1$. For $h \le R_1$ the position of the centre of mass of the water is

$$\begin{cases} x_{\mathrm{CS}_1} = t = 0.3R, \\ z_{\mathrm{CS}_1} = \dfrac{\displaystyle\int_{-R}^{-(R-h)} \rho_2 \pi (R^2 - z^2) z \,\mathrm{d}z}{\displaystyle\int_{-R}^{-(R-h)} \rho_2 \pi (R^2 - z^2) \,\mathrm{d}z} = -\dfrac{3(2R - h)^2}{4(3R - h)} \end{cases} \tag{12}$$

The mass m of the qi vessel and the mass m_{S_1} of the water then in the vessel are respectively

$$\begin{aligned} m &= \rho_1 \left[\left(\frac{2}{3}\pi R_2^3 + \pi R_2^2 l \right) - \left(\frac{2}{3}\pi R_1^3 + \pi R_1^2 l \right) \right] = \frac{1}{3}\rho_1 \pi [3l(R_2^2 - R_1^2) + 2(R_2^3 - R_1^3)] \\ &= \frac{1}{3}\rho_1 \pi (R_2 - R_1)[3l(R_2 + R_1) + 2(R_2^2 + R_1 R_2 + R_1^2)] \end{aligned}$$

$$m_{S_1} = \int_{-R}^{-(R-h)} \rho_2 \pi (R^2 - z^2) \,\mathrm{d}z = \frac{\rho_2 \pi h^2 (3R - h)}{3}$$

The coordinates of the centre of mass C_t of the whole device are

$$\begin{aligned} x_{Ct1} &= \frac{m x_\mathrm{C} + m_\mathrm{S} x_\mathrm{CS}}{m + m_\mathrm{S}} = \frac{[\rho_1 R_1^2(3l + 2R_1) + \rho_2 h^2 (h - 3R_1)]t}{\rho_1[3l(R_1^2 - R_2^2) + 2(R_1^3 - R_2^3)] + \rho_2 h^2 (h - 3R_1)} \\ &= \frac{3R(24R^3 - 3h^2 R + h^3)}{10[(h - 3R)h^2 - 6(5 + 3\sqrt{3})R^3]} \end{aligned} \tag{13}$$

When the body of the *qi* vessel stands upright, the line joining the suspension point and the centre of mass of the system must be vertical, so[a]

$$x_{Ct} = x_Q = -0.1h, \tag{14}$$

and solving,

$$h = -0.8394R, \qquad h = 1.2389R, \qquad h = 2.5996R \tag{15}$$

None of these satisfies $0 < h < R$, so there is no solution in this case. Therefore, when the *qi* vessel stands upright one must have

$$h > R_1$$

When $h > R_1$, comparison with (3) and (4) (in which one may regard $l = h - R_1$) gives the position of the centre of mass of the water as

$$\begin{aligned} x_{\mathrm{CS}} &= t = 0.3R, \\ z_{\mathrm{CS}} &= \frac{\displaystyle\int_{-R_1}^{0} \rho_2\pi(R_1^2 - z^2)z\,\mathrm{d}z + \int_0^{h-R_1} \rho_2\pi R_1^2 z\,\mathrm{d}z}{\displaystyle\int_{-R_1}^{0} \rho_2\pi(R_1^2 - z^2)\,\mathrm{d}z + \int_0^{h-R_1} \rho_2\pi R_1^2\,\mathrm{d}z} = \frac{3(2h^2 - 4hR + R^2)}{4(3h - R)} \end{aligned} \tag{16}$$

The mass m_{S_2} of the water then in the vessel is[b]

$$m_{S_2} = \rho_1\pi R_1^2\left(h - \frac{R_1}{3}\right)$$

and the position of the centre of mass of the whole device is

$$\begin{aligned} x_{Ct2} &= \frac{m x_{\mathrm{C}} + m_{S_2} x_{\mathrm{CS}_2}}{m + m_{S_2}} = \frac{R_1^2[\rho_1(3l + 2R_1) + \rho_2(R_1 - 3h)]t}{\rho_1[3l(R_1^2 - R_2^2) + 2(R_1^3 - R_2^3)] + \rho_2 R_1^2(R_1 - 3h)} \\ &= \frac{3(3h - 25R)R}{10[3h + (29 + 18\sqrt{3})R]} \end{aligned} \tag{17}$$

$$\begin{aligned} z_{Ct2} &= \frac{m z_C + m_{S2} z_{Cs}}{m + m_{S2}} = \frac{3\{[\rho_1(R_2^2 - R_1^2)(2l^2 - R_1^2 - R_2^2) + \rho_2 R_1^2(2h^2 - 4hR_1 + R_1^2)]\}}{4\{\rho_1[3l(R_2^2 - R_1^2) + 2(R_1^3 - R_1^3)] + \rho_2 R_1^2(3h - R_1)\}} \\ &= \frac{3(2h^2 - 4hR + 25R^2)}{4[3h + (29 + 18\sqrt{3})R]} \end{aligned} \tag{18}$$

When the body of the *qi* vessel stands upright, the line joining the suspension point and the centre of mass of the system must be vertical, so

$$x_{Ct2} = x_Q = -0.1h$$

and moreover

$$z_{Ct2} < z_{\mathrm{Q}} = \frac{2\sqrt{3}}{11}h$$

When $h > R_1$, (18) gives[c]

$$h = \frac{1}{3}\left[R_1 + \frac{\rho_1}{\rho_2}\left(3l + 2R_1 - \frac{x_Q R_2^2(3l + 2R_2)}{R_1^2(t - x_Q)}\right)\right] = \frac{(23 - 9\sqrt{3})}{6}R = 1.23526R \tag{19}$$

while $z_{Ct2} < z_{\mathrm{Q}} = \dfrac{2\sqrt{3}}{11}h$ gives

$$0 < h < 2.67979R \qquad \text{or} \qquad h < -20.059R$$

so the *qi* vessel stands upright when $h = 1.23526R$.]

(4) The critical condition for the *qi* vessel to "topple when brimful" is that the centre of mass of the system be just above the suspension point; that is,

$$x_{\mathrm{CH}} = -x_Q, \qquad z_{\mathrm{CH}} = z_Q \tag{20}$$

[a] *Translator's note:* Equation (14) is printed as shown. Since $x_Q = -R/10$, its right-hand side should be $-0.1R$; the same substitution of h for R appears in $z_Q = \frac{2\sqrt{3}}{11}h$ below.

[b] *Translator's note:* Printed as shown; the density here should be ρ_2, the density of water.

[c] *Translator's note:* The relation actually used here is (17), the equation for x_{Ct2}.