The 39th Chinese Physics Olympiad

Final — Theoretical Examination (Problems)

29 October 2022, 9:00–12:00

Six problems, 320 points in total. Figures are reproduced from the Chinese original; a glossary is given below any figure that carries Chinese labels.

Problem 1 (40 points)

As shown in Fig. 1a, a piece of rigid metal wire bent into the shape of a parabola is fixed in a vertical plane. The equation of the parabola is $y = ax^2$ (the y axis points vertically upward, and a is a constant to be determined). A uniform rigid thin rod of length $2l$ has a small round hole at each of its two ends A and B, and both holes are threaded on the wire. The holes and the metal wire are extremely smooth, the friction being so small that it may be neglected in parts (1), (2) and (3). If the rod is given an impulse so that it starts to move, then after a sufficiently long time the rod comes to rest at an equilibrium position, at which the angle between the rod and the horizontal direction is $\theta = 30°$. The magnitude of the gravitational acceleration is g.

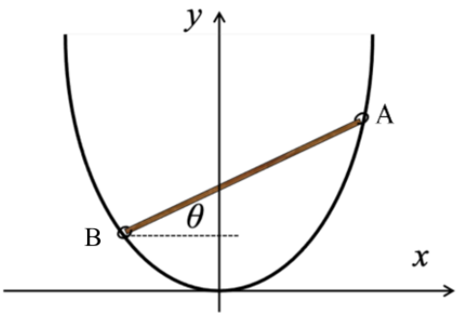

Figure 1a

(1) Find the undetermined constant a.

(2) If the rod performs small-amplitude oscillations about the above equilibrium position, find the frequency of the oscillation.

(3) The rod is at rest at the above equilibrium position. A small white mouse starts from rest at the lower end of the rod and climbs up along the rod. During the whole climb the rod always remains at rest. Assume that the mouse may be treated as a point mass, and that the mouse does not touch the metal wire at the ends of the rod. Find the displacement $s(t)$ of the mouse along the rod at time t (taking the instant at which the mouse starts to climb as time zero). Can the mouse climb to the top end of the rod? If it can, what is the shortest time in which the mouse reaches the top end of the rod?

Problem 2 (60 points)

A particle 1 (the incident particle) with kinetic energy K_1 comes in from infinity and makes an elastic collision with a particle 2 (the target particle) that is at rest. After the collision both the kinetic energy and the direction of motion of particle 1 have changed. Gravity is not to be considered.

(1) After the scattering (at infinity) the kinetic energy of particle 1 is K_1'. The quantity $k = \dfrac{K_1'}{K_1}$ is called the kinematic factor. Give the range of values of k.

(2) In this part Newtonian mechanics is to be used.

(i) Denote by θ the angle (the scattering angle) between the direction of motion of particle 1 after the scattering (the scattering direction) and the incident direction. Denote by R the ratio of the mass of particle 2 to the mass of particle 1. For any given R, k is a function of θ. For each of the three cases $R > 1$, $R = 1$ and $R < 1$, derive the dependence $k(\theta)$ of k on θ, and give the range of θ.

(ii) For some ranges of R, k may be a multivalued function of θ (each functional form is called a branch of the function k). In order to fix the value of k one needs a further parameter that describes some of the details of the interaction between the two particles. Take the simplest hard sphere model, i.e. regard both colliding particles as uniform rigid spheres with smooth surfaces, which interact only at the moment of the collision. Let the sum of the radii of the two particles be A, and let the distance from the centre of mass of the target particle 2 to the straight line along which the velocity through the centre of the incident particle 1 lies be b (the impact parameter), as shown in Fig. 2a. Find k (expressed in terms of R, A and b), and use the two special cases of a "grazing pass" and a "head-on collision" to verify that the branch on which $k(\theta)$ lies at one and the same θ can be judged from the value of b. Find the range of b corresponding to each branch of $k(\theta)$.

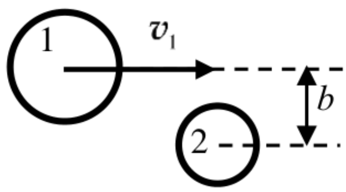

Figure 2a. Geometrical sketch of the impact parameter b.

(3) Suppose the rest masses of the two particles (which may be regarded as point masses) are both m_0. The incident speed of particle 1 is so large that relativistic effects have to be taken into account. The speed of light in vacuum is c.

(i) Find the kinematic factor $k(\theta)$ for a scattering angle θ, and compare it with the result of Newtonian mechanics.

(ii) Find the relation between the angle α between the velocities of particles 1 and 2 after the collision and K_1'. Under what circumstances does α take an extreme value? Judge the nature of the extreme value (maximum or minimum), and give this extreme value together with the corresponding value of θ.

(Note: when the solution involves physical quantities that have to be distinguished, use the subscripts 1 and 2 to distinguish the particles, and quantities without a prime and with a prime to denote the quantities before and after the collision respectively. All trigonometric functions appearing in the final expressions must be cosines, and must not contain half angles or double angles.)

Problem 3 (60 points)

A uniform rigid thin circular ring of radius R and mass M rolls on horizontal ground (the x–y coordinate plane). Assume that the coefficient of friction is large enough that there is never any slipping between the ring and the ground. At time t, let $\theta(t)$ be the angle between the plane in which the ring lies and the vertical direction (the z direction), let $\phi(t)$ be the angle between the line of intersection of the plane of the ring with the ground and the x direction, and let the Cartesian coordinates of the instantaneous contact point of the ring with the ground be $(x(t), y(t), 0)$, as shown in Fig. 3a. The magnitude of the gravitational acceleration is g.

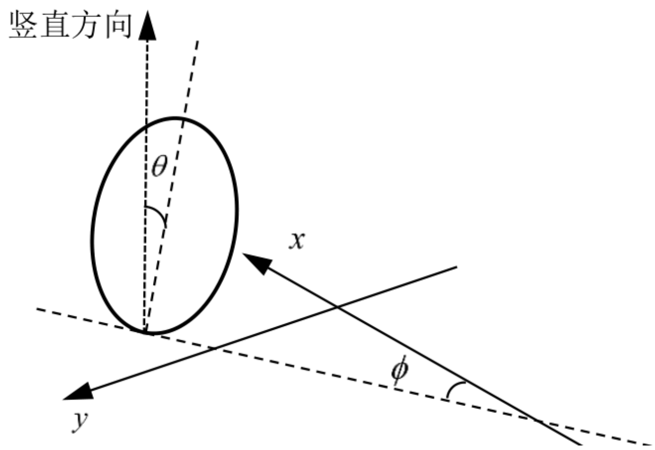

Figure 3a
Labels: 竖直方向 = *vertical direction.*

(1) The quantities $x(t)$, $y(t)$, $\theta(t)$, $\phi(t)$ that describe the spatial position of the ring at an arbitrary time t are not completely independent of one another. Find all the constraints between them and their first derivatives with respect to time.

(2) Suppose the ring performs a "uniform circular motion": the contact point of the ring with the ground traces out, at a constant speed, a circle of radius r about the z axis; $\theta(t) = \theta$ (θ a constant), and $\phi(t)$ is a linear function of the time t. In the laboratory frame Σ we may, without loss of generality, set

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$$

where ω is the angular velocity to be found. In the rotating frame Σ' that rotates about the z axis with the constant angular velocity ω, both θ and ϕ are constants, and the ring rotates about its own central axis perpendicular to its plane. In the frame Σ' every mass element of the ring "feels" a centrifugal force and a Coriolis force. Regarding the ring as a collection of many mass elements, find the resultant centrifugal force $\boldsymbol{F}_{\text{cent}}$ acting on all the mass elements, the resultant Coriolis force $\boldsymbol{F}_{\text{Cor}}$, and also the resultant torque $\boldsymbol{\tau}_{\text{cent}}$ of the centrifugal forces and the resultant torque $\boldsymbol{\tau}_{\text{Cor}}$ of the Coriolis forces about the centre of mass of the ring (the results may contain the as yet unknown ω).

(3) Find the ω of part (2) and the force exerted by the ground on the ring (the results must not contain ω).

(4) When the ring rolls fast it can wobble near a vertical plane without falling over, so that it has good stability, and the centre of mass of the ring may then be regarded approximately as moving along a straight line. Let this straight line be along the x axis, and let the magnitude of the velocity of the centre of mass be approximately the constant V, so that the position coordinate of the centre of mass is $x(t) = Vt + \delta x(t)$, while $\delta x(t)$, $y(t)$, $\theta(t)$, $\phi(t)$ are all small quantities (note that they need not be small quantities of the same order). Derive the equations of motion of the ring, keeping the lowest-order small quantities. Assuming

that the lowest-order small quantities vary with time as cosine functions, find the angular frequency Ω of the variation; and determine the minimum value $V_{\min}$ of the constant V required to keep the motion of the ring stable.

Problem 4 (60 points)

A free electron laser is a device that uses a free electron beam as its working substance and converts the kinetic energy of a relativistic electron beam into coherent radiation energy; it has great prospects of application in scientific research, production and other fields. As shown in Fig. 4a, the basic structure of a free electron laser has three parts: the electron beam accelerator, the undulator (wiggler) and the optical resonant cavity. The undulator is the core part of the free electron laser. It consists of permanent magnets arranged along the z direction with the spatial period Λ, and produces a periodic transverse static magnetic field whose magnetic induction is directed along the x axis and has the magnitude

$$B = B_0 \cos\left(\frac{2\pi}{\Lambda} z\right).$$

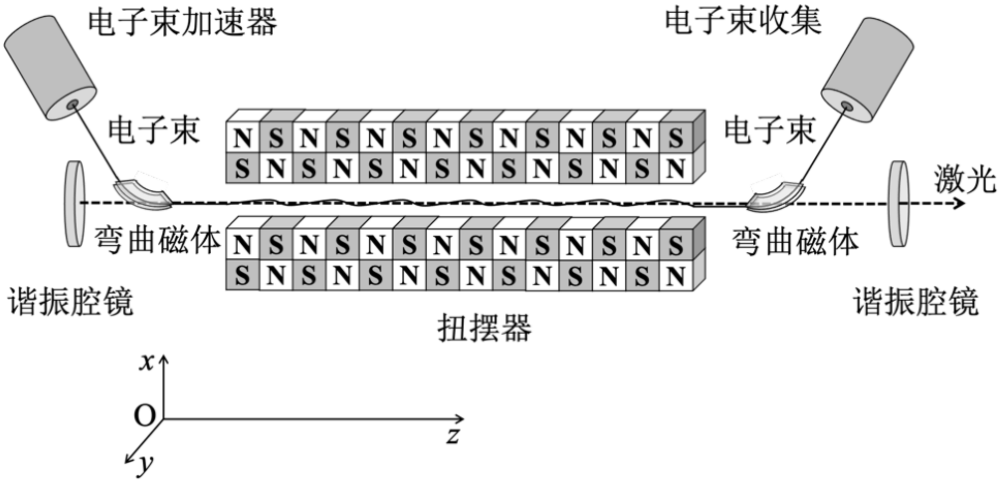

Figure 4a

Labels: 电子束加速器 = *electron beam accelerator;* 电子束 = *electron beam;* 电子束收集 = *electron beam collector;* 弯曲磁体 = *bending magnet;* 谐振腔镜 = *resonant cavity mirror;* 扭摆器 = *undulator;* 激光 = *laser.*

The electron beam is accelerated by the accelerator to a prescribed speed v_0, is guided by the bending magnets, and is injected into the undulator along the positive z direction. The fast-moving electrons in the undulator are acted on by the alternating magnetic field and perform a wiggling motion, radiating coherent electromagnetic waves at the same time. Assume that B_0 is not too strong, so that the change of the electron speed produced by the magnetic field is much smaller in magnitude than v_0, and that the loss of electron kinetic energy caused by the coherent electromagnetic radiation of the electron beam may be neglected. The rest mass of the electron is $m_{\rm e}$ and its charge is $-e$. Gravity is not to be considered.

(1) Set up a reference frame $\mathrm{S}'(x', y', z')$ which moves uniformly in a straight line with respect to the laboratory frame $\mathrm{S}(x, y, z)$ along the positive z direction with speed of magnitude v_0; the x' axis is parallel to the x axis and the y' axis is parallel to the y axis, while the z' axis coincides with the z axis. Using the approximation stated above, find in the frame S' the position coordinates (x', y', z') of the electron at time t', and draw a sketch of the trajectory of the electron.

(2) If the spatial period of the magnetic field of the undulator is $\Lambda = 1\,\text{mm}$ (which may be regarded as exact), and the wavelength of the X-ray laser radiated along the positive z direction is $4.00\,\text{Å}$, find the accelerating voltage of the electron beam accelerator. The rest mass of the electron is $m_\text{e} = 9.11\times10^{-31}\,\text{kg}$, and the elementary charge is $e = 1.60\times10^{-19}\,\text{C}$.

(3) Use a laser of wavelength $4.00\,\text{Å}$ as the incident light (which may be regarded as a plane wave), as shown in Fig. 4b. In the x–z plane there is a coplanar two-dimensional crystal made of equilateral rhombi of side $d = 8.00\,\text{Å}$ whose two vertex angles are $60°$ and $120°$. Assume that the scattering of the incident wave by the two-dimensional crystal is weak, so that the effect of a scattered wave being scattered a second time may be neglected. How many principal diffraction maxima can be observed far away in the x–z plane? Find also the directions of the corresponding principal maxima (expressed by the angle θ of Fig. 4b).

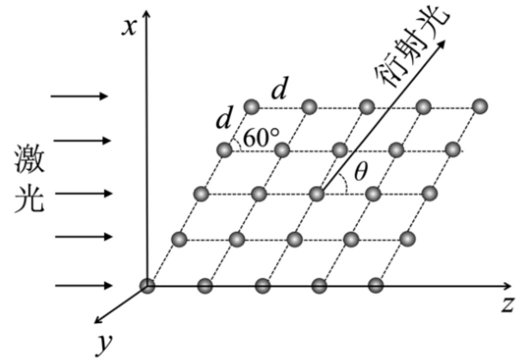

Figure 4b
Labels: 激光 = laser; 衍射光 = diffracted light.

Given: the transformation relations of the electromagnetic field between the two inertial frames $\text{S}'(x', y', z')$ and $\text{S}(x, y, z)$ are

$$\begin{cases} E_{x'} = \dfrac{E_x - v_0 B_y}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}, \\ E_{y'} = \dfrac{E_y + v_0 B_x}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}, \\ E_{z'} = E_z, \end{cases} \qquad \begin{cases} B_{x'} = \dfrac{B_x + \frac{v_0}{c^2} E_y}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}, \\ B_{y'} = \dfrac{B_y - \frac{v_0}{c^2} E_x}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}, \\ B_{z'} = B_z, \end{cases}$$

where $c = 3.00 \times 10^8\,\text{m/s}$ is the speed of light in vacuum.

Problem 5 (50 points)

The propagation of light in a uniaxial crystal is anisotropic. According to the state of polarization the light can be divided into the o ray (the ordinary ray) and the e ray (the extraordinary ray). Inside a uniaxial crystal there is one particular direction along which the o ray and the e ray propagate with the same speed and their propagation directions do not separate; this direction is called the optic axis of the crystal. The refractive index felt by the o ray inside the crystal is isotropic and equal to n_o, and its wave speed is isotropic and equal to $v_o = c/n_o$ (c being the speed of light in vacuum). The refractive index felt by the e ray when it propagates in the direction perpendicular to the optic axis is n_e, and its wave speed is $v_e = c/n_e$. When the propagation direction of the e ray lies between the directions parallel to and perpendicular to the optic axis, the refractive index it feels varies monotonically with the angle between that direction and the optic axis. For a point source inside the crystal, or for a secondary wave source in Huygens' principle, the wavefront of the light it emits is an ellipsoid of revolution whose axis of symmetry is the optic axis. As in Fig. 5a, set up a Cartesian coordinate system in which the y axis coincides with the direction of the optic axis; the following analysis of the propagation of the o ray and the e ray is restricted to the case within the y–x cross-section.

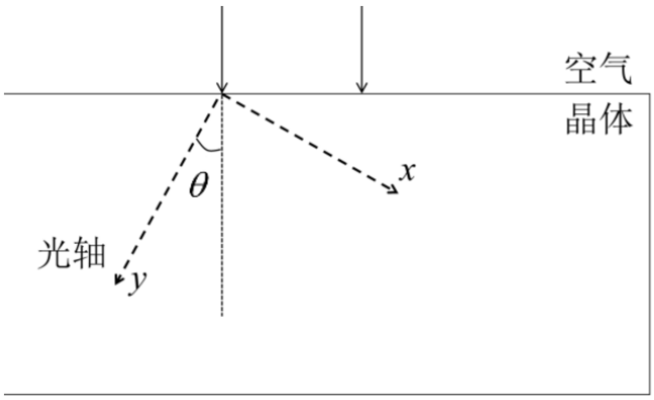

Figure 5a
Labels: 空气 *= air;* 晶体 *= crystal;* 光轴 *= optic axis.*

(1) Draw a construction based on Huygens' principle.

In Fig. 5a, a plane light wave of vacuum wavelength λ falls normally from air onto the surface of the crystal, and the angle between the normal to the crystal surface and the optic axis is θ. Draw the wavefronts of the o ray and of the e ray propagating inside the crystal. Mark the directions of the wave vectors $\boldsymbol{k}_o$ and $\boldsymbol{k}_e$ that correspond to the o ray and the e ray respectively (the normal direction of the envelope of the equal-phase surfaces of the secondary waves, i.e. the propagation direction of the equal-phase surface), as well as the propagation directions $\boldsymbol{N}_o$ and $\boldsymbol{N}_e$ of the corresponding light rays as determined by Huygens' principle. Take $n_o > n_e$, and show the propagation of the o ray and of the e ray in one and the same figure.

(2) Denote by ξ the angle between the propagation direction $\boldsymbol{N}_e$ of the e light ray and the optic axis. Derive the relation between the tangents of the two angles ξ and θ.

(3) The dispersion curve equations of the refractive indices of a uniaxial crystal called BBO are (λ being the vacuum wavelength, in units of µm):

$$\begin{cases} n_o^2(\lambda) = 2.7359 + \dfrac{0.01878}{\lambda^2 - 0.01822} - 0.01354\lambda^2 \\ n_e^2(\lambda) = 2.3753 + \dfrac{0.01224}{\lambda^2 - 0.01667} - 0.01516\lambda^2 \end{cases}$$

Find n_o and n_e in the BBO uniaxial crystal for the vacuum wavelengths 800.0 nm and 400.0 nm respectively.

(4) One of the important uses of an optical crystal is to realize frequency conversion, for example the optical frequency-doubling process, in which two fundamental-frequency photons of the same frequency are converted into one frequency-doubled photon of twice the frequency. This process must satisfy both energy conservation and momentum conservation. Inside a crystal the momentum of a photon of wave vector $\boldsymbol{k}$ can also be written as $\boldsymbol{p} = \hbar\boldsymbol{k}$, where $\hbar$ is the reduced Planck constant.

(4.1) Express the relation between the fundamental-frequency photon wave vector $\boldsymbol{k}_1$ and the frequency-doubled photon wave vector $\boldsymbol{k}_2$ in the frequency-doubling process.

(4.2) Using a BBO crystal, a light wave of vacuum wavelength 800.0 nm is frequency-doubled into a light wave of vacuum wavelength 400.0 nm. The fundamental-frequency light and the frequency-doubled light may each be chosen to be an o ray or an e ray. Give the choice that realizes this frequency-doubling process.

(5) Set up the BBO crystal in the way shown in Fig. 5a, so that the angle between the normal to the crystal surface and the optic axis is θ, and let fundamental-frequency light of vacuum wavelength 800.0 nm fall perpendicularly onto this crystal in the form of a plane wave. How large must the angle θ be so that the momentum conservation condition is satisfied and frequency-doubled light of vacuum wavelength 400.0 nm is thereby produced?

(6) Calculate the angle α between the wave vector $\boldsymbol{k}_e$ and the propagation direction $\boldsymbol{N}_e$ of the e light in part (5).

Problem 6 (50 points)

Consider only a non-relativistic gas made of a single component of monatomic molecules.

For such a gas, if the size of the molecules and the interactions between the molecules except at the instants of collision are neglected, the gas can be described by the ideal-gas model; if the actual size of the molecules and the non-collisional interactions between them are taken into account, the ideal-gas model is no longer applicable. For this case van der Waals constructed the van der Waals model, in which the equation of state of 1 mol of gas (called the van der Waals equation) is

$$\left(p+\frac{a}{v^2}\right)(v-b)=RT$$

where T, p and v denote respectively the temperature, the pressure and the volume of the gas (that is, the volume of the container), R is the universal gas constant, and a and b are constants greater than zero. In the dilute limit (for any given T, $v\to\infty$) this model degenerates into the ideal-gas model.

(1) State directly the physical meaning of b (the analysis need not be given). Assuming that a gas molecule may be regarded as a rigid small sphere of radius r (in the van der Waals model there is a weak attraction between the small spheres), estimate the value of b. The Avogadro constant N_{A} is known.

(2) Write down directly the equation of state of N moles of a van der Waals gas occupying the volume V (the molar volume v must not appear in the expression).

(3) Let the molar heat capacity of the gas at constant volume be C_V, which satisfies

$$\left(\frac{\partial C_V}{\partial v}\right)_T=T\left(\frac{\partial^2 p}{\partial T^2}\right)_v$$

Here $\left(\frac{\partial C_V}{\partial v}\right)_T$ denotes the first derivative of C_V with respect to v when T is regarded as a constant, $\left(\frac{\partial^2 p}{\partial T^2}\right)_v$ denotes the second derivative of p with respect to T when v is regarded as a constant, and so on. Prove that the molar heat capacity at constant volume C_V of a van der Waals gas is a constant, and determine this constant. (The explicit form of this constant need not be substituted into the later calculations.)

(4) Given that the internal energy u of 1 mol of van der Waals gas is

$$u=C_VT-\frac{a}{v},$$

derive the expression for its molar entropy $s(T, v)$, which may contain undetermined constants. Give also the equation of a quasi-static adiabatic process undergone by this gas (expressed with T and v; the expression may contain undetermined constants).

(5) 1 mol of van der Waals gas undergoes the following reversible Carnot cycle:

- Process I — isothermal expansion: the temperature is T_1 and the volume changes from v_1 to v_2;
- Process II — adiabatic cooling: the temperature falls from T_1 to T_2 and the volume changes from v_2 to v_3;
- Process III — isothermal compression: the temperature is T_2 and the volume changes from v_3 to v_4;
- Process IV — adiabatic heating: the temperature rises from T_2 to T_1 and the volume changes from v_4 to v_1.

Calculate the heat absorbed Q_1, the heat released Q_2 and the efficiency η of this cycle (finally express η as a function of T_1 and T_2 only).

(6) Define the isothermal compressibility as

$$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$$

Derive the expression for the isothermal compressibility $\kappa_T(T, v)$ of a van der Waals gas. Taking as an example a van der Waals gas that satisfies the condition $a \ll pv^2$, find the range of values of κ_T, and explain its intuitive physical meaning.

Hint: for a function of two variables $y = y(x_1, x_2)$, when x_1 and x_2 change independently by the small amounts $\mathrm{d}x_1$ and $\mathrm{d}x_2$, the total differential of y (the small change of y) is

$$\mathrm{d}y = \left(\frac{\partial y}{\partial x_1}\right)_{x_2} \mathrm{d}x_1 + \left(\frac{\partial y}{\partial x_2}\right)_{x_1} \mathrm{d}x_2$$