The 38th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2021 — official reference solutions

Equation numbers (1), (2), … in each problem follow the circled numbers of the Chinese original. Passages that the original encloses in square brackets (alternative methods and alternative branches) are kept as "Method 2", "Method 3", …. Per-step marks are given only where the original prints them (Problem 1). Obvious misprints of the source are transcribed as printed, with a translator's note giving the correct form.

Problem 1 (zone plate: foci, dark points and imaging)

Solution

(1) *(8 points)*

For the focus P the difference of the distances PO and PM is fixed at half a wavelength. Let r_0 be the distance from P to the centre of the hole (the position coordinate of P being x). The half-wave-zone method gives

$$\sqrt{R^2+r_0^2}=r_0+\frac{\lambda}{2} \tag{1}$$

where R is the radius of the small circular hole. With the data of the problem,

$$1.00^2+x_0^2=\left(x_0+0.3164\times10^{-3}\right)^2,$$

where $r_0=x_0\,\text{mm}$. Hence

$$r_0=1580\,\text{mm} \tag{2}$$

Let r_1 be the distance from the 3-zone focus P_1 to the centre of the hole (position coordinate x_1). The half-wave-zone method gives

$$\sqrt{R^2+r_1^2}=r_1+(2\times1+1)\frac{\lambda}{2} \tag{3}$$

With the data of the problem,

$$1.00^2+x_1^2=\left(x_1+3\times0.3164\times10^{-3}\right)^2,$$

where $r_1=x_1\,\text{mm}$. Hence

$$r_1=527\,\text{mm} \tag{4}$$

(2)(i) *(8 points)*

Because the grey parts add half a wavelength, when all the zones reach the fixed point P_0' their optical path differences relative to that of the central zone become $0,\ \lambda,\ \lambda$; hence the four zones do not cancel in pairs at the fixed point P_0', but reinforce one another and form a focus.

[Translator's note: The values $0,\ \lambda,\ \lambda$ are as printed. The four zones in fact give the four values $0,\ 0,\ \lambda,\ \lambda$; the first "0" appears to have been dropped in the source.]

In computing the geometric lengths, the geometric length of the outer rim of the outermost zone exceeds the geometric length along the central axis by 2λ, so

$$\sqrt{R^2 + r_0'^2} = r_0 + 2\lambda \tag{5}$$

[Translator's note: As printed; the right-hand side should read $r_0' + 2\lambda$, which is what the numerical equation below actually uses.]

With the data of the problem,

$$1.00^2 + x'^2 = \left(x' + 4 \times 0.3164 \times 10^{-3}\right)^2,$$

where $r_0' = x'$ mm. Hence

$$r_0' = 395\,\text{mm} \tag{6}$$

(ii) *(8 points)*

The dark point P'_{-1} immediately to the left of the focus P'_0 appears when the zone plate is further divided into an 8-zone structure:

$$\sqrt{R^2 + r_{-1}'^2} = r_0 + 4\lambda \tag{7}$$

With the data of the problem,

$$1.00^2 + x_{-1}'^2 = \left(x'_{-1} + 8 \times 0.3164 \times 10^{-3}\right)^2,$$

where $r'_{-1} = x'_{-1}$ mm. The coordinate of the dark point P'_{-1} is therefore

$$r'_{-1} = 198\,\text{mm} \tag{8}$$

(iii) *(8 points)*

The dark point P'_{+1} immediately to the right of the focus P'_0 appears when the four zones are merged in pairs and the whole is regarded as two zones:

$$\sqrt{R^2 + r_{+1}'^2} = r_0 + \lambda \tag{9}$$

With the data of the problem,

$$1.00^2 + x_{+1}'^2 = \left(x'_{+1} + 2 \times 0.3164\right)^2,$$

[Translator's note: As printed; the factor $\times 10^{-3}$ is missing in the source. The correct numerical equation is $1.00^2 + x_{+1}'^2 = (x'_{+1} + 2 \times 0.3164 \times 10^{-3})^2$.]

where $r'_{+1} = x'_{+1}$ mm. The coordinate of the dark point P'_{+1} is

$$r'_{+1} = 790\,\text{mm} \tag{10}$$

(3)(i) *(10 points)*

When a ray leaves the outermost edge of the zone plate, is deflected and reaches the image point, its geometric path length exceeds that along the axis by 2λ. The equation is

$$\sqrt{s_{\text{obj}}^2 + R^2} + \sqrt{s_{\text{img}}^2 + R^2} = s_{\text{obj}} + s_{\text{img}} + 4 \times \frac{\lambda}{2} \tag{11}$$

where $s_{\text{obj}} = \mathrm{OS}$, $s_{\text{img}} = \mathrm{OS}'$. With the data of the problem,

$$\sqrt{3000^2 + 1.00^2} + \sqrt{x_2^2 + 1.00^2} = 3000 + x_2 + 2\lambda$$

where $s_{\text{img}} = x_2$ mm. This gives

$$s_{\text{img}} = 455\,\text{mm} \tag{12}$$

The imaging formula is

$$\frac{1}{s_{\text{obj}}} + \frac{1}{s_{\text{img}}} = \frac{1}{f} \tag{13}$$

Substituting the data of the problem,

$$f = 395\,\text{mm} \tag{14}$$

which agrees with the focal length r_0' of the 4-zone plate, i.e. with Eq. (6); so the imaging formula holds.

(ii) *(8 points)*

For the next image point the 4-zone plate must be split into a 12-zone plate. The path difference described in the previous part is therefore 6λ, and the equation is

$$\sqrt{s_{\text{obj}}^2 + R^2} + \sqrt{s_{\text{img}}'^2 + R^2} = s_{\text{obj}} + s_{\text{img}}' + 3 \times 4 \times \frac{\lambda}{2} \tag{15}$$

With the data of the problem,

$$\sqrt{3000^2 + 1.00^2} + \sqrt{x_3^2 + 1.00^2} = 3000 + x_3 + 6\lambda$$

where $s_{\text{img}}' = x_3$ mm. This gives

$$s_{\text{img}}' = 137\,\text{mm} \tag{16}$$

The corresponding secondary focus is the focus obtained when the 4-zone plate is split into a 12-zone plate. Every zone plate gives a new secondary focus when it is split an odd number of times, so each zone of the 4-zone plate must be split into 3 zones in order to obtain this secondary focus:

$$\sqrt{f'^2 + R^2} = f' + 3 \times 4 \times \frac{\lambda}{2} \tag{17}$$

where f' is the focal length of the secondary focus. With the data of the problem,

$$\sqrt{f'^2 + 1.00^2} = f' + 6\lambda$$

which gives

$$f' = 132\,\text{mm} \tag{18}$$

that is, the secondary focal length is 132 mm.

(iii) *(14 points)*

Consider first the image formed when the principal focus of the 4-zone plate is used as the focus. Since the object point is on the left, between the principal focus and the zone plate, it is still on the left. From Eq. (6) or Eq. (14),

$$f = r_0' = 395\,\text{mm},$$

and by the problem the object distance is

$$s_{\text{obj}} = \frac{f}{2} = 198\,\text{mm} \tag{19}$$

Let the image distance be s_{img}''; then

$$\sqrt{s_{\text{obj}}^2 + R^2} + \sqrt{s_{\text{img}}''^2 + R^2} = s_{\text{obj}} + s_{\text{img}}'' + 4 \times \frac{\lambda}{2} \tag{20}$$

With the data of the problem,

$$\sqrt{198^2 + 1.00^2} + \sqrt{x_4^2 + 1.00^2} = 198 + x_4 + 2\lambda$$

where $s''_{\text{img}} = x_4\,\text{mm}$. This gives

$$s''_{\text{img}} = -396\,\text{mm}, \tag{21}$$

that is, this image point is virtual and lies at the position of the focus on the left. (22)
Now consider the image formed when the secondary focus described in (3)(ii) is used as the focus. Let the image distance be s'''_{img}; then

$$\sqrt{s_{\text{obj}}^2 + R^2} + \sqrt{s'''^2_{\text{img}} + R^2} = s_{\text{obj}} + s'''_{\text{img}} + 3 \times 4 \times \frac{\lambda}{2} \tag{23}$$

With the data of the problem,

$$\sqrt{198^2 + 1^2} + \sqrt{x_5^2 + 1^2} = 198 + x_5 + 6\lambda$$

where $s'''_{\text{img}} = x_5\,\text{mm}$. This gives

$$s'''_{\text{img}} = 393\,\text{mm}, \tag{24}$$

that is, this image point is real and lies 393 mm to the right of the zone plate.

Problem 2 (Hohmann transfer and orbital rendezvous)

Solution

(1) From the orbit equation,

$$r_{\text{peri}} = \frac{R}{1+\varepsilon}, \qquad r_{\text{apo}} = \frac{R}{1-\varepsilon} \tag{1}$$

The total energy E and the orbital angular momentum L are conserved:

$$L = m\,v_{\text{peri}}\,r_{\text{peri}} = m\,v_{\text{apo}}\,r_{\text{apo}} = mvr$$

or

$$v = \frac{L}{mr} \tag{2}$$

The total energy of the system is

$$E = -\frac{C}{r_{\text{peri}}} + \frac{L^2}{2mr_{\text{peri}}^2} = -\frac{C}{r_{\text{apo}}} + \frac{L^2}{2mr_{\text{apo}}^2} \tag{3}$$

where

$$C = GMm = mr_0 v_0^2,$$

here using

$$\frac{GMm}{r_0^2} = m\frac{v_0^2}{r_0}.$$

From Eqs. (1), (2), (3) and the expression for C,

$$R = \frac{L^2}{m^2 r_0 v_0^2} \tag{4}$$

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{m^3 r_0^2 v_0^4}} = \sqrt{1 + \frac{2ER}{m r_0 v_0^2}} \tag{5}$$

(2) After firing, the elliptical orbit of the chaser c becomes a rotated ellipse. With the angles defined in the problem, this rotated elliptical orbit is

$$r_\mathrm{c}(\theta_\mathrm{c}) = \frac{R}{1 + \varepsilon \cos(\theta_\mathrm{c} + \phi)} \tag{6}$$

The initial condition is

$$r_\mathrm{c} = r_0 \quad \text{when } \theta_\mathrm{c} = 0.$$

Substituting into Eq. (6),

$$R = r_0(1 + \varepsilon \cos\phi) \tag{7}$$

so

$$r_\mathrm{c}(\theta_\mathrm{c}) = \frac{r_0(1 + \varepsilon\cos\phi)}{1 + \varepsilon \cos(\theta_\mathrm{c} + \phi)} \tag{8}$$

(3) From Eq. (6), the length $2a$ of the major axis of the elliptical orbit after firing is

$$2a = r_\mathrm{peri} + r_\mathrm{apo} = \frac{r_0(1+\varepsilon\cos\phi)}{1+\varepsilon} + \frac{r_0(1+\varepsilon\cos\phi)}{1-\varepsilon} = \frac{2r_0(1+\varepsilon\cos\phi)}{1-\varepsilon^2} \tag{9}$$

The major axis of the orbit of the target A is $2r_0$. Kepler's third law gives

$$\frac{T_\mathrm{c}}{T_\mathrm{A}} = \left(\frac{a}{r_0}\right)^{3/2} = \left(\frac{1+\varepsilon\cos\phi}{1-\varepsilon^2}\right)^{3/2} \tag{10}$$

(4) Express the velocity after firing in terms of the firing parameters. Take polar coordinates. The radial velocity (along the radius vector from the centre of the circular orbit, i.e. the focus, to the position of the chaser c) is

$$v_r = v_0 \delta \sin\alpha,$$

and the tangential velocity (perpendicular to the radius vector) is

$$v_\theta = v_0(1 + \delta\cos\alpha) \tag{11}$$

Conservation of the angular momentum L gives

$$L = m r_0 v_\theta = m r_0 v_0 (1 + \delta\cos\alpha) \tag{12}$$

From Eq. (12) and

$$r_\mathrm{peri} = \frac{r_0(1+\varepsilon\cos\phi)}{1+\varepsilon}, \qquad r_\mathrm{apo} = \frac{r_0(1+\varepsilon\cos\phi)}{1-\varepsilon},$$

the speeds at perigee and apogee are

$$\begin{aligned} v_\mathrm{peri} &= \frac{L}{m r_\mathrm{peri}} = \frac{r_0 v_0 (1+\delta\cos\alpha)}{\dfrac{r_0(1+\varepsilon\cos\phi)}{1+\varepsilon}} = v_0 \frac{(1+\varepsilon)(1+\delta\cos\alpha)}{1+\varepsilon\cos\phi}, \\ v_\mathrm{apo} &= \frac{L}{m r_\mathrm{apo}} = \frac{r_0 v_0 (1+\delta\cos\alpha)}{\dfrac{r_0(1+\varepsilon\cos\phi)}{1-\varepsilon}} = v_0 \frac{(1-\varepsilon)(1+\delta\cos\alpha)}{1+\varepsilon\cos\phi} \end{aligned} \tag{13}$$

Conservation of energy gives

$$E = -\frac{C}{r} + \frac{1}{2}mv^2 = -\frac{C}{r} + \frac{L^2}{2mr^2} \tag{14}$$

At apogee and perigee,

$$-\frac{C}{r_{\text{apo}}} + \frac{1}{2}mv_{\text{apo}}^2 = -\frac{C}{r_{\text{peri}}} + \frac{1}{2}mv_{\text{peri}}^2,$$

that is

$$\begin{aligned}-&\frac{(1-\varepsilon)C}{r_0(1+\varepsilon\cos\phi)} + \frac{1}{2}m\left[v_0\frac{(1-\varepsilon)(1+\delta\cos\alpha)}{1+\varepsilon\cos\phi}\right]^2 \\ &= -\frac{(1+\varepsilon)C}{r_0(1+\varepsilon\cos\phi)} + \frac{1}{2}m\left[v_0\frac{(1+\varepsilon)(1+\delta\cos\alpha)}{1+\varepsilon\cos\phi}\right]^2.\end{aligned}$$

After simplification,

$$\frac{C}{r_0} = mv_0^2\frac{(1+\delta\cos\alpha)^2}{1+\varepsilon\cos\phi},$$

and using the expression for C this reads

$$1+\varepsilon\cos\phi = (1+\delta\cos\alpha)^2,$$

or

$$\varepsilon\cos\phi = \delta\cos\alpha\,(2+\delta\cos\alpha) \tag{15}$$

Just after firing ($t=0$) the ratio of the radial to the tangential speed along the orbit is

$$\left.\frac{v_r}{v_\theta}\right|_{t=0} = \frac{\delta\sin\alpha}{1+\delta\cos\alpha} = \frac{1}{r_0}\left(\frac{\mathrm{d}r_{\mathrm{c}}}{\mathrm{d}\theta_c}\right)_{\theta_c=0} = \frac{\varepsilon\sin\phi}{1+\varepsilon\cos\phi} \tag{16}$$

that is

$$\begin{aligned}\varepsilon\sin\phi &= (1+\varepsilon\cos\phi)\frac{\delta\sin\alpha}{1+\delta\cos\alpha} \\ &= (1+2\delta\cos\alpha+\delta^2\cos^2\alpha)\frac{\delta\sin\alpha}{1+\delta\cos\alpha} \\ &= \delta\sin\alpha\,(1+\delta\cos\alpha)\end{aligned} \tag{17}$$

From Eqs. (15) and (17),

$$\varepsilon = \delta\sqrt{1+3\cos^2\alpha+2\delta\cos\alpha(1+\cos^2\alpha)+\delta^2\cos^2\alpha} \tag{18}$$

Method 2. Express the velocity after firing in terms of the firing parameters. Take polar coordinates; the radial velocity (along the radius vector from the centre of the circular orbit, i.e. the focus, to the position of the chaser c) is

$$v_r = v_0\delta\sin\alpha,$$

the tangential velocity (perpendicular to the radius vector) is

$$v_\theta = v_0(1+\delta\cos\alpha) \tag{11}$$

Conservation of the angular momentum L gives

$$L = mr_0v_\theta = mr_0v_0(1+\delta\cos\alpha) \tag{12}$$

From Eqs. (4) and (12),

$$R = \frac{[mr_0v_0(1+\delta\cos\alpha)]^2}{m^2r_0v_0^2} = r_0(1+\delta\cos\alpha)^2 \tag{13$'$}$$

From Eqs. (7) and (13′),

$$(1+\varepsilon\cos\phi) = (1+\delta\cos\alpha)^2 \tag{14$'$}$$

so

$$\varepsilon\cos\phi = \delta\cos\alpha\,(2+\delta\cos\alpha) \tag{15}$$

The energy after the change of velocity is

$$E = -\frac{GMm}{r_0} + \frac{1}{2}m\left[v_0^2(1+\delta\cos\alpha)^2 + v_0^2\delta^2\sin^2\alpha\right] \tag{16$'$}$$

Using $C = GMm = mr_0v_0^2$,

$$E = -mv_0^2 + \frac{1}{2}m\left[v_0^2(1+\delta\cos\alpha)^2 + v_0^2\delta^2\sin^2\alpha\right] \tag{17$'$}$$

Substituting Eq. (17′) into Eq. (5),

$$\varepsilon = \sqrt{1+\frac{2EL^2}{m^3r_0^2v_0^4}} = \sqrt{1+\frac{2ER}{mr_0v_0^2}} = \delta\sqrt{1+3\cos^2\alpha+2\delta\cos\alpha(1+\cos^2\alpha)+\delta^2\cos^2\alpha} \tag{18}$$

(5) The two meet at the time t, which requires

$$t = n_cT_c = n_AT_A - \frac{\theta_0}{2\pi}T_A \tag{19}$$

From Eqs. (10) and (19),

$$n_\mathrm{A} = \frac{\theta_0}{2\pi} + n_\mathrm{c}\frac{T_\mathrm{c}}{T_\mathrm{A}} = \frac{\theta_0}{2\pi} + n_\mathrm{c}\left(\frac{1+\varepsilon\cos\phi}{1-\varepsilon^2}\right)^{3/2} \tag{20}$$

(6) Substituting Eqs. (15) and (18) into Eq. (20),

$$n_\mathrm{A}(\alpha) = \frac{\theta_0}{2\pi} + n_\mathrm{c}\left[f(\alpha)\right]^{\frac{3}{2}},$$

where

$$f(\alpha) = \frac{1+\delta\cos\alpha(2+\delta\cos\alpha)}{1-\delta^2\left[1+3\cos^2\alpha+2\delta\cos\alpha(1+\cos^2\alpha)+\delta^2\cos^2\alpha\right]} \tag{21}$$

Differentiating with respect to α and setting

$$\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha} = 0 \tag{22}$$

that is

$$\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha} = -\sin\alpha\,\frac{\mathrm{d}f(\cos\alpha)}{\mathrm{d}\cos\alpha} = 0,$$

we obviously have the solutions

$$\alpha = 0,\ \pi \tag{23}$$

(7)(i) If $\alpha = \alpha_0 = 0$, Eq. (18) gives

$$\varepsilon = \delta\sqrt{4 + 4\delta + \delta^2} = \delta(\delta + 2) \le 1 \tag{24}$$

hence

$$0 < \delta \le \delta_{\max} = \sqrt{2} - 1 \qquad (\text{or } 0 < \delta < 0.5) \tag{25}$$

Alternative branch. If $\alpha = \alpha_0 = \pi$, Eq. (18) gives

$$\varepsilon = \delta\sqrt{\delta^2 - 4\delta + 4} = \delta\,|\delta - 2| \le 1 \tag{24}$$

For $\delta \le 2$,

$$\varepsilon = \delta(2 - \delta) \le 1,$$

which gives

$$-(\delta - 1)^2 \le 0,$$

always true. For $\delta > 2$,

$$\varepsilon = \delta(\delta - 2) \le 1,$$

which gives

$$2 < \delta \le \sqrt{2} + 1.$$

Taken together, if $\alpha_0 = \pi$,

$$0 < \delta \le \delta_{\max} = \sqrt{2} + 1 \tag{25}$$

(ii) If $\alpha = \alpha_0 = 0$, Eqs. (20) and (21) give

$$\left(\frac{2\pi n_{\mathrm{A}} - \theta_0}{2\pi n_{\mathrm{c}}}\right)^{2/3} = \frac{1 + \delta(2 + \delta)}{1 - \delta^2\left[4 + 4\delta + \delta^2\right]} = \frac{1}{1 - \delta(2 + \delta)} \tag{26}$$

or

$$\delta^2 + 2\delta + \left(\frac{2\pi n_{\mathrm{c}}}{2\pi n_{\mathrm{A}} - \theta_0}\right)^{2/3} - 1 = 0.$$

The condition for it to have a solution is that the discriminant satisfies

$$\Delta = 2^2 - 4\left[\left(\frac{2\pi n_{\mathrm{c}}}{2\pi n_{\mathrm{A}} - \theta_0}\right)^{2/3} - 1\right] \ge 0.$$

For $\theta_0 = \dfrac{\pi}{2}$, $n_{\mathrm{A}} = 2$, $n_{\mathrm{c}} = 1$ (it is easy to check that $\Delta \ge 0$), the above becomes

$$\delta^2 + 2\delta + \left(\frac{4}{7}\right)^{2/3} - 1 = 0$$

whose solution is

$$0 < \delta(\alpha = 0) = -1 + \sqrt{2 - \left(\frac{4}{7}\right)^{2/3}} = 0.145 < \sqrt{2} - 1 \tag{27}$$

the solution $\delta < 0$ having been discarded.

Alternative branch. If $\alpha = \alpha_0 = \pi$, Eqs. (20) and (21) give

$$\left(\frac{2\pi n_{\mathrm{A}} - \theta_0}{2\pi n_{\mathrm{c}}}\right)^{2/3} = \frac{1 - \delta(2 - \delta)}{1 - \delta^2\left[4 - 4\delta + \delta^2\right]} = \frac{1}{1 + \delta(2 - \delta)} \tag{26}$$

or

$$\delta^2 - 2\delta + \left(\frac{2\pi n_\mathrm{c}}{2\pi n_\mathrm{A} - \theta_0}\right)^{2/3} - 1 = 0.$$

The condition for it to have a solution is that the discriminant satisfies

$$\Delta = (-2)^2 - 4\left[\left(\frac{2\pi n_\mathrm{c}}{2\pi n_\mathrm{A} - \theta_0}\right)^{2/3} - 1\right] \geq 0.$$

For $\theta_0 = \frac{\pi}{2}$, $n_\mathrm{A} = 2$, $n_\mathrm{c} = 1$ (it is easy to check that $\Delta \geq 0$), the above becomes

$$\delta^2 - 2\delta + \left(\frac{4}{7}\right)^{2/3} - 1 = 0$$

whose solution is

$$0 < \delta(\alpha = \pi) = 1 + \sqrt{2 - \left(\frac{4}{7}\right)^{2/3}} = 2.145 < \sqrt{2} + 1 \tag{27}$$

the solution $\delta < 0$ having been discarded.

Problem 3 (dark-matter detection by nuclear recoil)

Solution

(1)(i)

Method 1. Note that the masses of the dark-matter particle and of the nucleus are equal, so in the centre-of-mass frame the momenta of the dark-matter particle and of the nucleus are both $p_0' = Mv_0'$:

$$\boldsymbol{p}_\mathrm{X}' = -\boldsymbol{p}_\mathrm{Xe}', \qquad |\boldsymbol{p}_\mathrm{X}'| = p_0' \tag{1}$$

Transforming the particle momenta from the centre-of-mass frame (quantities primed) to the laboratory frame (quantities unprimed),

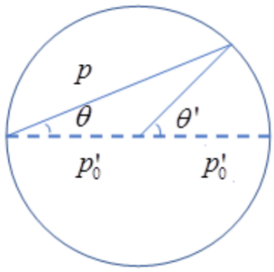
$$p_x = p_x', \quad p_y = p_y', \quad p_z = p_z' + p_0 = p_z' + Mv_0' \tag{2}$$

The momentum relation between the centre-of-mass frame and the laboratory frame is shown in solution Fig. 3a. From the figure it is easy to see that if the scattering angle in the centre-of-mass frame is θ' and the scattering angle in the laboratory frame is θ, then in the laboratory frame

$$\theta = \frac{\theta'}{2}, \tag{3}$$

so the relation between the momentum of the ^{132}Xe nucleus and its outgoing angle is

$$\begin{aligned} &p(\theta) = 2p_0' \cos\theta = 2Mv_0' \cos\theta = Mv_0 \cos\theta, \quad 0 \leq \theta \leq \frac{\pi}{2},\ v_0 = 2v_0'; \\ &p(\theta) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \frac{\pi}{2} < \theta \leq \pi. \end{aligned} \tag{4}$$



解题图 3a

Labels: 解题图 3a = *Solution Fig. 3a.*

Method 2. In the centre-of-mass frame X and Xe collide head-on with the same speed $v_0' = 100\,\mathrm{km/s}$, and separate again with the same speed, as shown in solution Fig. 3b. Let θ' be the angle between the direction of the recoil velocity of the Xe particle and the direction of incidence of the X particle (taken as the Z axis) (quantities in the centre-of-mass frame are primed, those in the laboratory frame unprimed). Going over to the laboratory frame, let v be the magnitude of the velocity of the Xe particle after the collision and θ the angle it makes with the Z axis, as shown in solution Fig. 3c. Then

$$\begin{aligned} v &= \sqrt{v_z^2 + v_x^2} \\ &= v_0'\sqrt{(1+\cos\theta')^2 + \sin^2\theta'} \\ &= 2v_0'\sqrt{\cos^4\frac{\theta'}{2} + \sin^2\frac{\theta'}{2}\cos^2\frac{\theta'}{2}} \\ &= 2v_0'\cos\frac{\theta'}{2}, \end{aligned} \tag{1}$$

and therefore

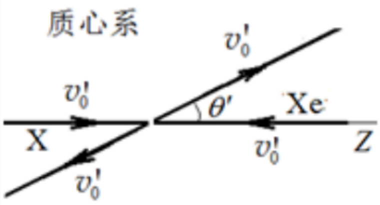
$$\cos\theta = \frac{v_z}{v} = \frac{1+\cos\theta'}{2\cos\dfrac{\theta'}{2}} = \cos\frac{\theta'}{2}, \tag{2}$$

whence

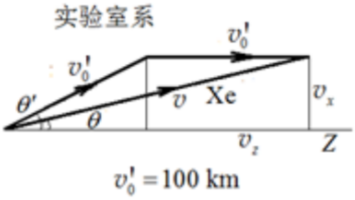
$$\theta = \frac{\theta'}{2}, \tag{3}$$

so the relation between the momentum of the ^{132}Xe nucleus and its outgoing angle is

$$\begin{aligned} &p(\theta) = 2Mv_0'\cos\theta = Mv_0\cos\theta, \quad 0 \le \theta \le \frac{\pi}{2},\ v_0 = 2v_0'; \\ &p(\theta) = 0, \qquad\qquad\qquad\qquad\qquad \frac{\pi}{2} < \theta \le \pi. \end{aligned} \tag{4}$$



解题图 3b



解题图 3c

Labels: 质心系 = *centre-of-mass frame;* 实验室系 = *laboratory frame;* 解题图 3b = *Solution Fig. 3b;* 解题图 3c = *Solution Fig. 3c.*

Method 3. By the problem, in the laboratory frame the initial speed of the incident dark-matter particle X is $v_0 = 200\,\mathrm{km/s}$ and its momentum has magnitude $p_0 = Mv_0$, M being the mass of the dark-matter particle X. Let the momentum of the ^{132}Xe nucleus after the collision have magnitude p and make the angle θ with the initial velocity v_0, as shown in solution Fig. 3d; let the momentum of the dark-matter particle X have magnitude p_X and make the angle θ_X with the initial velocity v_0. Conservation of momentum gives

$$p_0 = p\cos\theta + p_\mathrm{X}\cos\theta_\mathrm{X} \tag{1}$$

$$p\sin\theta = p_\mathrm{X}\sin\theta_\mathrm{X} \tag{2}$$

From Eqs. (1) and (2) (or directly from the vector triangle, see solution Fig. 3e),

$$p_\mathrm{X}^2 = p^2 + p_0^2 - 2pp_0\cos\theta.$$

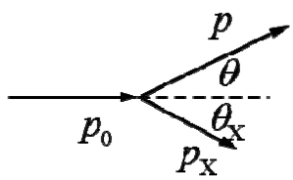
Conservation of energy gives

$$\frac{p_0^2}{2M} = \frac{p^2}{2M} + \frac{p_\mathrm{X}^2}{2M} \tag{3}$$

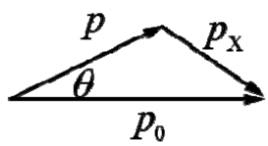
Dividing the previous equation by $2M$ and using Eq. (3), the relation between the momentum of the ^{132}Xe nucleus and its outgoing angle θ is

$$\begin{aligned} &p(\theta) = p_0\cos\theta = Mv_0\cos\theta, && 0 \le \theta \le \frac{\pi}{2}; \\ &p(\theta) = 0, && \frac{\pi}{2} < \theta \le \pi. \end{aligned} \tag{4}$$

Here the value of p at θ has been written $p(\theta)$.



解题图 3d



解题图 3e

Labels: 解题图 3d = *Solution Fig. 3d;* 解题图 3e = *Solution Fig. 3e.*

(ii)

Method 1. In the centre-of-mass frame the scattering is isotropic, i.e. the scattering probability is the same in every direction. From solution Fig. 3a one sees that the circumference of the circle traced out at the position of the scattering angle θ' in the centre-of-mass frame, going once round the z axis (as shown by the dashed line in solution Fig. 3a), is

$$2\pi p_0'\sin\theta', \tag{5}$$

so the scattering probability per unit angle θ' must be proportional to $\sin\theta'$. Therefore, in the centre-of-mass frame, the angular distribution of the scattering probability P is

$$\frac{\mathrm{d}P}{\mathrm{d}\theta'} = a\sin\theta' \tag{6}$$

The proportionality constant a follows from the normalization condition

$$\int_0^\pi \frac{\mathrm{d}P}{\mathrm{d}\theta'}\,\mathrm{d}\theta' = 1 \tag{7}$$

which gives

$$a = \frac{1}{2} \tag{8}$$

so that

$$\frac{\mathrm{d}P}{\mathrm{d}\cos\theta'} = \frac{\mathrm{d}P}{\mathrm{d}\theta'}\left|\frac{\mathrm{d}\theta'}{\mathrm{d}\cos\theta'}\right| = a\sin\theta'\frac{1}{\sin\theta'} = \frac{1}{2} \tag{9}$$

where $\left|\frac{\mathrm{d}\theta'}{\mathrm{d}\cos\theta'}\right|$ is the change-of-measure factor of the probability integral. Going over to the laboratory frame,

$$\mathrm{d}\cos\theta' = \mathrm{d}\cos 2\theta = \mathrm{d}(2\cos^2\theta - 1) = 4\cos\theta\,\mathrm{d}\cos\theta$$

so that

$$\frac{\mathrm{d}P}{\mathrm{d}\cos\theta} = 2\cos\theta, \qquad 0 \le \theta \le \frac{\pi}{2} \tag{10}$$

(Note: the angular distribution (10) may be written in the following equivalent ways,

$$\frac{\mathrm{d}P}{\mathrm{d}\theta} = \sin(2\theta), \qquad 0 \le \theta \le \frac{\pi}{2} \tag{10}$$

or

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mathrm{d}P}{\mathrm{d}\cos\theta\,\mathrm{d}\phi} = \frac{1}{\pi}\cos\theta, \qquad 0 \le \theta \le \frac{\pi}{2} \tag{10}$$

both are acceptable, Ω being the solid angle.)

Method 2. In the centre-of-mass frame the Xe particle is emitted completely isotropically after the collision, that is, its angular distribution function is a constant C:

$$\rho'(\theta',\phi') = C \tag{5}$$

The normalization condition

$$\int_0^{2\pi}\int_{-1}^{1} \rho'(\theta',\phi')\,\mathrm{d}\cos\theta'\,\mathrm{d}\phi' = 1$$

gives

$$C\int_0^{2\pi}\int_{-1}^{1} \mathrm{d}\cos\theta'\,\mathrm{d}\phi' = 4\pi C = 1$$

so

$$\rho'(\theta',\phi') = C = \frac{1}{4\pi} \tag{6}$$

The probability of a particular event does not depend on the reference frame, so in spherical coordinates

$$\rho'(\theta',\phi')\,\mathrm{d}\cos\theta'\,\mathrm{d}\phi' = \rho(\theta,\phi)\,\mathrm{d}\cos\theta\,\mathrm{d}\phi \tag{7}$$

Moreover, since the system is axially symmetric about the z axis, $\rho(\theta,\phi)$ does not depend on ϕ, so

$$\rho(\theta,\phi) = \rho(\theta) \tag{8}$$

In the transformation from the centre-of-mass frame to the laboratory frame (along the z axis) the azimuthal angle is unchanged:

$$\phi' = \phi.$$

From Eqs. (7), (8) and the last equation,

$$\rho(\theta) = \frac{1}{4\pi}\frac{\mathrm{d}\cos\theta'\,\mathrm{d}\phi'}{\mathrm{d}\cos\theta\,\mathrm{d}\phi} = \frac{1}{4\pi}\frac{\mathrm{d}\cos\theta'}{\mathrm{d}\cos\theta} \tag{9}$$

Using $\theta = \dfrac{\theta'}{2}$ and the above,

$$\begin{aligned}\rho(\theta) &= \frac{1}{4\pi}\frac{\mathrm{d}\cos(2\theta)/\mathrm{d}\theta}{\mathrm{d}\cos\theta/\mathrm{d}\theta} = \frac{1}{4\pi}\frac{2\sin(2\theta)}{\sin\theta} = \frac{1}{\pi}\cos\theta, && \left(0 \le \theta \le \frac{\pi}{2}\right)\\ \rho(\theta) &= 0, && \left(\frac{\pi}{2} < \theta \le \pi\right)\end{aligned} \tag{10}$$

Method 3. The momentum distribution function of the outgoing particle may be written $\rho'(p'_x, p'_y, p'_z)$, satisfying the normalization condition

$$\int \mathrm{d}p'_x\,\mathrm{d}p'_y\,\mathrm{d}p'_z\,\rho'(p'_x, p'_y, p'_z) = 1.$$

Since the scattering is isotropic, the probability density ρ here is a function only of the magnitude p' of the momentum, so

$$\rho'(p'_x, p'_y, p'_z) = \rho'(p').$$

For a two-body elastic collision the magnitude of the momentum of the outgoing particle is definite in the centre-of-mass frame; let it be p_0. In spherical coordinates the integral over the solid angle can be done at once:

$$\int p'^2\,\mathrm{d}p'\,\mathrm{d}\phi'\,\mathrm{d}\cos\theta'\,\rho'(p') = 4\pi\int \rho'(p')p'^2\,\mathrm{d}p' = 1.$$

Hence, in the centre-of-mass frame,

$$\rho'(p') = \frac{1}{4\pi p_0^2}\delta(p' - p_0) \tag{5}$$

or

$$\rho'(p') = \frac{1}{4\pi p'^2}\delta(p' - p_0).$$

(Since the δ function is defined inside an integral, the two forms above are in fact equivalent.) From the centre-of-mass frame (quantities unprimed) to the laboratory frame (quantities primed) we have

[Translator's note: As printed; here the source interchanges the primed/unprimed convention used elsewhere. The relations that follow are the usual ones: primed = centre-of-mass frame, unprimed = laboratory frame.]

$$p_x = p'_x, \quad p_y = p'_y, \quad p_z = p'_z + p_0.$$

Going over to laboratory spherical coordinates,

$$\phi = \phi', \quad p\sin\theta = p'\sin\theta', \quad p\cos\theta = p'\cos\theta' + p_0,$$

from which

$$p' = (p^2 - 2pp_0\cos\theta + p_0^2)^{1/2} \tag{6}$$

or

$$\cos\theta' = \frac{p\cos\theta - p_0}{(p^2 - 2pp_0\cos\theta + p_0^2)^{1/2}}.$$

The probability of a particular event does not depend on the reference frame, so

$$\rho'(p'_x, p'_y, p'_z)\,\mathrm{d}p'_x\mathrm{d}p'_y\mathrm{d}p'_z = \rho(p_x, p_y, p_z)\,\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z,$$

which in spherical coordinates becomes

$$\rho'(p')p'^2\,\mathrm{d}p'\,\mathrm{d}\phi'\,\mathrm{d}\cos\theta' = \rho(p)p^2\,\mathrm{d}p\,\mathrm{d}\phi\,\mathrm{d}\cos\theta \tag{7}$$

The integration measure $\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$ is invariant under a Galilean transformation, i.e.

$$\mathrm{d}p'_x\mathrm{d}p'_y\mathrm{d}p'_z = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z,$$

which can be written in spherical coordinates as

$$p'^2\,\mathrm{d}p'\,\mathrm{d}\cos\theta'\,\mathrm{d}\phi' = p^2\,\mathrm{d}p\,\mathrm{d}\cos\theta\,\mathrm{d}\phi \tag{8}$$

From Eqs. (5)–(8) we obtain, in the laboratory frame,

$$\rho(p,\cos\theta) = \frac{1}{4\pi p_0^2}\delta\Big[(p^2 - 2p_0p\cos\theta' + p_0^2)^{1/2} - p_0\Big] \tag{9}$$

[Translator's note: As printed; the angle inside the δ function should be θ, not θ', as the next equation shows.]

Therefore, in the laboratory frame, the angular distribution of the probability P that a nucleus is struck is

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}\cos\theta} &= \int p^2\,\mathrm{d}p\,\mathrm{d}\phi\,\rho(p, p\cos\theta) \\
&= \int p^2\,\mathrm{d}p\,\mathrm{d}\phi\,\frac{1}{4\pi p_0^2}\delta\Big[(p^2 - 2p_0p\cos\theta + p_0^2)^{1/2} - p_0\Big] \\
&= \int p^2\left|\frac{\mathrm{d}p}{\mathrm{d}p'}\right|\mathrm{d}p'\,\frac{1}{2p_0^2}\delta(p'-p_0) \\
&= \int p^2\left|\frac{p'}{p - p_0\cos\theta}\right|\mathrm{d}p'\,\frac{1}{2p_0^2}\delta(p'-p_0) \\
&= (2p_0\cos\theta)^2\left|\frac{p_0}{2p_0\cos\theta - p_0\cos\theta}\right|\frac{1}{2p_0^2} \\
&= 2\left|\cos\theta\right|,
\end{aligned} \qquad 0 \le \theta \le \frac{\pi}{2} \tag{10}$$

Here we have used Eq. (6) together with

$$\frac{\partial p}{\partial p'} = \frac{p'}{p - p_0\cos\theta}$$

and the fact that, when $p' = p_0$,

$$p = 0,\ 2p_0\cos\theta,$$

the former giving no contribution. Note that in the laboratory frame the outgoing angle θ' of the nucleus lies in $[0, \pi/2]$ and not in $[0, \pi]$; see the kinematic constraint $\theta' = \theta/2$ marked in Fig. 1.

[Translator's note: As printed. The intended statement is that the laboratory outgoing angle θ lies in $[0, \pi/2]$, because of the kinematic constraint $\theta = \theta'/2$ shown in solution Fig. 3a.]

(iii)
Method 1. Consider the kinetic-energy distribution of the outgoing particle. From $p = 2p_0' \cos\theta = p_0 \cos\theta$ we get

$$\cos\theta = \frac{p}{p_0},$$

so

$$\mathrm{d}\cos\theta = \frac{\mathrm{d}p}{p_0} \tag{11}$$

Substituting into Eq. (10) gives the momentum distribution of the struck nucleus,

$$\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{2p}{p_0^2} \tag{12}$$

The kinetic energy of the outgoing Xe nucleus in the laboratory frame is

$$E_\mathrm{k} = \frac{p^2}{2m_\mathrm{Xe}},$$

where m_Xe is the mass of the Xe nucleus. From this,

$$\frac{\mathrm{d}P}{\mathrm{d}E_\mathrm{k}} = \frac{2m_\mathrm{Xe}}{p_0^2} = \frac{2M}{p_0^2} = \frac{1}{E_\mathrm{k}^\mathrm{in}} = \frac{2M}{p_0^2} = \frac{1}{\frac{1}{2}Mv_0^2} \tag{13}$$

Here we used

$$m_\mathrm{Xe} = M,$$

while

$$E_\mathrm{k}^\mathrm{in} = \frac{p_0^2}{2m_\mathrm{X}} = \frac{p_0^2}{2M} \tag{14}$$

is the kinetic energy of the incident dark-matter particle in the laboratory frame. Thus, for the dark-matter model assumed in the problem, in the laboratory frame the kinetic energy of the outgoing nucleus is uniformly distributed between 0 and E_k^in.
Method 2. The kinetic-energy distribution function of the Xe particle satisfies

$$\rho(\theta)\, 2\pi \sin\theta \,\mathrm{d}\theta = \rho(E_\mathrm{k})\,\mathrm{d}E_\mathrm{k} \tag{11}$$

The relation between the kinetic energy E_k of the Xe particle and its outgoing angle θ is

$$\begin{aligned} &E_\mathrm{k}(\theta) = \frac{1}{2}Mv_0^2\cos^2\theta, && \left(0 \le \theta \le \frac{\pi}{2}\right) \\ &E_\mathrm{k}(\theta) = 0, && \left(\frac{\pi}{2} < \theta \le \pi\right) \end{aligned} \tag{12}$$

From Eqs. (11) and (12),

$$\rho(E_\mathrm{k}) = \frac{2\cos\theta\sin\theta}{Mv_0^2\cos\theta\sin\theta} = \frac{1}{\frac{1}{2}Mv_0^2}, \qquad \left(0 \le E_\mathrm{k} \le \frac{1}{2}Mv_0^2\right) \tag{13}$$

Therefore the kinetic energy of the Xe particle is uniformly distributed over the range $0 \le E_\mathrm{k} \le \frac{1}{2}Mv_0^2$ (corresponding to $0 \le \theta \le \frac{\pi}{2}$).
(2) The total number of nuclei is

$$N_\mathrm{Xe} = \frac{M_\mathrm{total}}{m_\mathrm{Xe}} = \frac{1.0\,\mathrm{t}}{132\,\mathrm{g/mol}} \times N_\mathrm{A} \approx 4.6 \times 10^{27} \tag{15}$$

where N_{A} is Avogadro's number. The probability that one nucleus is hit by a dark-matter particle within one year is

$$P_1 = \sigma v t \, \frac{\rho_{\mathrm{X}}}{m_{\mathrm{X}}} \tag{16}$$

where σ is the collision cross-section, v is the speed of the dark matter, t is the total time, and ρ_{X} and m_{X} are respectively the density of dark matter and the mass of a dark-matter particle X. Substituting the numbers,

$$P_1 \approx 1.9 \times 10^{-26} \tag{17}$$

This probability is far smaller than 1, so we may neglect the case in which one nucleus is hit by two dark-matter particles. Hence, in one year, the total number of collisions in this detector may be estimated as

$$N_{\mathrm{C}} = P_1 N_{\mathrm{Xe}} \approx 87 \tag{18}$$

so about 87 collisions occur.

(3) Since the kinetic energy of the outgoing nuclei produced by the collisions is uniformly distributed between 0 and $E_{\mathrm{k}}^{\mathrm{in}}$, the number with outgoing kinetic energy larger than a threshold E_{thr} is

$$N_{\mathrm{S}} = N_{\mathrm{C}} \, \frac{E_{\mathrm{k}}^{\mathrm{in}} - E_{\mathrm{thr}}}{E_{\mathrm{k}}^{\mathrm{in}}} \tag{19}$$

Substituting the data of the problem into Eq. (14),

$$E_{\mathrm{k}}^{\mathrm{in}} \approx 28\,\mathrm{keV} \tag{20}$$

By the problem,

$$E_{\mathrm{thr}} = 10\,\mathrm{keV}.$$

From Eqs. (17), (18) and (19),

$$N_{\mathrm{S}} = 56 \tag{21}$$

The number of dark-matter signals produced in this detector per year is 56.

(4) This cut-off is caused by the escape speed: a dark-matter particle whose speed exceeds the escape speed may escape from the gravitational binding of the Milky Way.

(5) Let m_{X} be the mass of the dark-matter particle and v_{X} its speed in the laboratory frame. The speed of the centre of mass c of the dark-matter particle and the stationary xenon nucleus relative to the laboratory frame is

$$v_{\mathrm{c}} = v_{\mathrm{X}} \, \frac{m_{\mathrm{X}}}{m_{\mathrm{X}} + m_{\mathrm{Xe}}}. \tag{22}$$

In the laboratory frame the maximum recoil speed that the xenon nucleus can obtain in a collision with a dark-matter particle is

$$v_{\mathrm{r\,max}} = 2 v_{\mathrm{c}} = 2 v_{\mathrm{X}} \frac{m_{\mathrm{X}}}{m_{\mathrm{X}} + m_{\mathrm{Xe}}}. \tag{23}$$

The maximum recoil kinetic energy that the xenon nucleus can obtain is

$$E_{\mathrm{r\,max}} = \frac{1}{2} m_{\mathrm{Xe}} v_{\mathrm{r\,max}}^2 = \frac{1}{2} m_{\mathrm{X}} v_{\mathrm{X}}^2 \, \frac{4 m_{\mathrm{X}} m_{\mathrm{Xe}}}{(m_{\mathrm{X}} + m_{\mathrm{Xe}})^2} \tag{24}$$

The condition for the laboratory not to be able to detect a dark-matter signal is

$$E_{\mathrm{r\,max}} < E_{\mathrm{thr}}, \quad \text{when } v_{\mathrm{X}} = v_{\max}, \tag{25}$$

From Eqs. (23), (24) and (25), when

$$m_{\mathrm{X}} < \frac{m_{\mathrm{Xe}}}{v_{\max}\sqrt{\dfrac{2m_{\mathrm{Xe}}}{E_{\mathrm{thr}}}} - 1} = \frac{m_{\mathrm{Xe}}}{\dfrac{v_{\max}}{c}\sqrt{\dfrac{2m_{\mathrm{Xe}}c^2}{E_{\mathrm{thr}}}} - 1} = 14\ \mathrm{GeV}/c^2 \tag{26}$$

the detector cannot detect a dark-matter signal.

Problem 4 (charged cell membranes and the counter-ion cloud)

Solution

(1)(i) Near the point x outside the biological-membrane surface take a small volume with unit-area base and thickness $x + \mathrm{d}x$ (its top and bottom both parallel to the membrane surface). By symmetry the electric field strength outside the biological-membrane surface depends only on x, so the field strengths at the top and the bottom of the small volume may be written $E(x + \mathrm{d}x)$ and $E(x)$, the positive x direction being taken as positive. Gauss's theorem gives

$$E(x + \mathrm{d}x) - E(x) = \frac{1}{\varepsilon} en(x)\,\mathrm{d}x,$$

while

$$E(x) = -\frac{\mathrm{d}\varphi}{\mathrm{d}x},$$

so that the two equations give

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = -\frac{e}{\varepsilon} n(x). \tag{1}$$

The electrostatic potential energy of a positive ion of charge $+e$ in the above field is $e\varphi$. Boltzmann statistics gives

$$n(x) = n_0 \mathrm{e}^{-\frac{e\varphi(x)}{kT}}, \tag{2}$$

where n_0 is the number density of the positive ions where $\varphi = 0$, so that the equation satisfied by $\varphi(x)$ is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = -\frac{en_0}{\varepsilon}\mathrm{e}^{-\frac{e\varphi}{kT}}. \tag{3}$$

(ii) The boundary conditions of $\varphi(x)$ are

$$\varphi|_{x=0} = 0, \tag{4}$$

$$\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=0} = -E|_{x=0} = -\frac{(-\sigma)}{\varepsilon}, \qquad (\sigma > 0) \tag{5}$$

Here we have noted that the surface charge density of the outer membrane surface is $-\sigma$ and that the interior of the membrane may be regarded as a conductor.

(iii) Substituting the trial solution $\varphi(x) = A\ln(1 + Bx)$ (which already satisfies Eq. (4)) into Eq. (3),

$$-\frac{AB^2}{(1 + Bx)^2} = -\frac{en_0}{\varepsilon}(1 + Bx)^{-\frac{eA}{kT}},$$

and comparing the powers of $(1 + Bx)$ and the coefficients on the two sides,

$$A = \frac{2kT}{e}, \tag{6}$$

$$B = \sqrt{\frac{n_0 e^2}{2\varepsilon kT}}, \tag{7}$$

Substituting the trial solution $\varphi(x) = A\ln(1+Bx)$ into Eq. (5),

$$AB = \frac{\sigma}{\varepsilon},$$

so

$$B = \frac{\sigma e}{2\varepsilon kT}, \tag{8}$$

From Eqs. (7) and (8),

$$n_0 = \frac{\sigma^2}{2\varepsilon kT}. \tag{9}$$

From the trial solution and Eqs. (6), (8),

$$\varphi(x) = \frac{2kT}{e}\ln\left(1 + \frac{\sigma e}{2\varepsilon kT}x\right), \tag{10}$$

and from Eqs. (2) and (9),

$$n(x) = \frac{\sigma^2}{2\varepsilon kT}\frac{1}{\left(1 + \frac{\sigma e}{2\varepsilon kT}x\right)^2}. \tag{11}$$

(iv) The total charge in the infinitely tall cylindrical volume whose base is a unit area of the outer membrane surface is

$$Q = e\int_0^\infty n(x)\,\mathrm{d}x = \frac{\sigma^2 e}{2\varepsilon kT}\int_0^\infty \frac{\mathrm{d}x}{\left(1 + \frac{\sigma e}{2\varepsilon kT}x\right)^2} = \sigma\int_0^\infty \frac{\mathrm{d}y}{(1+y)^2} = \sigma. \tag{12}$$

Hence the whole system (including the outer membrane surface) is electrically neutral.

(v) The electrostatic field energy density satisfies

$$u_{\mathrm{f}} = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2, \tag{13}$$

where

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{2kT}{e}\frac{\frac{\sigma e}{2\varepsilon kT}}{1 + \frac{\sigma e}{2\varepsilon kT}x}, \tag{14}$$

so the total electrostatic field energy in a sufficiently tall cylindrical volume whose base is a unit area of the charged outer membrane surface is

$$\begin{aligned} U_{\mathrm{f}} &= \frac{1}{2}\varepsilon\int_0^\infty\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2\mathrm{d}x = \frac{\sigma kT}{e}\int_0^\infty \frac{\frac{\sigma e}{2\varepsilon kT}}{\left(1 + \frac{\sigma e}{2\varepsilon kT}x\right)^2}\,\mathrm{d}x \\ &= \frac{\sigma kT}{e}\int_0^\infty \frac{\mathrm{d}y}{(1+y)^2} = \frac{\sigma kT}{e}. \end{aligned} \tag{15}$$

On the other hand, the total electrostatic potential energy of all the positive ions in the same volume is

$$\begin{aligned} U_{\mathrm{e}} &= \int_0^\infty en(x)\varphi(x)\,\mathrm{d}x = \frac{\sigma^2}{2\varepsilon kT}2kT\int_0^\infty \frac{\ln\left(1 + \frac{\sigma e}{2\varepsilon kT}x\right)}{\left(1 + \frac{\sigma e}{2\varepsilon kT}x\right)^2}\,\mathrm{d}x = \frac{2\sigma kT}{e}\int_0^\infty \frac{\ln(1+y)}{(1+y)^2}\,\mathrm{d}y \\ &= -\frac{2\sigma kT}{e}\int_0^\infty \ln(1+y)\,\mathrm{d}\frac{1}{1+y} = \frac{2\sigma kT}{e}\int_0^\infty \frac{\mathrm{d}y}{(1+y)^2} = \frac{2\sigma kT}{e}. \end{aligned} \tag{16}$$

Here we have noted that the positive ions considered are only a very small part of all the positive ions, so the charge of the positive ions considered may be regarded as a test charge in the field produced by all the charges (including the whole positive-ion charge and the surface charge on the outer membrane surface).

(2)(i) By analogy with Eq. (3), the differential equation satisfied by φ is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = -\frac{en_0}{\varepsilon}\mathrm{e}^{-\frac{e\varphi}{kT}}, \qquad (-D < x < +D,\ \varphi|_{x=0} = 0) \tag{17}$$

Substituting the trial function $\varphi(x) = A\ln[\cos(Bx)]$ (which satisfies the boundary condition $\varphi|_{x=0} = 0$),

$$AB^2\frac{1}{\cos^2 Bx} = \frac{en_0}{\varepsilon}(\cos Bx)^{-\frac{eA}{kT}},$$

so that

$$A = \frac{2kT}{e}, \tag{18}$$

$$B = \sqrt{\frac{n_0e^2}{2\varepsilon kT}}. \tag{19}$$

(ii) By analogy with Eq. (5), the other boundary condition is

$$\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=-D} = -\frac{(-\sigma)}{\varepsilon}, \qquad (\sigma > 0)$$

Substituting the trial solution $\varphi(x) = A\ln[\cos(Bx)]$,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = -AB\tan Bx, \tag{20}$$

so that

$$AB\tan BD = \frac{\sigma}{\varepsilon},$$

and inserting the expression (18) for A,

$$BD\tan BD = \frac{\sigma eD}{2\varepsilon kT}. \tag{21}$$

Writing $\theta \equiv BD$, it satisfies the transcendental equation

$$\theta\tan\theta = \frac{e\sigma D}{2\varepsilon kT} \equiv \kappa, \qquad 0 < \theta < \frac{\pi}{2}.$$

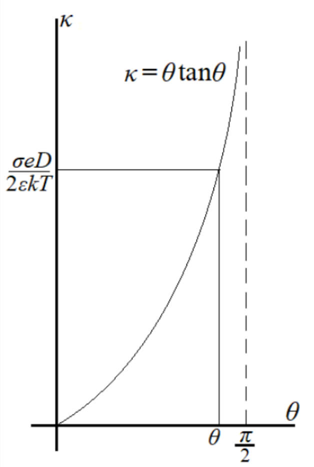
Once D, σ, ε, T are given, $\kappa \equiv \dfrac{\sigma eD}{2\varepsilon kT}$ is a definite constant. In the case considered in this problem σ is very small, even equal to 0; and when $\sigma = 0$ the transcendental equation above obviously has the solution $\theta = 0$. The physical solution must be continuous and cannot cross a singular point. Therefore the physical region of the transcendental equation considered is $0 \le \theta < \dfrac{\pi}{2}$. From solution Fig. 4a one sees that in the physical region $0 \le \theta < \dfrac{\pi}{2}$ the transcendental equation considered has one and only one root θ (indeed, since θ and $\tan\theta$ are both monotonic functions of θ in the region $0 \le \theta < \dfrac{\pi}{2}$, their product $\kappa(\theta)$ is also a monotonic function of θ), which is exactly in accordance with the uniqueness theorem of electrostatics. Once θ has been determined,

$$B = \frac{\theta}{D},$$

so that

$$n_0 = \frac{2\varepsilon kT}{e^2}B^2 = \frac{2\varepsilon kT\theta^2}{e^2D^2}. \tag{22}$$

Thus, given D, σ, ε, T, the value of θ is fixed, and n_0 is then completely determined.



解题图 4a

Labels: 解题图 4a = *Solution Fig. 4a.*

(iii) By analogy with Eq. (2), and using the trial-solution form $\varphi(x) = A\ln[\cos(Bx)]$ together with Eq. (18),

$$n(x) = n_0 \mathrm{e}^{-\frac{e\varphi(x)}{kT}} = n_0 [\cos(Bx)]^{-\frac{eA}{kT}} = n_0 \frac{1}{\cos^2 Bx}, \tag{23}$$

where n_0 and B are as above, so that

$$n(\pm D) = n_0 \frac{1}{\cos^2 BD} = n_0(1 + \tan^2 BD) = n_0 + n_0 \tan^2\theta,$$

while

$$n_0 \tan^2\theta = \frac{2\varepsilon kT}{D^2e^2}(\theta\tan\theta)^2 = \frac{2\varepsilon kT}{D^2e^2}\left(\frac{\sigma eD}{2\varepsilon kT}\right)^2 = \frac{\sigma^2}{2\varepsilon kT},$$

so

$$n(\pm D) - n_0 = \frac{\sigma^2}{2\varepsilon kT}. \tag{24}$$

(iv) Using the expression (23) for $n(x)$,

$$\int_{-D}^{D} en(x)\,\mathrm{d}x = en_0 \int_{-D}^{D} \frac{\mathrm{d}x}{\cos^2 Bx} = \frac{2e}{B}\frac{2\varepsilon kT\theta^2}{e^2D^2}\tan\theta = \frac{2e}{B}\frac{2\varepsilon kT\theta}{e^2D^2}\frac{\sigma eD}{2\varepsilon kT} = 2\sigma, \tag{25}$$

where $\theta\tan\theta = \dfrac{\sigma eD}{2\varepsilon kT}$ and $\theta = BD$. Since the two uniformly negatively charged infinite parallel planes each carry the surface charge density $-\sigma$, the whole system is electrically neutral.

(v) For one of the two charged planes, the other charges of the system are the negative charge on the other plane and the positive charge of the positive-ion cloud; their total charge is exactly equal in magnitude and opposite in sign to the total charge of this plane, which is equivalent to the other plate of a parallel-plate capacitor, so the electrostatic force per unit area that they exert on this plane is

$$f_{\mathrm{e}} = \frac{\sigma^2}{2\varepsilon},$$

directed outwards from the biological membrane, i.e. towards the other plane. As for the pressure produced by the collisions of the positive ions, according to the hint of the problem it is completely analogous to the pressure of an ideal gas (a "positive-ion gas"), so

$$f_{\mathrm{h}} = n(\pm D)kT = \left(n_0 + \frac{\sigma^2}{2\varepsilon kT}\right)kT = \frac{2\varepsilon k^2T^2\theta^2}{e^2D^2} + \frac{\sigma^2}{2\varepsilon}, \tag{26}$$

directed into the biological membrane. The resultant force is

$$f = f_{\mathrm{h}} - f_{\mathrm{e}} = n_0kT = \frac{2\varepsilon k^2T^2\theta^2}{e^2D^2}, \tag{27}$$

directed into the biological membrane. That is to say, the resultant of the electrostatic force and the pressure of the positive-ion gas produces a repulsion between the two charged planes. It is worth pointing out that both inside and outside the membrane there is water, and the water pressures on the two sides of the membrane cancel and are not considered.

Method 2. The plane $x = 0$ divides the system of positive ions (including the corresponding liquid) between the two charged planes into system I and system II. Take for system I the positive ions (including the corresponding liquid) between the plane $x = 0$ and one of the two charged planes; system II is then equivalent to a plate of surface charge density $+\sigma$, and its action on I is exactly cancelled by the action on I of the adjacent charged plate of charge density $-\sigma$, so it need not be considered. Therefore, besides the force exerted on I by the charged plane and directed into I (whose magnitude per unit area f equals the magnitude per unit area of the resultant of the electrostatic force and of the positive-ion-gas pressure experienced by the charged plane), system I also feels the pressure of the positive-ion gas at the plane $x = 0$ (directed into I), whose magnitude per unit area is

$$f_{\mathrm{h0}} = n_0kT \tag{26}$$

Since system I is in mechanical equilibrium,

$$f = f_{\mathrm{h0}}$$

so

$$f = n_0kT = \frac{2\varepsilon k^2T^2\theta^2}{e^2D^2} \tag{27}$$

It is worth pointing out that both inside and outside the membrane there is water, and the water pressures on the two sides of the membrane cancel and are not considered.

(vi) From Eq. (26),

$$f \propto \left(\frac{\theta}{D}\right)^2, \tag{28}$$

and θ satisfies

$$\theta\tan\theta = \frac{\sigma eD}{2\varepsilon kT},$$

so it too depends on D. In the small-distance limit

$$\frac{\sigma eD}{2\varepsilon kT} \ll 1,$$

so one may take $\tan\theta \approx \theta$, and then

$$\theta \approx \sqrt{\frac{\sigma e D}{2\varepsilon k T}} \propto \sqrt{D},$$

so that

$$f \propto \left(\frac{\sqrt{D}}{D}\right)^2 \propto \frac{1}{D}. \tag{29}$$

In the large-distance limit

$$\frac{\sigma e D}{2\varepsilon k T} \gg 1,$$

so one may take

$$\theta \approx \frac{\pi}{2} = \text{constant},$$

and then

$$f \propto \frac{1}{D^2}, \tag{30}$$

Equations (29) and (30) show that f decays as the distance increases.

Problem 5 (paramagnetism, ferromagnetism, antiferromagnetism and vortices)

Solution

(1)(i) By the problem, the magnetic moment of a molecule equals the spin magnetic moment of an electron,

$$\boldsymbol{\mu} = \frac{-e}{m}\boldsymbol{S}.$$

Its energy in the external field $\boldsymbol{B}$ is

$$E = -\boldsymbol{\mu}\cdot\boldsymbol{B} = \frac{e}{m}S_z B = \pm\frac{e\hbar}{2m}B = \pm m_0 B$$

where $m_0 = \dfrac{e\hbar}{2m}$, and we have taken $\boldsymbol{B} = B\hat{z}$, $\hat{z}$ being the unit direction vector.
According to Boltzmann statistics, the number of molecules (the occupation number) in the state of energy E satisfies

$$N_\pm \propto \mathrm{e}^{\mp\frac{m_0 B}{kT}} \tag{1}$$

The total number of molecules is

$$N = N_+ + N_-.$$

The two equations above give

$$\frac{N_\pm}{N} = \frac{\mathrm{e}^{\mp m_0 B/kT}}{\mathrm{e}^{m_0 B/kT} + \mathrm{e}^{-m_0 B/kT}} \tag{2}$$

The average magnetic moment of each molecule is

$$\langle m\rangle = \frac{-m_0 N_+ + m_0 N_-}{N} \tag{3}$$

By the definition of the magnetization,

$$M = n\langle m\rangle = nm_0 \frac{\dfrac{\mathrm{e}^{m_0B/kT} - \mathrm{e}^{-m_0B/kT}}{2}}{\dfrac{\mathrm{e}^{m_0B/kT} + \mathrm{e}^{-m_0B/kT}}{2}} = nm_0 \frac{\mathrm{sh}\,\dfrac{m_0B}{kT}}{\mathrm{ch}\,\dfrac{m_0B}{kT}} = nm_0 \tanh\frac{m_0B}{kT} \tag{4}$$

In the high-temperature approximation

$$kT \gg \frac{e\hbar}{2m}B = m_0B$$

we have

$$\tanh\frac{m_0B}{kT} = \frac{m_0B}{kT}.$$

From

$$M = \chi H = \frac{\chi}{1+\chi}\frac{B}{\mu_0} \approx \frac{\chi}{\mu_0}B \tag{5}$$

where $\chi \ll 1$ has been used, and from Eqs. (4) and (5), in the high-temperature approximation

$$\chi = \frac{\mu_0 n m_0^2}{kT} = \frac{C}{T} \tag{6}$$

and hence

$$C = \frac{\mu_0 n m_0^2}{k} = \frac{\mu_0 n}{k}\left(\frac{e\hbar}{2m}\right)^2 \tag{7}$$

(ii) Above the Curie temperature θ, the Weiss law

$$\chi = \frac{C}{T-\theta}$$

gives

$$M = \chi H = \frac{C}{T-\theta}H,$$

that is

$$\frac{T}{C}M = H + \frac{\theta}{C}M = H + H_m \tag{8}$$

Therefore there exists an additional magnetic field (the so-called “molecular field”)

$$H_\mathrm{m} = \frac{\theta}{C}M,$$

and comparing with the expression given in the problem,

$$H_m = \gamma M,$$

we obtain

$$\gamma = \frac{\theta}{C} \tag{9}$$

In the case in which all the magnetic moments are aligned, Eq. (9) gives

$$H_\mathrm{m} = \gamma M = \frac{\theta}{C}M = \frac{\theta}{\mu_0 n m_0^2/k}\,nm_0 = \frac{k\theta}{\mu_0 m_0} \tag{10}$$

(2)(i) If this molecular field is regarded as being supplied by the nearest-neighbour spins, then the molecular field (effective field) that each spin supplies to its nearest-neighbour molecules is

$$H'_{\mathrm{m}} = \frac{k\theta}{2\mu_0 m_0} \tag{11}$$

Hereafter, for brevity, H'_{m} will be written H_{m}, but understood as an effective field. In this molecular field the spin energy is

$$E = -m_0 B_{\mathrm{m}} = -m_0\mu_0 H_{\mathrm{m}} = -\frac{1}{2}k\theta \tag{12}$$

The energy of the whole system is

$$E_N = -\frac{1}{2}(N-1)k\theta \tag{13}$$

(ii) Before the spin on this non-end lattice site flips, the energy of the molecular field it supplies to its left and right nearest-neighbour spins is

$$\left(-\frac{1}{2}k\theta - \frac{1}{2}k\theta\right) = (-k\theta).$$

After the spin on this non-end lattice site flips, the energy of the molecular field it supplies to its left and right nearest-neighbour spins is

$$\left(+\frac{1}{2}k\theta + \frac{1}{2}k\theta\right) = (+k\theta).$$

Compared with the case of question (2)(i), the energy of the system increases by

$$(+k\theta) - (-k\theta) = 2k\theta \tag{14}$$

(3)(i) The A-site spin energy is

$$\begin{aligned} -\mu_0\boldsymbol{\mu}_{\mathrm{A}}\cdot\boldsymbol{H}_{\mathrm{mA}} &= -\mu_0\boldsymbol{\mu}_{\mathrm{A}}\cdot(-\alpha_{\mathrm{AB}}\boldsymbol{\mu}_{\mathrm{B}} - \alpha_{\mathrm{AA}}\boldsymbol{\mu}_{\mathrm{A}}) \\ &= \mu_0(\alpha\boldsymbol{\mu}_{\mathrm{A}}\cdot\boldsymbol{\mu}_{\mathrm{B}} - \beta\mu_{\mathrm{A}}^2) = -\mu_0(\beta m_0^2 - \alpha m_0^2) = -\mu_0 m_0^2(\beta-\alpha) \end{aligned} \tag{15}$$

[Translator's note: As printed. In the middle expression α and β are interchanged: with $\alpha_{\mathrm{AB}} = \beta$ and $\alpha_{\mathrm{AA}} = \alpha$ it should read $\mu_0(\beta\boldsymbol{\mu}_{\mathrm{A}}\cdot\boldsymbol{\mu}_{\mathrm{B}} + \alpha\mu_{\mathrm{A}}^2)$, which with $\boldsymbol{\mu}_{\mathrm{A}}\cdot\boldsymbol{\mu}_{\mathrm{B}} = -m_0^2$ gives the printed final result.]

Similarly, the B-site spin energy is the same as that of the A site. The total energy is

$$E_N = -\frac{1}{2}(N-1)\mu_0 m_0^2\beta + \frac{1}{2}(N-2)\mu_0 m_0^2\alpha \quad \left(\text{or } E_N = -\frac{1}{2}N\mu_0 m_0^2(\beta-\alpha)\right) \tag{16}$$

(ii) The molecular number densities of the A and B sites are each $\dfrac{n}{2}$. For the A site,

$$M_{\mathrm{A}} = \frac{C}{2T}H_{\mathrm{A}}, \quad \text{i.e.} \quad \boldsymbol{M}_{\mathrm{A}} \equiv \frac{n}{2}\bar{\boldsymbol{\mu}}_{\mathrm{A}} = \frac{C}{2T}(\boldsymbol{H} + \boldsymbol{H}_{\mathrm{mA}}) \tag{17}$$

where $\bar{\boldsymbol{\mu}}_{\mathrm{A}}$ is the average value of $\boldsymbol{\mu}_{\mathrm{A}}$ on the A site and $\boldsymbol{M}_{\mathrm{A}}$ is the corresponding magnetization; similarly, for the B site,

$$\boldsymbol{M}_{\mathrm{B}} \equiv \frac{n}{2}\bar{\boldsymbol{\mu}}_{\mathrm{B}} = \frac{C}{2T}(\boldsymbol{H} + \boldsymbol{H}_{\mathrm{mB}}) \tag{18}$$

When the external field $\boldsymbol{H}$ is zero, if spontaneous magnetization occurs this corresponds to the antiferromagnetic phase transition, i.e.

$$\boldsymbol{M}_\mathrm{A} = \frac{C}{nT}(-\alpha_\mathrm{AB}\boldsymbol{M}_\mathrm{B} - \alpha_\mathrm{AA}\boldsymbol{M}_\mathrm{A}) \tag{19}$$

$$\boldsymbol{M}_\mathrm{B} = \frac{C}{nT}(-\alpha_\mathrm{BA}\boldsymbol{M}_\mathrm{A} - \alpha_\mathrm{BB}\boldsymbol{M}_\mathrm{B}) \tag{20}$$

that is

$$\left(1 + \frac{C}{nT}\alpha\right)\boldsymbol{M}_\mathrm{A} + \frac{C}{nT}\beta\boldsymbol{M}_\mathrm{B} = 0$$

$$\frac{C}{nT}\beta\boldsymbol{M}_\mathrm{A} + \left(1 + \frac{C}{nT}\alpha\right)\boldsymbol{M}_\mathrm{B} = 0$$

For a non-zero solution the determinant of the coefficients must vanish:

$$\det\begin{pmatrix} 1 + \frac{C}{nT}\alpha & \frac{C}{nT}\beta \\ \frac{C}{nT}\beta & 1 + \frac{C}{nT}\alpha \end{pmatrix} = 0 \tag{21}$$

that is

$$T = \frac{C}{n}(\beta - \alpha) \tag{22}$$

This is the Néel temperature; substituting the expression $C = \dfrac{\mu_0 n m_0^2}{k}$,

$$T = \frac{\mu_0 m_0^2}{k}(\beta - \alpha) \tag{23}$$

(4)(i)

Method 1. Consider a closed loop of radius r around the vortex centre; on this loop there are altogether

$$N = \frac{2\pi r}{a}$$

magnetic moments. Since the total change of the angle θ in going once round the loop is $2\pi l$, the average value of the change of the angle θ between neighbouring magnetic moments (the difference of the θ angles) is

$$\langle\theta_i - \theta_j\rangle = \frac{2\pi l}{N} = \frac{2\pi l}{2\pi r/a} = \frac{la}{r} \tag{24}$$

Method 2. Take any two neighbouring lattice points $\boldsymbol{r}$ and $\boldsymbol{r} + \mathrm{d}\boldsymbol{r}$ on a closed loop of radius r around the vortex centre, and let the directions of their spins $\boldsymbol{S}(\boldsymbol{r})$, $\boldsymbol{S}(\boldsymbol{r} + \mathrm{d}\boldsymbol{r})$ be $\theta(\boldsymbol{r})$, $\theta(\boldsymbol{r} + \mathrm{d}\boldsymbol{r})$; then

$$\theta(\boldsymbol{r} + \mathrm{d}\boldsymbol{r}) - \theta(\boldsymbol{r}) = \nabla\theta(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = |\nabla\theta(\boldsymbol{r})|\,\hat{\boldsymbol{n}}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

Because of the symmetry of the vortex about the axis through its centre, $|\nabla\theta(\boldsymbol{r})|$ at every point $\boldsymbol{r}$ depends only on r, and the direction of $\nabla\theta(\boldsymbol{r})$ at every point $\boldsymbol{r}$ is $\hat{\boldsymbol{n}}_\theta$. (Note: let the angle between the spin direction at the lattice point at $\boldsymbol{r}$ and the x axis of the two-dimensional plane be θ_r; then

$$\mathrm{d}\theta_r = \nabla\theta_r \cdot \mathrm{d}\boldsymbol{r} = \left(\frac{\partial\theta_r}{\partial\theta}\hat{\boldsymbol{n}}_\theta + \frac{\partial\theta_r}{\partial r}\hat{\boldsymbol{n}}_r\right) \cdot r\,\mathrm{d}\hat{\boldsymbol{n}}_r$$

$$= \frac{\partial\theta_r}{\partial\theta}\hat{\boldsymbol{n}}_\theta \cdot (r\,\mathrm{d}\theta)\hat{\boldsymbol{n}}_\theta = \frac{\partial\theta_r}{\partial\theta}r\,\mathrm{d}\theta \equiv |\nabla\theta_r|\,r\,\mathrm{d}\theta$$

), and $\mathrm{d}\boldsymbol{r} = (r\,\mathrm{d}\theta)\hat{\boldsymbol{n}}_\theta$, so that going once round a closed loop about the vortex centre the change of the angle θ is

$$2\pi l = \oint \nabla\theta(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = |\nabla\theta(\boldsymbol{r})| \oint r\,\mathrm{d}\theta = 2\pi r\,|\nabla\theta(\boldsymbol{r})|$$

hence

$$|\nabla\theta(\boldsymbol{r})| = \frac{l}{r}.$$

Similarly

$$\theta_i(\boldsymbol{r}_i) - \theta_j(\boldsymbol{r}_j) = |\nabla\theta(\boldsymbol{r}_{ij})|\,a,$$

so

$$\theta_i(\boldsymbol{r}_i) - \theta_j(\boldsymbol{r}_j) = \frac{la}{r},$$

and therefore

$$\langle\theta_i - \theta_j\rangle = \frac{la}{r} \tag{24}$$

(ii) Since the change of the angle of the magnetic moments along the circumference is very small, the energy of the system is

$$\begin{aligned} E(\{\theta_i\}) &= -J\sum_{\langle i,j\rangle}\cos(\theta_i - \theta_j) \approx -J\sum_{\langle i,j\rangle}\left[1 - \frac{1}{2}(\theta_i-\theta_j)^2\right] \\ &= E_0 + \frac{J}{2}\sum_{r\in[a,L]}\sum_{\langle i,j\rangle\in\{r\}}(\theta_i-\theta_j)^2 \end{aligned} \tag{25}$$

Method 1. The energy of one vortex is

$$\begin{aligned} E_{\text{vortex}} &\equiv E(\{\theta_i\}) - E_0 \\ &= \frac{J}{2}\sum_{r\in[a,L]}\sum_{\langle i,j\rangle\in\{r\}}(\theta_i-\theta_j)^2 = \frac{J}{2}\int_a^L \mathrm{d}r\int_0^{2\pi} r\,\mathrm{d}\theta\,|\nabla\theta(\boldsymbol{r})|^2 \end{aligned} \tag{26}$$

hence

$$E_{\text{vortex}} = \frac{J}{2}\int_a^L \mathrm{d}r\int_0^{2\pi} r\,\mathrm{d}\theta\,|\nabla\theta(\boldsymbol{r})|^2 = \frac{2\pi J}{2}\int_a^L \mathrm{d}r\,\frac{l^2}{r} = \pi J l^2 \ln\frac{L}{a} \tag{27}$$

Method 2. The energy of one vortex is

$$\begin{aligned} E_{\text{vortex}} &\equiv E(\{\theta_i\}) - E_0 \\ &= \frac{J}{2}\sum_{r\in[a,L]}\sum_{\langle i,j\rangle\in\{r\}}(\theta_i-\theta_j)^2 = \frac{J}{2}\sum_{r\in[a,L]} N\langle\theta_i-\theta_j\rangle^2 \\ &= \frac{J}{2}\sum_{r\in[a,L]}\frac{2\pi r}{a}\left(\frac{la}{r}\right)^2 = \pi J l^2\sum_{r\in[a,L]}\frac{a}{r} \end{aligned} \tag{26}$$

so

$$E_{\text{vortex}} = \pi J l^2 \sum_{r\in[a,L]}\frac{\Delta r}{r} = \pi J l^2\int_a^L\frac{\mathrm{d}r}{r} = \pi J l^2 \ln\frac{L}{a} \tag{27}$$

Method 3. Consider a thin two-dimensional annulus centred on the vortex centre O with radii $r \to r + \Delta r$ ($r \gg a$, $r\Delta r \gg a^2$). The angle between two neighbouring magnetic moments, $\Delta\theta = \theta_i - \theta_j \ll 1$, allows the approximation

$$-J(\cos\Delta\theta - 1) \approx \frac{J}{2}(\Delta\theta)^2.$$

Let $l = 1$. In solution Fig. 5a, a, b, c, d are the nearest-neighbour lattice sites of the site e, and Ox is the horizontal axis of the two-dimensional x–y (square lattice) plane. Let the length Oe be r and the angle between Oe and Ox be α; by the problem, Oe is perpendicular to the spin orientation at the site e (which is parallel to the line fg) (similarly for the others). Let the angles between the spin orientations at the sites b, c and the spin orientation at the site e be θ_{be}, θ_{ce} (and similarly for the rest); since O$x \perp$ be and O$x \parallel$ ca (or be $\perp$ ce), we have

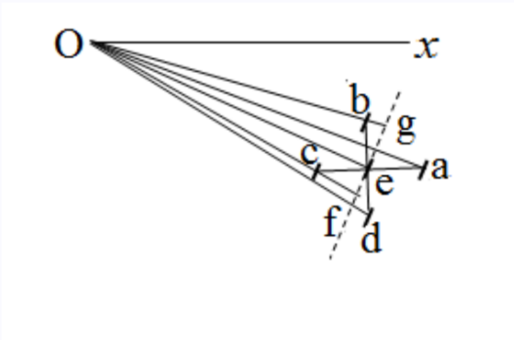
$$\angle\text{beg} = \angle x\text{Oe} = \alpha, \qquad \angle\text{cef} = \frac{\pi}{2} - \angle\text{beg} = \frac{\pi}{2} - \alpha$$

so that

$$\theta_{\text{be}} = \frac{a\cos\alpha}{r}, \qquad \theta_{\text{ce}} = \frac{a\sin\alpha}{r}$$

and similarly

$$\theta_{\text{de}} = \frac{a\cos\alpha}{r}, \qquad \theta_{\text{ae}} = \frac{a\sin\alpha}{r}.$$



Solution Fig. 5a.

The interaction energy between the magnetic moment of the site e and the magnetic moments of its nearest-neighbour sites is

$$\langle E_{\text{a}}\rangle_{l=1} = \frac{J}{2}\left[\frac{1}{2}(\theta_{\text{be}}^2 + \theta_{\text{de}}^2 + \theta_{\text{ae}}^2 + \theta_{\text{ce}}^2)\right]$$

where the factor $\frac{1}{2}$ in front of the round bracket appears because the interaction energy between magnetic moments on adjacent sites is shared by the magnetic moments of the two adjacent sites. Hence

$$\langle E_{\text{a}}\rangle_{l=1} = \frac{J}{4}\left[2\left(\frac{a\cos\alpha}{r}\right)^2 + 2\left(\frac{a\sin\alpha}{r}\right)^2\right] = \frac{Ja^2}{2r}.$$

For arbitrary l, since $\Delta\theta$ is proportional to l while the interaction energy is proportional to $(\Delta\theta)^2$,

$$\langle E_{\text{a}}\rangle = \frac{Jl^2a^2}{2r}$$

which depends only on r.

The number of lattice magnetic moments contained in a thin annulus centred on the vortex centre O with radii $r \to r + \Delta r$ ($r \gg a$, Δr small enough, but $r\Delta r \gg a^2$) is

$$\Delta n = \frac{2\pi r\Delta r}{a^2}.$$

The interaction energy of all the lattice magnetic moments in the thin annulus $r \to r + \Delta r$ is

$$\Delta E_r = \langle E_i \rangle \Delta n = \frac{Jl^2 a^2}{2r^2} \frac{2\pi r \Delta r}{a^2} = \pi J l^2 \frac{\Delta r}{r} \tag{26}$$

It is worth pointing out that in a three-dimensional model the nearest neighbours of the site e should also include the two nearest-neighbour sites m, n in front of and behind it (in the direction through e perpendicular to the original plane); but by the problem the directions of the magnetic moments at m and n are both parallel to that at e, so they do not contribute to the interaction energy.

The energy of one vortex is

$$E_{\text{vortex}} = \sum_{r=a}^{L} \Delta E_r = \pi J l^2 \sum_{r=a}^{L} \frac{\Delta r}{r} = \pi J l^2 \int_a^L \frac{\mathrm{d}r}{r} = \pi J l^2 \ln \frac{L}{a} \tag{27}$$

Method 4. First compute the energy of a spin-magnetic-moment vortex with $l = 1$, see Fig. 5e; the lattice constant is a.

By the problem: (1) only the interaction energy between nearest-neighbour magnetic moments is considered, i.e.

$$E(\{\theta_i\}) = -J \sum_{\langle i,j \rangle} \left[\cos(\theta_i - \theta_j) - 1\right],$$

where $\langle i, j \rangle$ denotes that i, j are nearest neighbours (note: here the energy has already been “renormalized”, that is, the energy of the ferromagnetic ground state has been taken as 0); (2) for the $l = 1$ vortex the direction of every magnetic moment (assumed all clockwise) is always perpendicular to the radius from the vortex centre to that magnetic moment; (3) on average, the area occupied by each lattice site is a^2, so the number of magnetic moments contained in any microscopically large but macroscopically small area ΔS is $\Delta S / a^2$.

Now consider a thin annulus centred on the vortex centre with radii $r \to r + \Delta r$ ($r\Delta r \gg a^2$); its area is $2\pi r \Delta r$, so the number of magnetic moments it contains is

$$\Delta n = \frac{2\pi r \Delta r}{a^2}.$$

Consider the region far from the vortex centre, so that the angle between two neighbouring magnetic moments $\Delta\theta = \theta_i - \theta_j \ll 1$ and one may take the approximation

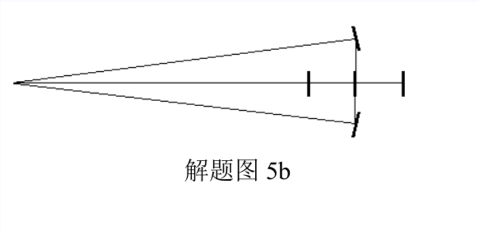
$$-J(\cos\Delta\theta - 1) \approx \frac{J}{2}(\Delta\theta)^2.$$

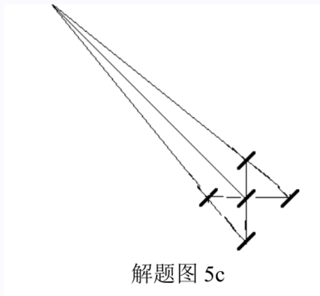
A magnetic moment has 4 nearest neighbours. Consider two extreme cases. The first is that this unit cell is exactly perpendicular to the ray from the vortex centre to this unit cell, see solution Fig. 5b. Then the angle between a given magnetic moment and its two transverse neighbouring magnetic moments is 0 and the interaction energy is also 0, while the angle with its two longitudinal neighbouring magnetic moments is a/r, so the average value of its interaction energy with the nearest-neighbour magnetic moments is

$$\langle E_i \rangle = \frac{J}{2} \cdot \frac{1}{2} \cdot 2 \cdot \frac{a^2}{r^2} = \frac{J}{2} \frac{a^2}{r^2}.$$

The second is that this unit cell makes an angle of 45° with the ray from the vortex centre to this unit cell, see solution Fig. 5c. Then the angles between the given magnetic moment and its 4 neighbours are all $a/(\sqrt{2}\, r)$, so the average value of its interaction energy with the nearest-neighbour magnetic moments is

$$\langle E_i \rangle = \frac{J}{2} \cdot \frac{(a/\sqrt{2})^2}{r^2} \times \frac{4}{2} = \frac{J}{2} \frac{a^2}{r^2}.$$





Labels: 解题图 5b = *Solution Fig. 5b;* 解题图 5c = *Solution Fig. 5c.*

We find that the two are exactly the same! Extending the above analysis to the other cases, it is easy to see that this conclusion applies to a unit cell in any position (orientation), because we always have $\sin^2\alpha + \cos^2\alpha = 1$. So the general conclusion is

$$\langle E_i\rangle = \frac{J}{2}\frac{a^2}{r^2}.$$

The interaction energy in the thin annulus $r \to r + \Delta r$ is

$$\Delta E = \langle E_i\rangle \Delta n = \frac{J}{2}\frac{a^2}{r^2}\frac{2\pi r\Delta r}{a^2} = \pi J\frac{\Delta r}{r}, \tag{26}$$

and integrating r from $a/2$ to $L/2$ gives the total energy of a single vortex,

$$E_{\text{vortex}} = \pi J\int_{a/2}^{L/2}\frac{\mathrm{d}r}{r} = \pi J\ln\frac{L}{a}.$$

The extension of the above conclusion to arbitrary l is immediate: $\Delta\theta$ is proportional to l while the energy is proportional to $(\Delta\theta)^2$, so

$$E_{l\text{-vortex}} = \pi J\, l^2 \ln\frac{L}{a}. \tag{27}$$

(iii) The vortex centre can be placed at

$$\Omega = \frac{L^2}{a^2}$$

possible positions, so the entropy of a single vortex can be estimated as

$$S = k\ln\Omega = k\ln\frac{L^2}{a^2} = 2k\ln\frac{L}{a} \tag{28}$$

(iv) The Helmholtz free energy of a single vortex is

$$F = E_0 + (\pi J l^2 - 2kT)\ln\frac{L}{a} \tag{29}$$

From Eq. (29), the possible phase-transition temperature of the system is

$$T = \frac{\pi J l^2}{2k} \tag{30}$$

When $T < \dfrac{\pi J l^2}{2k}$, the free energy of a single vortex diverges to $+\infty$ as the size L increases, which shows that in this case the system does not favour the formation of a single vortex; when $T > \dfrac{\pi J l^2}{2k}$, the free energy of a single vortex diverges to $-\infty$ as the size L increases, which shows that increasing the number of vortices can lower the free energy, and in this case the system tends to form single vortices.