The 39th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

Official reference solutions and mark scheme

Equation numbers (1), (2), … in each problem follow the circled numbers of the Chinese original; primed variants (5′), (5″) follow the primed circled numbers. Per-step marks printed in the original are reproduced with a right-aligned italic "(N points)". Passages that the original encloses in the brackets 【 】 or [ ] (alternative methods and alternative branches) are kept as "Method 2", "Method 3", …. The mark-scheme summary printed at the end of each problem (评分标准) is translated as a short closing paragraph. Obvious misprints of the source are transcribed as printed, with a translator's note giving the correct form.

Problem 1 (rod threaded on a parabolic wire; a mouse climbing it)

Solution

(1) Let the coordinates of the centre of mass of the rod be (x_c, y_c) and the length of the rod be $2l$. Then the coordinates of the ends A and B of the rod are

$$\begin{cases} x_A = x_c + l\cos\theta \\ y_A = y_c + l\sin\theta \end{cases} \qquad \begin{cases} x_B = x_c - l\cos\theta \\ y_B = y_c - l\sin\theta \end{cases} \tag{1}$$

(2 points)

Substituting into the equation of the parabola gives

$$y_c + l\sin\theta = a(x_c + l\cos\theta)^2 \tag{2}$$

$$y_c - l\sin\theta = a(x_c - l\cos\theta)^2 \tag{3}$$

Solving (2) and (3) together gives

$$x_c = \frac{\sin\theta}{2a\cos\theta} \tag{4}$$

(1 point)

$$y_c = \frac{\sin^2\theta}{4a\cos^2\theta} + al^2\cos^2\theta \tag{5}$$

(1 point)

The gravitational potential energy of the rod is

$$E_p = mgy_c = mg\frac{\sin^2\theta}{4a\cos^2\theta} + mgal^2\cos^2\theta \tag{6}$$

(2 points)

The equilibrium point is an extremum of the potential energy:

$$\begin{aligned} \frac{\mathrm{d}E_p}{\mathrm{d}\theta} &= mg\frac{\cos^3\theta\sin\theta + \cos\theta\sin^3\theta}{2a\cos^4\theta} - 2mgal^2\cos\theta\sin\theta \\ &= mg\sin\theta\left(\frac{1}{2a\cos^3\theta} - 2al^2\cos\theta\right) = 0 \end{aligned} \tag{7}$$

(2 points)

The solutions are:

I. $\sin\theta = 0$, i.e. $\theta = 0$, which does not fit the problem and is discarded. (8)

II.

$$\frac{1}{2a\cos^3\theta} - 2al^2\cos\theta = 0 \tag{9}$$

which gives

$$\cos\theta = \sqrt{\frac{1}{2al}} \tag{10}$$

Since $\theta = 30°$,

$$\sqrt{\frac{1}{2al}} = \frac{\sqrt{3}}{2} \tag{11}$$

that is,

$$a = \frac{2}{3l} \tag{12}$$

(2 points)

(2) For small oscillations of the rod about the equilibrium position, conservation of mechanical energy gives

$$\tfrac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2) + \tfrac{1}{2}I_c\dot{\theta}^2 + mg\,\frac{\sin^2\theta}{4a\cos^2\theta} + mgal^2\cos^2\theta = \text{constant} \tag{13}$$

(4 points)

that is,

$$\begin{aligned}&\frac{1}{2}m\left[\left(\frac{1}{2a\cos^2\theta}\right)^2 + \left(\frac{\sin\theta}{2a\cos^3\theta} - 2al^2\sin\theta\cos\theta\right)^2\right]\dot{\theta}^2 + \frac{1}{6}ml^2\dot{\theta}^2\\ &\quad + mg\,\frac{\sin^2\theta}{4a\cos^2\theta} + mgal^2\cos^2\theta = \text{constant}\end{aligned} \tag{14}$$

(2 points)

Put $\theta = \theta_0 + \Delta\theta$ ($|\Delta\theta| \ll \theta_0$), with $\theta_0 = 30°$ the equilibrium position. Expanding (14) about θ_0 in the small quantity $\Delta\theta$, and keeping in each of the three parts (the kinetic energy of the centre of mass, the rotational kinetic energy about the centre of mass, and the potential energy) up to the largest term that is not a constant, one obtains the approximate relation

$$\frac{1}{2}m\left[\left(\frac{1}{2a\cos^2\theta_0}\right)^2\right]\Delta\dot{\theta}^2 + \frac{1}{6}ml^2\Delta\dot{\theta}^2 + \frac{mg}{2}\sin^2\theta_0\left(\frac{3}{2a\cos^4\theta_0} + 2al^2\right)\Delta\theta^2 = \text{constant} \tag{15}$$

(6 points)

Differentiating both sides with respect to time gives

$$\left\{\left[\left(\frac{1}{2a\cos^2\theta_0}\right)^2\right] + \frac{1}{3}l^2\right\}\Delta\ddot{\theta} + g\sin^2\theta_0\left(\frac{3}{2a\cos^4\theta_0} + 2al^2\right)\Delta\theta = 0 \tag{16}$$

(2 points)

or

$$\Delta\ddot{\theta} + \frac{g}{l}\Delta\theta = 0$$

The small oscillation can be treated approximately as simple harmonic motion, with angular frequency

$$\omega = \sqrt{\frac{g}{l}} \tag{17}$$

and frequency

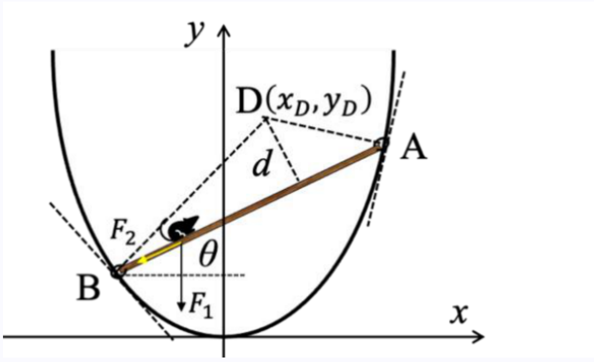
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \tag{18}$$

(2 points)

(3) Let the mass of the mouse be m_s, its displacement along the rod be s (with $s = 0$ at the point B), and its acceleration be d^2s/dt^2. Then the forces exerted by the mouse on the rod are

$$F_1 = m_s g \text{ (vertically downward);} \qquad F_2 = m_s \frac{d^2 s}{dt^2} \text{ (along the rod).} \tag{19}$$

(4 points)



Solution Figure 1a

For the rod to stay in equilibrium the resultant torque on it must vanish. In order to avoid having to analyse the torques of the constraint forces, we choose a particular reference point: at A and at B draw respectively the tangent and the normal to the parabola; the two normals meet at the point D, and we choose D as the reference point, as shown in Solution Fig. 1a. We now find the coordinates of D. The equations of the two normals are

$$y - y_B = -\frac{1}{2ax_B}(x - x_B) \tag{20}$$

$$y - y_A = -\frac{1}{2ax_A}(x - x_A) \tag{21}$$

Solving for the coordinates of the intersection D of the two normals:

$$x_D = x_c = \frac{\sqrt{3}}{4} l \tag{22}$$

(1 point)

$$y_D = \frac{13}{8} l \tag{23}$$

(1 point)

The equation of the straight line of the rod is

$$y - y_A = \frac{1}{\sqrt{3}}(x - x_A) \tag{24}$$

that is,

$$y = \frac{1}{\sqrt{3}} x + \frac{3}{8} l \tag{25}$$

The distance from D to the rod (written as $Ax + By + C = 0$) is

$$d = \frac{|Ax_D + By_D + C|}{\sqrt{A^2 + B^2}}$$

From this and (25),

$$d=\frac{\left|\frac{1}{\sqrt{3}}x_{\rm D}-y_{\rm D}+\frac{3}{8}l\right|}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+(-1)^2}}=\frac{\left|\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{4}l-\frac{13}{8}l+\frac{3}{8}l\right|}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+(-1)^2}}=\frac{\sqrt{3}}{2}l \tag{26}$$

(2 points)

【**Or:** From (22) we know that the point D and the centre of mass C of the rod lie on the same vertical line. Let the perpendicular dropped from D to the rod meet the rod at E; then

$$\angle\mathrm{CDE}=\theta \tag{25$'$}$$

and hence

$$d=(y_{\rm D}-y_{\rm C})\cos\theta=\left(\frac{13}{8}l-\frac{5}{8}l\right)\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}l \tag{26$'$}$$

(2 points)

】

Take D as the reference point. Because the weight of the rod passes through D, and the supporting forces of the parabola at A and at B are along the corresponding normals, i.e. also pass through D, their torques are zero. Hence the torque balance condition of the rod can be written as

$$L=m_s\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\,d-m_sg(l-s)\cos\theta_0=0 \tag{27}$$

(2 points)

From (26) and (27),

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+\frac{g}{l}(s-l)=0 \tag{28}$$

whose solution is

$$s=l+A\cos\left(\sqrt{\frac{g}{l}}\,t+\varphi_0\right) \tag{29}$$

(2 points)

where A and φ_0 are undetermined constants. From the initial conditions, at $t=0$, $s=0$ and $\dot{s}=0$, we get

$$l+A\cos\varphi_0=0 \tag{30}$$

$$-\sqrt{\frac{g}{l}}\sin\varphi_0=0 \tag{31}$$

[Translator's note: as printed; the left-hand side of (31) should read $-A\sqrt{g/l}\,\sin\varphi_0$.]

Hence

$$\varphi_0=0,\quad A=-l \tag{32}$$

or

$$\varphi_0=\pi,\quad A=l \tag{33}$$

So we obtain

$$s=l-l\cos\left(\sqrt{\frac{g}{l}}\,t\right) \tag{34}$$

(2 points)

It is worth noting that the solution (34) satisfies the equilibrium condition (27) over the whole range $0\le s\le 2l$. This shows that the mouse can climb all the way up to the top end of the rod.

When $s = 2l$ the mouse reaches the top end of the rod; let the total time be t. From (34),

$$\cos\left(\sqrt{\frac{g}{l}}\,t\right) = -1 \tag{35}$$

that is,

$$t = \pi\sqrt{\frac{l}{g}} \tag{36}$$

(2 points)

It is also worth noting that the solution (34) is a periodic function whose period is

$$T' = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}} = 2T$$

The amplitude is l, and the mouse moves back and forth between $s = 0$ and $s = 2l$. The times t at which the mouse reaches the top end of the rod are

$$t = T,\ 3T,\ 5T,\ \ldots \tag{37}$$

Therefore the shortest time for the mouse to climb to the top end of the rod is T.

[Translator's note: here $T = T'/2 = \pi\sqrt{l/g}$, in agreement with (36).]

Mark scheme (40 points in total). Part (1), 10 points: (1) 2 points, (4) 1 point, (5) 1 point, (6) 2 points, (7) 2 points, (12) 2 points. Part (2), 16 points: (13) 4 points, (14) 2 points (if (13) is missing and (14) is given directly, 6 points), (15) 6 points, (16) 2 points, (18) 2 points. Part (3), 14 points: (19) 2 points, (22) and (23) 1 point each, (26) (or (26′)) 2 points, (27) 2 points, (29) 2 points, (34) 2 points, (36) 2 points.

Problem 2 (elastic scattering: the kinematic factor, Newtonian and relativistic)

Solution

(1) In an elastic collision the total kinetic energy of the system is unchanged, i.e.

$$K_1 = K_1' + K_2' \tag{1}$$

(2 points)

From the definition of k,

$$0 \leq k = \frac{K_1'}{K_1} \leq 1 \tag{2}$$

(2 points)

(2)(i) Because particle 2 is at rest and there is a scattering angle, the process can be treated as a two-dimensional collision. Writing the momentum of a particle as $\boldsymbol{p}$, the collision process obeys conservation of momentum:

$$\boldsymbol{p}_1 = \boldsymbol{p}_1' + \boldsymbol{p}_2' \tag{3}$$

(2 points)

The angle between $\boldsymbol{p}_1'$ and $\boldsymbol{p}_1$ is the scattering angle θ. Hence (3) can be written as

$$p_2'^2 = p_1^2 + p_1'^2 - 2p_1p_1'\cos\theta \tag{4}$$

(2 points)

From the definitions of momentum and of kinetic energy their relation is

$$p^2 = 2mK \tag{5}$$

Substituting (5) into (4) and combining with (1) to eliminate K_2' gives

$$K_1 - 2\sqrt{K_1 K_1'}\cos\theta + K_1' = R(K_1 - K_1')$$

$$\frac{K_1'}{K_1}(1+R) - 2\cos\theta\sqrt{\frac{K_1'}{K_1}} + 1 - R = 0$$

Solving,

$$\sqrt{k} = \frac{\cos\theta \pm \sqrt{\cos^2\theta - (1-R^2)}}{1+R} \tag{6}$$

$$k = \left[\frac{\cos\theta \pm \sqrt{\cos^2\theta - (1-R^2)}}{R+1}\right]^2 \tag{7}$$

(2 points)

Clearly we must now discuss the multivaluedness of (6) or (7) and the domain of the corresponding independent variable θ. From (2) and (6),

$$0 \leq \frac{\cos\theta \pm \sqrt{\cos^2\theta - (1-R^2)}}{1+R} \leq 1 \tag{8}$$

Discussion:

a. If $R > 1$, the mass of the target particle 2 is larger than that of the incident particle 1, and

$$\sqrt{\cos^2\theta - (1-R^2)} > |\cos\theta|$$

The result with the minus sign in front of the root in (6) or (7) must be discarded in order to satisfy the first half of the inequality (8); the second half of (8) is automatically satisfied because $\cos\theta \leq 1$. Hence

$$\sqrt{k} = \frac{\cos\theta + \sqrt{\cos^2\theta - (1-R^2)}}{1+R} \tag{9}$$

$$k = \left[\frac{\cos\theta + \sqrt{\cos^2\theta - (1-R^2)}}{R+1}\right]^2 \tag{10}$$

(2 points)

From the symmetry of the collision geometry and from (10) one sees that in this case

$$0 \leq \theta \leq \pi \tag{11}$$

(2 points)

b. If $R = 1$, the target particle 2 and the incident particle 1 have equal masses; substituting into (7) gives

$$k = \cos^2\theta \quad \text{and} \quad 0 \tag{12}$$

(2 points)

The solution $k = 0$ means that the incident particle 1 stops at the place of the collision, is not scattered out, and there is then no scattering angle. The first solution requires $\cos\theta > 0$, so that θ is restricted to the first quadrant, i.e.

$$0 \leq \theta < \frac{\pi}{2} \tag{13}$$

(2 points)

Therefore the kinematic factor of the scattered particles that can actually be detected has only one branch, namely $k = \cos^2\theta$.

c. If $R < 1$, the mass of the target particle 2 is smaller than that of the incident particle 1, and

$$\sqrt{\cos^2\theta - (1-R^2)} < |\cos\theta|$$

From this we see that $\theta \neq \pi/2$. When $\theta > \pi/2$, (8) cannot be satisfied; for $\theta < \pi/2$ one requires

$$\cos^2\theta - (1-R^2) \geq 0$$

that is,

$$\cos\theta \geq \sqrt{1-R^2}$$

Then (8) is satisfied. Hence

$$k = \left[\frac{\cos\theta \pm \sqrt{\cos^2\theta - (1-R^2)}}{R+1}\right]^2 \tag{14}$$

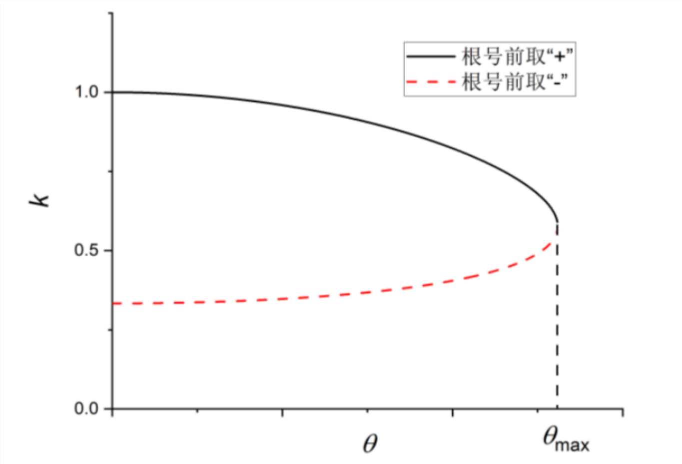
(2 points)

and correspondingly

$$0 \leq \theta \leq \arccos\sqrt{1-R^2} \tag{15}$$

(2 points)

i.e. there exists a maximum scattering angle $\theta_{\max} = \arccos\sqrt{1-R^2}$.

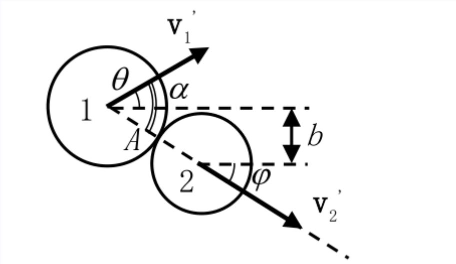


Solution Figure 2a. The k–θ relation for $R < 1$.

Labels: 根号前取“+” *= “+” sign taken in front of the root;* 根号前取“−” *= “−” sign taken in front of the root.*

(2)(ii) The conclusion (14) means that for $R < 1$ two values of k may occur at one and the same scattering angle. Energy and momentum conservation alone cannot determine which of the two values k takes; the details of the interaction during the collision must be considered. Here the hard-sphere interaction model is used: the sum A of the radii of the two particles is known, and the impact parameter b is used as the parameter that determines the branch on which k lies.

Since there is no scattering when $b > A$, only the case $0 \leq b \leq A$ is discussed.



Solution Figure 2b. Sketch of the outgoing directions of the particles in an oblique collision in the hard-sphere model.

Let the angle between the momentum $\boldsymbol{p}_2'$ of particle 2 (the recoil particle) after the collision and the incident momentum $\boldsymbol{p}_1$ be φ. Then

$$b = A\sin\varphi \tag{16}$$

(1 point)

From (3),

$$p_1'^2 = p_1^2 + p_2'^2 - 2p_1p_2'\cos\varphi \tag{17}$$

Substituting (1) and (5) into (17) gives

$$(1+R)\sqrt{(1-k)} = 2\sqrt{R}\cos\varphi \tag{18}$$

Substituting (16) into (18) and rearranging,

$$k = 1 - \frac{4R\cos^2\varphi}{(1+R)^2} = 1 - \frac{4R(1-\sin^2\varphi)}{(1+R)^2} = \frac{(1-R)^2 + 4R\left(\frac{b}{A}\right)^2}{(1+R)^2} \tag{19}$$

(4 points)

Clearly k increases monotonically with b.

For the hard-sphere model, $b = A$ is a grazing pass; then $\theta = 0$, but the two particles do not really interact, the kinetic energy of the scattered particle is unchanged, $k = 1$, and this corresponds to taking the "+" sign in front of the root in (14).

(1 point)

(20)

$b = 0$ is a head-on collision; then $\theta = 0$ as well, and $k = \left(\frac{1-R}{1+R}\right)^2$, which corresponds to taking the "−" sign in front of the root in (14).

(1 point)

(21)

From the above analysis, the change from the grazing pass to the head-on collision corresponds to b decreasing monotonically from A to 0, and correspondingly to k decreasing monotonically from 1 to $\left(\frac{1-R}{1+R}\right)^2$.

We now examine how θ varies with b.

【Method 1】 From (14) and (15) we know that $\theta = \theta_{\max} = \sqrt{1-R^2}$ is a branch point of $k(\theta)$, where the two branches of $k(\theta)$ merge into one.

[Translator's note: as printed; it should read $\cos\theta_{\max} = \sqrt{1-R^2}$.]

Denoting $k(\theta_{\max})$ by k_c,

$$\sqrt{k_c} = \frac{\cos\theta_{\max}}{1+R} = \sqrt{\frac{1-R}{1+R}} \tag{22}$$

(1 point)

The corresponding impact parameter is

$$b_{\mathrm{c}} = \sqrt{\frac{1-R}{2}}\,A \tag{23}$$

(2 points)

Taking into account that $\theta < \theta_{\max}$, the numerator of (14) satisfies

$$\begin{cases} \cos\theta + \sqrt{\cos^2\theta - (1-R^2)} > \cos\theta_{\max} \\ \cos\theta - \sqrt{\cos^2\theta - (1-R^2)} < \sqrt{1-R^2} = \cos\theta_{\max} \end{cases} \tag{24}$$

(2 points)

(Regarding $\cos\theta$, $\sqrt{\cos^2\theta-(1-R^2)}$ and $\sqrt{1-R^2}$ as the three side lengths of a right triangle, one obtains the second inequality.)

From the analysis of (20) and (21) we know that the first line of (24) corresponds to $A \geq b \geq b_c$ and the second line to $b_c > b \geq 0$. Therefore

$$\begin{cases} \sqrt{k} = \dfrac{\cos\theta + \sqrt{\cos^2\theta-(1-R^2)}}{1+R}, & A \geq b \geq \sqrt{\dfrac{1-R}{2}}\,A \\ \sqrt{k} = \dfrac{\cos\theta - \sqrt{\cos^2\theta-(1-R^2)}}{1+R}, & \sqrt{\dfrac{1-R}{2}}\,A > b \geq 0 \end{cases} \tag{25}$$

(4 points)

【Method 2】 Differentiating (6) with respect to $\cos\theta$,

$$\frac{\mathrm{d}\sqrt{k}}{\mathrm{d}(\cos\theta)} = \frac{1}{R+1}\left(1 \pm \frac{\cos\theta}{\sqrt{\cos^2\theta-(1-R^2)}}\right)$$

Substituting back into (6) and rearranging,

$$\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}k} = \begin{cases} \dfrac{\sqrt{\cos^2\theta-(1-R^2)}}{2k} > 0, & \text{“+” sign in front of the root in (6)} \\ 0, & \cos\theta = \cos\theta_{\max} = \sqrt{1-R^2} \\ -\dfrac{\sqrt{\cos^2\theta-(1-R^2)}}{2k} < 0, & \text{“−” sign in front of the root in (6)} \end{cases} \tag{22$'$}$$

(2 points)

Combining (19) with the analysis of (20), (21) and (22′), we obtain: at $b = A$ the “+” sign is taken in front of the root in (14) (and likewise in (6)); the scattering angle θ starts from 0 and at first increases as b decreases, until $\theta = \theta_{\max}$; after that the sign in front of the root in (14) (and in (6)) reverses to “−”, and θ decreases as b decreases, until $\theta = 0$ at $b = 0$.

When $\theta = \theta_{\max} = \arccos\sqrt{1-R^2}$, (14) gives at the branch point

$$k_{\mathrm{c}} = \frac{1-R}{1+R} \tag{23$'$}$$

(1 point)

Substituting (23′) into (19) and rearranging, at this point

$$b_{\mathrm{c}} = \sqrt{\frac{1-R}{2}}\,A \tag{24$'$}$$

(2 points)

Hence, for $R < 1$, the kinematic factor is

$$k = \begin{cases} \left[\dfrac{\cos\theta + \sqrt{\cos^2\theta - (1-R^2)}}{1+R}\right]^2, & A \geq b \geq \sqrt{\dfrac{1-R}{2}}\,A \\ \left[\dfrac{\cos\theta - \sqrt{\cos^2\theta - (1-R^2)}}{1+R}\right]^2, & \sqrt{\dfrac{1-R}{2}}\,A > b \geq 0 \end{cases} \tag{25'}$$

(4 points)

(3) Let E denote the energy of a moving particle and E_0 the energy of a particle at rest:

$$E_0 = m_0 c^2 \tag{26}$$

The kinetic energy of the particle is

$$K = E - E_0 \tag{27}$$

(2 points)

In the relativistic case equations (1)–(4) all still hold. But the energy–momentum relation becomes

$$p^2 c^2 = E^2 - E_0^2 \tag{28}$$

(2 points)

Substituting (28) into (4),

$$E_2'^2 = E_1^2 + E_1'^2 - E_0^2 - 2\sqrt{(E_1^2 - E_0^2)(E_1'^2 - E_0^2)}\cos\theta \tag{29}$$

(2 points)

Solving (27) and (29) together and rearranging,

$$\cos\theta = \sqrt{\frac{(E_1 + E_0)(E_1' - E_0)}{(E_1 - E_0)(E_1' + E_0)}} = \sqrt{\frac{(K_1 + 2m_0c^2)K_1'}{K_1(K_1' + 2m_0c^2)}} \tag{30}$$

(2 points)

Rearranging gives

$$k = \frac{K_1'}{K_1} = \frac{\cos^2\theta}{K_1(1-\cos^2\theta) + 2m_0c^2}\,2m_0c^2 \tag{31}$$

(2 points)

Comparing (31) with (12), their ratio is seen to be

$$\frac{2m_0c^2}{K_1(1-\cos^2\theta) + 2m_0c^2} \leq 1 \tag{32}$$

(1 point)

i.e. for the scattering of high-energy particles the classical model overestimates the kinetic energy of the scattered particle.

Or: comparing (31) with (12), one sees that when $K_1 \ll m_0c^2$, (31) degenerates into (12).

(1 point)

(32′)

(4) From (2) we obtain

[Translator's note: as printed; the momentum relation used here is (3), not (2).]

$$p_1^2 = p_1'^2 + p_2'^2 + 2p_1'p_2'\cos\alpha \tag{33}$$

(2 points)

Substituting (1), (27) and (28) into (33) gives

$$(E_1' - E_0)(E_2' - E_0) = \sqrt{(E_1'^2 - E_0^2)(E_2'^2 - E_0^2)}\cos\alpha$$

$$\cos\alpha = \sqrt{\frac{(E_1' - E_0)(E_2' - E_0)}{(E_1' + E_0)(E_2'^2 + E_0)}} = \sqrt{\frac{K_1'(K_1 - K_1')}{(K_1' + 2m_0c^2)(K_1 - K_1' + 2m_0c^2)}} \tag{34}$$

(3 points)

[Translator's note: as printed; the second factor of the denominator under the first root should read $(E_2' + E_0)$.]

Differentiating (33) with respect to K_1' and rearranging,

[Translator's note: as printed; it is (34) that is differentiated.]

$$\frac{\mathrm{d}(\cos\alpha)}{\mathrm{d}K_1'} = \frac{(K_1 + 2m_0c^2)(K_1 - 2K_1')E_0}{\sqrt{(K_1' + 2m_0c^2)(K_1 - K_1' + 2m_0c^2)}^3}$$

If $\dfrac{\mathrm{d}(\cos\alpha)}{\mathrm{d}K_1'} = 0$, then

$$K_1' = \frac{1}{2}K_1 \tag{35}$$

(2 points)

At this point $\dfrac{\mathrm{d}^2(\cos\alpha)}{\mathrm{d}K_1'^2} < 0$, so α takes a minimum value. Substituting (35) into (34) gives

$$\cos\alpha_{\min} = \frac{K_1}{K_1 + 4m_0c^2}$$

$$\alpha_{\min} = \arccos\frac{K_1}{K_1 + 4m_0c^2} \tag{36}$$

(2 points)

Substituting (35) into (30) gives

$$\cos^2\theta = \frac{K_1 + 2m_0c^2}{K_1 + 4m_0c^2} = \frac{1}{2}(\cos\alpha_{\min} + 1)$$

$$\theta = \frac{\alpha_{\min}}{2} = \frac{1}{2}\arccos\frac{K_1}{K_1 + 4m_0c^2} \tag{37}$$

(2 points)

This shows that when the scattered particle and the recoil particle each take half of the kinetic energy of the incident particle, the angle between their velocities is smallest, and the outgoing directions of the two particles are symmetric.

【From (34) we know that when $K_1' = 0$ or K_1, which correspond respectively to a head-on collision and to a grazing pass, $\cos\alpha = 0$, and α has the maximum limiting value

$$\alpha_{\max} = \frac{\pi}{2}$$

Correspondingly, from (38) we know that then $\cos\theta = 0$ or 1, i.e. $\theta = \dfrac{\pi}{2}$ or 0.】

[Translator's note: as printed; there is no equation (38) — the reference is to (30).]

Mark scheme (60 points in total). Part (1), 4 points: (1) and (2) 2 points each. Part (2), 34 points, of which (i) 18 points and (ii) 16 points. (i) (3), (4), (6) (or (7)), (10), (11), (12), (13) (if written as $0 \le \theta \le \pi/2$, no marks are deducted), (14), (15): 2 points each. (ii) (16) 1 point, (19) 4 points, (20) and (21) 1 point each; 【Method 1】: (22) 1 point, (23) and (24) 2 points each, (25) 4 points; 【Method 2】: (22′) 2 points, (23′) 1 point, (24′) 2 points, (25′) 4 points. Note: in (25) the value $\mathrm{d}(\cos\theta)/\mathrm{d}k = 0$ may be attached to either the upper or the lower branch, as long as an equality sign is added to the corresponding inequality. Part (3), 11 points: (27), (28), (29), (30), (31) 2 points each, (32) (or (32′)) 1 point. Part (4), 11 points: (33) 2 points, (34) 3 points, (35), (36), (37) 2 points each. (Note: if, besides (37), the content inside 【 】 is also written, i.e. if the maximum limiting value is taken as an extremum, no marks are deducted.)

Problem 3 (a rigid ring rolling on the ground: steady precession and stability)

Solution

(1) Denote the position vector of the instantaneous contact point by

$$\boldsymbol{A} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \tag{1}$$

The position vector of the centre O of the ring (its centre of mass) is

$$\boldsymbol{O} = \boldsymbol{A} + R\begin{pmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} \tag{2}$$

The velocity of the centre of mass O is

$$\boldsymbol{v}_{\mathrm{O}} = \dot{\boldsymbol{O}} = \begin{pmatrix} \dot{x} + R(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) \\ \dot{y} + R(-\dot{\theta}\cos\theta\cos\phi + \dot{\phi}\sin\theta\sin\phi) \\ -R\dot{\theta}\sin\theta \end{pmatrix} = \dot{\boldsymbol{A}} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{O} - \boldsymbol{A}) \tag{3}$$

The condition for rolling without slipping is that $\boldsymbol{v}_O$ is perpendicular to $(\boldsymbol{O} - \boldsymbol{A})$. Because the length of $(\boldsymbol{O} - \boldsymbol{A})$ is fixed, $(\boldsymbol{O} - \boldsymbol{A})$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{O} - \boldsymbol{A})$ are always perpendicular, so the condition of rolling without slipping reduces to $\dot{\boldsymbol{A}}$ being perpendicular to $(\boldsymbol{O} - \boldsymbol{A})$:

$$(\dot{x},\ \dot{y},\ 0)\begin{pmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} = 0 \tag{4}$$

that is,

$$\dot{x}\sin\theta\sin\phi - \dot{y}\sin\theta\cos\phi = 0 \tag{5}$$

In general θ is not zero, so

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \parallel \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{6}$$

That is, the ring should roll within the plane in which it instantaneously lies; the velocity of the contact point is along the direction of the line of intersection of that instantaneous plane with the horizontal plane.

(2) In this part we use the XY coordinate system which rotates uniformly with angular velocity ω about the z axis through the coordinate origin. In this coordinate system the centre of mass of the ring is at rest and its centre-of-mass momentum is always 0. According to the constraint of rolling without slipping, the magnitude of the angular velocity of rotation of the ring about its own central axis is

$$\omega' = \omega \frac{r}{R} \tag{7}$$

The angular velocity vector is

$$\boldsymbol{\omega}' = \omega \frac{r}{R} \begin{pmatrix} 0 \\ -\cos\theta \\ -\sin\theta \end{pmatrix} \tag{8}$$

The angular momentum is always

$$\boldsymbol{L} = MR^2 \boldsymbol{\omega}' = MRr\omega \begin{pmatrix} 0 \\ -\cos\theta \\ -\sin\theta \end{pmatrix}$$

The position vector of the contact point A of the ring with the ground is

$$\boldsymbol{A} = \begin{pmatrix} X_A \\ Y_A \\ z_A \end{pmatrix} = \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}$$

The position vector of the centre of mass O is

$$\boldsymbol{O} = \begin{pmatrix} X_O \\ Y_O \\ z_O \end{pmatrix} = \begin{pmatrix} 0 \\ r - R\sin\theta \\ R\cos\theta \end{pmatrix} \tag{9}$$

Then the position vector of the point B on the ring whose radius OB makes $90°$ with the radius OA is

$$\boldsymbol{B} = \begin{pmatrix} X_B \\ Y_B \\ z_B \end{pmatrix} = \begin{pmatrix} X_O \\ Y_O \\ z_O \end{pmatrix} + \begin{pmatrix} -R \\ 0 \\ 0 \end{pmatrix}$$

Consider an arbitrary point C on the ring and put $\angle COA \equiv \varphi$; then the position vector of C in the XY coordinate system is

$$\boldsymbol{C} = \begin{pmatrix} X_C \\ Y_C \\ z_C \end{pmatrix} = \begin{pmatrix} X_O \\ Y_O \\ z_O \end{pmatrix} + \begin{pmatrix} 0 \\ R\sin\theta \\ -R\cos\theta \end{pmatrix} \cos\varphi + \begin{pmatrix} -R \\ 0 \\ 0 \end{pmatrix} \sin\varphi = \begin{pmatrix} -R\sin\varphi \\ r - R\sin\theta\,(1-\cos\varphi) \\ R\cos\theta\,(1-\cos\varphi) \end{pmatrix} \tag{10}$$

The centrifugal force acting (in the XY frame) on the arc element $\mathrm{d}\varphi$ at the point C of the ring is

$$\mathrm{d}\boldsymbol{F}_{\text{cent}} = M\frac{\mathrm{d}\varphi}{2\pi}\omega^2 \begin{pmatrix} X_C \\ Y_C \\ 0 \end{pmatrix} \tag{11}$$

(2 points)

The total centrifugal force is

$$\boldsymbol{F}_{\text{cent}} = \int_{\varphi=0}^{2\pi} \mathrm{d}\boldsymbol{F}_{\text{cent}} = M\omega^2 \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \begin{pmatrix} X_C \\ Y_C \\ 0 \end{pmatrix} = M\omega^2 \begin{pmatrix} X_O \\ Y_O \\ 0 \end{pmatrix} = M\omega^2 \begin{pmatrix} 0 \\ r - R\sin\theta \\ 0 \end{pmatrix} \tag{12}$$

(2 points)

The Coriolis force acting (in the XY frame) on the arc element $\mathrm{d}\varphi$ at the point C of the ring is

$$\mathrm{d}\boldsymbol{F}_{\mathrm{Cor}} = M\frac{\mathrm{d}\varphi}{2\pi}\left[-2\omega\begin{pmatrix}0\\0\\1\end{pmatrix}\times\boldsymbol{v}_C\right] \tag{13}$$

(2 points)

where $\boldsymbol{v}_C$ is the velocity of the point C in the XY frame due to its rotation about the centre of the ring,

$$\boldsymbol{v}_C = \boldsymbol{\omega}'\times(\boldsymbol{C}-\boldsymbol{O}) = \omega\frac{r}{R}\begin{pmatrix}0\\-\cos\theta\\-\sin\theta\end{pmatrix}\times(\boldsymbol{C}-\boldsymbol{O}) = \omega r\begin{pmatrix}\cos\varphi\\ \sin\theta\sin\varphi\\ -\cos\theta\sin\varphi\end{pmatrix} \tag{14}$$

The total Coriolis force is

$$\boldsymbol{F}_{\mathrm{Cor}} = \int_{\varphi=0}^{2\pi}\mathrm{d}\boldsymbol{F}_{\mathrm{Cor}} = -2M\omega\begin{pmatrix}0\\0\\1\end{pmatrix}\times\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\boldsymbol{v}_C = -2M\omega\begin{pmatrix}0\\0\\1\end{pmatrix}\times\boldsymbol{v}_O = 0 \tag{15}$$

(3 points)

The torque of the centrifugal force on the arc element $\mathrm{d}\varphi$ is

$$\mathrm{d}\boldsymbol{\tau}_{\mathrm{cent}} = (\boldsymbol{C}-\boldsymbol{O})\times M\frac{\mathrm{d}\varphi}{2\pi}\omega^2\begin{pmatrix}X_C\\Y_C\\0\end{pmatrix} = M\frac{\mathrm{d}\varphi}{2\pi}\omega^2(\boldsymbol{C}-\boldsymbol{O})\times\left(\boldsymbol{C}-\begin{pmatrix}0\\0\\z_C\end{pmatrix}\right) \tag{16}$$

(2 points)

Note that

$$(\boldsymbol{C}-\boldsymbol{O})\times\boldsymbol{C} = (\boldsymbol{C}-\boldsymbol{O})\times(\boldsymbol{C}-\boldsymbol{O}) + (\boldsymbol{C}-\boldsymbol{O})\times\boldsymbol{O} = (\boldsymbol{C}-\boldsymbol{O})\times\boldsymbol{O}$$

and therefore

$$\int_{\varphi=0}^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}(\boldsymbol{C}-\boldsymbol{O})\times\boldsymbol{C} = \left[\int_{\varphi=0}^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}(\boldsymbol{C}-\boldsymbol{O})\right]\times\boldsymbol{O} = 0$$

The total torque $\boldsymbol{\tau}_{\mathrm{cent}}$ of the centrifugal forces is

$$\begin{aligned}\boldsymbol{\tau}_{\mathrm{cent}} &= -\int_{\varphi=0}^{2\pi}M\frac{\mathrm{d}\varphi}{2\pi}\omega^2(\boldsymbol{C}-\boldsymbol{O})\times\begin{pmatrix}0\\0\\z_C\end{pmatrix} = -M\omega^2\int_{\varphi=0}^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}z_C\begin{pmatrix}Y_C-Y_O\\-(X_C-X_O)\\0\end{pmatrix}\\ &= -M\omega^2\int_{\varphi=0}^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}R\cos\theta\,(1-\cos\varphi)\begin{pmatrix}R\sin\theta\cos\varphi\\R\sin\varphi\\0\end{pmatrix}\\ &= \frac{1}{2}M\omega^2R^2\begin{pmatrix}\cos\theta\sin\theta\\0\\0\end{pmatrix}\end{aligned} \tag{17}$$

(4 points)

The torque $\mathrm{d}\boldsymbol{\tau}_{\mathrm{Cor}}$ of the Coriolis force on the arc element $\mathrm{d}\varphi$ is

$$\begin{aligned}\mathrm{d}\boldsymbol{\tau}_{\mathrm{Cor}} &= -M\frac{\mathrm{d}\varphi}{2\pi}(\boldsymbol{C}-\boldsymbol{O})\times\left[\begin{pmatrix}0\\0\\2\omega\end{pmatrix}\times\boldsymbol{v}_C\right]\\ &= -M\frac{\mathrm{d}\varphi}{2\pi}\begin{pmatrix}-R\sin\varphi & R\sin\theta\cos\varphi & -R\cos\theta\cos\varphi\end{pmatrix}\times\begin{pmatrix}-2\omega^2r\sin\theta\sin\varphi\\2\omega^2r\cos\varphi\\0\end{pmatrix}\end{aligned} \tag{18}$$

(2 points)

The total torque $\boldsymbol{\tau}_{\text{Cor}}$ of the Coriolis forces is

$$\boldsymbol{\tau}_{\text{Cor}} = \int_{\varphi=0}^{2\pi} \mathrm{d}\boldsymbol{\tau}_{\text{Cor}} = \begin{pmatrix} -M\omega^2 Rr\cos\theta \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

(4 points)

In the above calculation the following integral formulas have been used:

$$\int_{\varphi=0}^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\sin\varphi = \int_{\varphi=0}^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\cos\varphi = \int_{\varphi=0}^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\sin\varphi\cos\varphi = 0$$

$$\int_{\varphi=0}^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}(\cos\varphi)^2 = \int_{\varphi=0}^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}(\sin\varphi)^2 = \frac{1}{2}$$

(3)

Method 1. In the rotating frame the position of the centre of mass of the ring is fixed and the ring rotates uniformly about its axis of symmetry, so the resultant force and the resultant torque on the ring must both be zero. Besides the centrifugal force and the Coriolis force considered in part (2), the ring is also acted on by gravity

$$\boldsymbol{F}_{\text{grav}} = \begin{pmatrix} 0 \\ 0 \\ -Mg \end{pmatrix}$$

as well as by the normal force $\boldsymbol{N}$ of the ground on the ring (vertical) and the friction $\boldsymbol{f}$ (horizontal). The force balance condition is

$$\boldsymbol{F}_{\text{grav}} + \boldsymbol{F}_{\text{cent}} + \boldsymbol{F}_{\text{Cor}} + \boldsymbol{N} + \boldsymbol{f} = 0$$

that is,

$$\begin{pmatrix} 0 \\ 0 \\ -Mg \end{pmatrix} + M\omega^2 \begin{pmatrix} 0 \\ r - R\sin\theta \\ 0 \end{pmatrix} + 0 + \boldsymbol{N} + \boldsymbol{f} = 0 \tag{20}$$

(1 point)

Clearly $\boldsymbol{N}$ should be along the z direction and $\boldsymbol{f}$ should lie in the xy plane. So in the XY coordinate system

$$\boldsymbol{N} = \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix} \tag{21}$$

(2 points)

$$\boldsymbol{f} = M\omega^2 \begin{pmatrix} 0 \\ R\sin\theta - r \\ 0 \end{pmatrix} \tag{22}$$

(1 point)

The torque of gravity about the centre of mass is zero. The torque $\boldsymbol{\tau}_{\text{ground}}$ about the centre of mass O of the resultant of the normal force and the friction of the ground on the ring (in the XY frame) is

$$\begin{aligned} \boldsymbol{\tau}_{\text{ground}} &= (\boldsymbol{A} - \boldsymbol{O}) \times (\boldsymbol{N} + \boldsymbol{f}) = R\begin{pmatrix} 0 & \sin\theta & -\cos\theta \end{pmatrix} \times \begin{pmatrix} 0 \\ M\omega^2(R\sin\theta - r) \\ Mg \end{pmatrix} \\ &= M\begin{pmatrix} gR\sin\theta + \omega^2 R(R\sin\theta - r)\cos\theta \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{23}$$

(2 points)

The torque balance condition is

$$\boldsymbol{\tau}_{\text{cent}} + \boldsymbol{\tau}_{\text{Cor}} + \boldsymbol{\tau}_{\text{ground}} = 0$$

from which

$$\frac{\omega^2 R^2}{2}\cos\theta\sin\theta - \omega^2 Rr\cos\theta + gR\sin\theta + \omega^2 R(R\sin\theta - r)\cos\theta = 0 \tag{24}$$

(1 point)

which gives

$$\omega = \sqrt{\frac{g\sin\theta}{\cos\theta\left(2r - \frac{3}{2}R\sin\theta\right)}} \tag{25}$$

(3 points)

The friction of the ground on the ring equals

$$\boldsymbol{f} = -\frac{2Mg\sin\theta}{(4r - 3R\sin\theta)\cos\theta}\begin{pmatrix} 0 \\ r - R\sin\theta \\ 0 \end{pmatrix} \tag{26}$$

(2 points)

The force exerted by the ground on the ring is given by (21) and (26).

Method 2: using the concepts of the principal axes, the inertia tensor and the angular velocity vector of a rigid body.
This method considers the equations of motion of the ring in the non-rotating xyz coordinate system. Note that the relation between the xyz frame and the rotating XYz frame is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X\cos(\omega t) - Y\sin(\omega t) \\ X\sin(\omega t) + Y\cos(\omega t) \end{pmatrix}$$

For convenience define the unit vectors of the rotating XYz frame,

$$\hat{X} = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix}, \quad \hat{Y} = \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}, \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

The centre of mass of the ring performs uniform circular motion of radius $r - R\sin\theta$ with angular velocity ω; from the theorem of the motion of the centre of mass, the normal force and the friction of the ground on the ring are respectively

$$\boldsymbol{N} = Mg\hat{z} \tag{21}$$

(2 points)

$$\boldsymbol{f} = -M\omega^2(r - R\sin\theta)\hat{Y} \tag{22}$$

(1 point)

The angular velocity of rotation of the ring about its centre of mass in the rotating XYz frame is

$$\boldsymbol{\omega}'_{XYz} = -\omega\frac{r}{R}\left(\sin\theta\,\hat{z} + \cos\theta\,\hat{Y}\right)$$

so the angular velocity of rotation of the ring about its centre of mass in the xyz frame is

$$\boldsymbol{\omega}'_{xyz} = \boldsymbol{\omega}'_{XYz} + \omega\hat{z} = \omega\left(1 - \frac{r}{R}\sin\theta\right)\hat{z} - \omega\frac{r}{R}\cos\theta\,\hat{Y} \tag{23}$$

The unit vectors of the principal axis directions of the ring are respectively

$$\hat{a}_1 = \sin\theta\,\hat{z} + \cos\theta\,\hat{Y}, \qquad \hat{a}_2 = \hat{X}, \qquad \hat{a}_3 = -\cos\theta\,\hat{z} + \sin\theta\,\hat{Y} \tag{24}$$

The moments of inertia about the principal axes are respectively

$$I_1 = MR^2, \qquad I_2 = I_3 = \tfrac{1}{2}MR^2 \tag{25}$$

(1 point)

Therefore the angular momentum of the ring about its centre of mass is

$$\begin{aligned}\boldsymbol{L} &= (\boldsymbol{\omega}'_{xyz}\cdot\hat{a}_1)\hat{a}_1 I_1 + (\boldsymbol{\omega}'_{xyz}\cdot\hat{a}_2)\hat{a}_2 I_2 + (\boldsymbol{\omega}'_{xyz}\cdot\hat{a}_3)\hat{a}_3 I_3 \\ &= MR^2\omega\left(\sin\theta - \frac{r}{R}\right)\hat{a}_1 - \frac{1}{2}MR^2\omega\cos\theta\,\hat{a}_3 \\ &= MR^2\omega\left[\frac{1}{2}\left(1+(\sin\theta)^2\right) - \frac{r}{R}\sin\theta\right]\hat{z} + MR^2\omega\left(\frac{1}{2}\sin\theta - \frac{r}{R}\right)\cos\theta\,\hat{Y}\end{aligned} \tag{26}$$

The rate of change of the angular momentum about the centre of mass equals the torque of the external forces not including gravity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{\tau}_{\text{ground}}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X} = \omega\hat{Y}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{Y} = -\omega\hat{X}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{z} = 0$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = -MR^2\omega^2\left(\frac{1}{2}\sin\theta - \frac{r}{R}\right)\cos\theta\,\hat{X} \tag{27}$$

(1 point)

and the torque $\boldsymbol{\tau}_{\text{ground}}$ about the centre of mass of the ring of the resultant force of the ground on the ring is

$$\boldsymbol{\tau}_{\text{ground}} = R\hat{a}_3 \times (\boldsymbol{N} + \boldsymbol{f}) = MgR\sin\theta\,\hat{X} - M\omega^2(r - R\sin\theta)R\hat{X} \tag{28}$$

(2 points)

[Translator's note: as printed; the second term should carry a factor $\cos\theta$, i.e. $-M\omega^2(r - R\sin\theta)R\cos\theta\,\hat{X}$, as the next line implies.]

Therefore

$$-MR^2\omega^2\left(\frac{1}{2}\sin\theta - \frac{r}{R}\right)\cos\theta = MgR\sin\theta - M\omega^2(r - R\sin\theta)R$$

$$M\omega^2 R\left(2r - \frac{3}{2}R\sin\theta\right)\cos\theta = MgR\sin\theta$$

which gives

$$\omega = \sqrt{\frac{g\sin\theta}{\cos\theta\left(2r - \frac{3}{2}R\sin\theta\right)}} \tag{29}$$

(3 points)

The friction of the ground on the ring equals

$$\boldsymbol{f} = -\frac{2Mg\sin\theta}{(4r - 3R\sin\theta)\cos\theta}\begin{pmatrix}0\\ r - R\sin\theta\\ 0\end{pmatrix} \tag{30}$$

(2 points)

(4) Note: in the small-quantity analysis below it is always assumed that a variable and its time derivative are small quantities of the same order, i.e. it is assumed that the motion contains no component whose oscillation frequency tends to infinity.
The kinematic constraint of part (1), kept to the lowest order in the small quantities, is

$$\dot{y} \approx V\phi \tag{31}$$

(2 points)

Hence y and ϕ are small quantities of the same order.
To first order in the small quantities the position vector of the centre of mass O of the ring is

$$\boldsymbol{O} = \begin{pmatrix} x + R\sin\theta\sin\phi \\ y - R\sin\theta\cos\phi \\ R\cos\theta \end{pmatrix} \approx \begin{pmatrix} Vt + \delta x \\ y - R\theta \\ R \end{pmatrix} \tag{32}$$

(2 points)

The friction of the ground on the ring, to first order in the small quantities, is

$$\begin{aligned} \boldsymbol{f} &= M\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_O = M\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \dot{x} + R(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) \\ \dot{y} + R(-\dot{\theta}\cos\theta\cos\phi + \dot{\phi}\sin\theta\sin\phi) \\ -R\dot{\theta}\sin\theta \end{pmatrix} \\ &\approx M\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} V + \dot{\delta x} \\ \dot{y} - R\dot{\theta} \\ 0 \end{pmatrix} = M\begin{pmatrix} \ddot{\delta x} \\ \ddot{y} - R\ddot{\theta} \\ 0 \end{pmatrix} \end{aligned} \tag{33}$$

(1 point)

The normal force of the ground on the ring, to first order, is

$$\boldsymbol{N} \approx \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix} \tag{34}$$

(1 point)

The angular momentum of the ring about its centre of mass is now computed approximately in two ways.

Calculation method 1: analysis by mass elements. By analogy with part (2), consider a point C on the ring; its instantaneous coordinates are approximately

$$\boldsymbol{C} \approx \boldsymbol{O} + R\begin{pmatrix} -\sin\varphi \\ \theta\cos\varphi - \phi\sin\varphi \\ -\cos\varphi \end{pmatrix} \tag{35}$$

To first order,

$$\dot{\phi} \approx \frac{(\dot{x}\cos\phi + \dot{y}\sin\phi)}{R} \approx \frac{V}{R} + \frac{1}{R}\dot{\delta x} \tag{36}$$

so the velocity of the point C relative to the centre of mass O is approximately

$$\boldsymbol{v}_C - \boldsymbol{v}_O \approx R\begin{pmatrix} 0 \\ \dot{\theta}\cos\varphi - \dot{\phi}\sin\varphi \\ 0 \end{pmatrix} + \left(V + \dot{\delta x}\right)\begin{pmatrix} -\cos\varphi \\ -\theta\sin\varphi - \phi\cos\varphi \\ \sin\varphi \end{pmatrix} \tag{37}$$

The angular momentum of the arc element $\mathrm{d}\varphi$ about the centre of mass is

$$\begin{aligned}\mathrm{d}\boldsymbol{L} &= (\boldsymbol{C}-\boldsymbol{O})\times M\frac{\mathrm{d}\varphi}{2\pi}(\boldsymbol{v}_C-\boldsymbol{v}_O)\\ &\approx \frac{\mathrm{d}\varphi}{2\pi}MR^2\begin{pmatrix}\dot\theta(\cos\varphi)^2-\dot\phi\cos\varphi\sin\varphi\\ 0\\ -\dot\theta\sin\varphi\cos\varphi+\dot\phi(\sin\varphi)^2\end{pmatrix}+\frac{\mathrm{d}\varphi}{2\pi}MR\left(V+\dot{\delta x}\right)\begin{pmatrix}-\phi\\ 1\\ \theta\end{pmatrix}\end{aligned}\tag{38}$$

The total angular momentum is

$$\boldsymbol{L}=\int_{\varphi=0}^{2\pi}(\boldsymbol{C}-\boldsymbol{O})\times M\frac{\mathrm{d}\varphi}{2\pi}(\boldsymbol{v}_C-\boldsymbol{v}_O)\approx MR^2\begin{pmatrix}-\frac{V}{R}\phi+\frac{1}{2}\dot\theta\\ \frac{V}{R}+\frac{1}{R}\dot{\delta x}\\ \frac{1}{2}\dot\phi+\frac{V}{R}\theta\end{pmatrix}\tag{39}$$

(3 points)

Calculation method 2: using the principal axes, the inertia tensor and the angular velocity vector of a rigid body. To first order the three principal axis directions of the ring are respectively (see Method 2 of part (2) above):
central axis direction

$$\hat a_1=\begin{pmatrix}-\cos\theta\sin\phi\\ \cos\theta\cos\phi\\ \sin\theta\end{pmatrix}\approx\begin{pmatrix}-\phi\\ 1\\ \theta\end{pmatrix}$$

approximate ϕ rotation axis

$$\hat a_2=\begin{pmatrix}\sin\theta\sin\phi\\ -\sin\theta\cos\phi\\ \cos\theta\end{pmatrix}\approx\begin{pmatrix}0\\ -\theta\\ 1\end{pmatrix}$$

θ rotation axis

$$\hat a_3=\begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}\approx\begin{pmatrix}1\\ \phi\\ 0\end{pmatrix}$$

The moments of inertia are respectively

$$I_1=MR^2,\qquad I_2=I_3=\tfrac{1}{2}MR^2\tag{35}$$

The angular velocities about the three principal axes are approximately:
angular velocity about the central axis direction

$$\omega_1'\approx\frac{1}{R}\left(V+\dot{\delta x}\right)\tag{36}$$

angular velocity about the approximate ϕ rotation axis

$$\omega_2'\approx\dot\phi\tag{37}$$

angular velocity about the θ rotation axis

$$\omega_3'\approx\dot\theta\tag{38}$$

The angular momentum of the ring about its centre of mass is approximately

$$\boldsymbol{L}\approx\frac{1}{R}\left(V+\dot{\delta x}\right)MR^2\begin{pmatrix}-\phi\\ 1\\ \theta\end{pmatrix}+\frac{1}{2}MR^2\dot\phi\begin{pmatrix}0\\ -\theta\\ 1\end{pmatrix}+\frac{1}{2}MR^2\dot\theta\begin{pmatrix}1\\ \phi\\ 0\end{pmatrix}\approx MR^2\begin{pmatrix}-\frac{V}{R}\phi+\frac{1}{2}\dot\theta\\ \frac{V}{R}+\frac{1}{R}\dot{\delta x}\\ \frac{1}{2}\dot\phi+\frac{V}{R}\theta\end{pmatrix}\tag{39}$$

The torque $\boldsymbol{\tau}_{\text{ground}}$ about the centre of mass of the ring of the resultant of the normal force and the friction of the ground on the ring is, to first order,

$$\boldsymbol{\tau}_{\text{ground}} \approx R\begin{pmatrix}0 & \theta & -1\end{pmatrix} \times M\begin{pmatrix}\ddot{\delta x} \\ \ddot{y} - R\ddot{\theta} \\ g\end{pmatrix} \approx MR\begin{pmatrix}g\theta + \ddot{y} - R\ddot{\theta} \\ -\ddot{\delta x} \\ 0\end{pmatrix} \tag{40}$$

(1 point)

From

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{\tau}_{\text{ground}}$$

the three components give the equations

$$-V\dot{\phi} + \frac{R}{2}\ddot{\theta} \approx g\theta + \ddot{y} - R\ddot{\theta} \tag{41}$$

(1 point)

$$\ddot{\delta x} \approx -\ddot{\delta x} \tag{42}$$

(1 point)

$$\frac{R}{2}\ddot{\phi} + V\dot{\theta} \approx 0 \tag{43}$$

(1 point)

Taking further account of the kinematic constraint $\dot{y} \approx V\phi$ found above, we see that $y(t)$, $\theta(t)$, $\phi(t)$ are small quantities of the same order.
Substituting (31) into (41) to eliminate y, we get

$$\frac{3R}{2}\ddot{\theta} - g\theta \approx 2V\dot{\phi}$$

Taking the derivative again and substituting (43), we obtain

$$\frac{3R}{2}\dddot{\theta} + \left(\frac{4V^2}{R} - g\right)\dot{\theta} = 0 \tag{44}$$

(2 points)

Assume that $y(t)$, $\theta(t)$, $\phi(t)$ all perform simple harmonic oscillations of angular frequency Ω, i.e. that they are all proportional to $\cos(\Omega t + \text{constant})$; then Ω must satisfy

$$-\frac{3R}{2}\Omega^2 + \left(\frac{4V^2}{R} - g\right) = 0$$

so

$$\Omega = \sqrt{\frac{8V^2}{3R^2} - \frac{2g}{3R}} \tag{45}$$

(2 points)

Stability of the motion of the ring requires Ω to be real, so

$$\frac{8V^2}{3R^2} - \frac{2g}{3R} \geq 0$$

and the minimum speed is

$$V_{\min} = \frac{\sqrt{gR}}{2} \tag{46}$$

(2 points)

Equation (42) shows that in the lowest-order small-quantity approximation $\ddot{\delta x} \approx 0$, so δx should be a small quantity of higher order than y, θ, ϕ.

Mark scheme (60 points in total). Part (1), 6 points: (2), (3), (5), (6) 1 point each; (4) 2 points. Part (2), 23 points: (7) 2 points, (11) 2 points, (12) 2 points, (13) 2 points, (15) 3 points, (16) 2 points, (17) 4 points, (18) 2 points, (19) 4 points. Part (3), 12 points: Method 1: (20) 1 point, (21) 2 points, (22) 1 point, (23) 2 points, (24) 1 point, (25) 3 points, (26) 2 points; Method 2: (21) 2 points, (22) 1 point, (25) 1 point, (27) 1 point, (28) 2 points, (29) 3 points, (30) 2 points. Part (4), 19 points: (31) 2 points, (32) 2 points, (33) and (34) 1 point each, (39) 3 points, (40), (41), (42), (43) 1 point each, (44), (45), (46) 2 points each.

Problem 4 (free electron laser: electron motion, accelerating voltage, 2-D diffraction)

Solution

(1)

【Method 1】 Find the motion of the electron in the frame S. Suppose the electron starts in the undulator from the coordinate origin with initial velocity

$$\boldsymbol{v_0} = v_0\boldsymbol{k}$$

By the relativistic momentum theorem for a particle,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -e\boldsymbol{v}\times\boldsymbol{B} \tag{1}$$

(2 points)

where the relativistic momentum of the electron is

$$\boldsymbol{p} = m\boldsymbol{v} = \frac{m_e}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\boldsymbol{v}$$

From (1) the magnetic force does no work, the energy is unchanged and correspondingly the speed is unchanged, so

$$v^2 = v_x^2 + v_y^2 + v_z^2 = v_0^2 \tag{2}$$

(2 points)

Therefore also

$$m = \frac{m_e}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}}$$

is a constant. Substituting

$$\boldsymbol{B} = B_0\cos\left(\frac{2\pi}{\Lambda}z\right)\boldsymbol{i}$$

into (1) and taking into account that m is a constant, we obtain

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{eB_0}{m}v_z\cos\left(\frac{2\pi}{\Lambda}z\right) \tag{3}$$

(1 point)

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{eB_0}{m}v_y\cos\left(\frac{2\pi}{\Lambda}z\right) \tag{4}$$

(1 point)

From $\mathrm{d}v_x/\mathrm{d}t = 0$ and the initial conditions ($x_0 = 0$, $v_{x0} = 0$),

$$v_x \equiv 0, \qquad x \equiv 0$$

From (3),

$$\mathrm{d}v_y = -\frac{eB_0}{m}\cos\left(\frac{2\pi}{\Lambda}z\right)\mathrm{d}z$$

Integrating (with $v_{y0} = 0$, $z_0 = 0$),

$$v_y = -\frac{eB_0\Lambda}{2\pi m}\sin\left(\frac{2\pi}{\Lambda}z\right) \tag{5}$$

(2 points)

Substituting $v_x \equiv 0$ and (5) into (2),

$$v_z = \sqrt{v_0^2 - \frac{e^2B_0^2\Lambda^2}{8\pi^2m^2}\left[1-\cos\left(\frac{4\pi}{\Lambda}z\right)\right]} \tag{6}$$

(1 point)

【Remark: one may also use the differential identity $\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}v_z^2}{\mathrm{d}z}$ and integrate (4) to obtain (6).】

By the statement of the problem the magnetic field is not strong and the change of the electron speed is much smaller in magnitude than v_0, so to leading order

$$v_z \approx v_0, \qquad z \approx v_0 t$$

Substituting this approximation into the magnetic field distribution function felt by the electron (and into the corresponding integrated results), (7) gives

[Translator's note: as printed; the equation used here is (5).]

$$v_y = -\frac{eB_0\Lambda}{2\pi m}\sin\left(\frac{2\pi}{\Lambda}v_0t\right)$$

Integrating (with $y_0 = 0$),

$$y = \frac{eB_0\Lambda^2}{4\pi^2v_0m}\left[\cos\left(\frac{2\pi}{\Lambda}v_0t\right)-1\right] \tag{7}$$

(1 point)

Moreover, accurate to the order of the small quantity B_0^2, (6) gives

$$v_z = v_0\left(1-\frac{e^2B_0^2\Lambda^2}{16\pi^2m^2v_0^2}\left[1-\cos\left(\frac{4\pi}{\Lambda}v_0t\right)\right]\right) \tag{8}$$

(1 point)

Integrating over time (with $z_0 = 0$),

$$z = v_0t - \frac{e^2B_0^2\Lambda^2}{16\pi^2m^2v_0}t + \frac{e^2B_0^2\Lambda^3}{64\pi^3m^2v_0}\sin\left(\frac{4\pi}{\Lambda}v_0t\right) \tag{9}$$

(1 point)

Using the Lorentz transformation to go over to the frame S′, we obtain

$$x' = x = 0$$

$$y' = y = \frac{eB_0\Lambda^2}{4\pi^2 v_0 m}\left[\cos\left(\frac{2\pi}{\Lambda}v_0\frac{t'+\frac{v_0}{c^2}z'}{\sqrt{1-\beta^2}}\right)-1\right] \tag{10}$$

(2 points)

$$z' = \frac{-\frac{e^2B_0^2\Lambda^2}{16\pi^2m^2v_0}\frac{t'+\frac{v_0}{c^2}z'}{\sqrt{1-\beta^2}}+\frac{e^2B_0^2\Lambda^3}{64\pi^3m^2v_0}\sin\left(\frac{4\pi}{\Lambda}v_0\frac{t'+\frac{v_0}{c^2}z'}{\sqrt{1-\beta^2}}\right)}{\sqrt{1-\beta^2}} \tag{11}$$

(2 points)

It is easy to see that the leading order of z' is proportional to the small quantity B_0^2. Expanding (10) and (11) each to its own leading order,

$$y' = \frac{eB_0v_0}{\omega'^2 m_e\sqrt{1-\beta^2}}\left[\cos(\omega' t')-1\right] \tag{12}$$

(1 point)

$$z' = \frac{e^2v_0B_0^2}{8\omega'^3m_e^2(1-\beta^2)}\left(\sin 2\omega' t' - 2\omega' t'\right) \tag{13}$$

(1 point)

where

$$\omega' = \frac{2\pi}{\Lambda}\frac{v_0}{\sqrt{1-\beta^2}}$$

【Method 2】 The electromagnetic field in the frame $\mathrm{S}'(x',y',z')$ (only the non-zero components are written):

$$E_{y'} = \frac{v_0}{\sqrt{1-\beta^2}}B_0\cos\left(\frac{2\pi}{\Lambda}z\right) \tag{1'}$$

(2 points)

$$B_{x'} = \frac{1}{\sqrt{1-\beta^2}}B_0\cos\left(\frac{2\pi}{\Lambda}z\right) \tag{2'}$$

(2 points)

where $\beta = v_0/c$. The Lorentz transformation gives

$$z = \frac{z'+v_0t'}{\sqrt{1-\beta^2}} \tag{3'}$$

(2 points)

Hence

$$E_{y'} = \frac{v_0}{\sqrt{1-\beta^2}}B_0\cos\left(\frac{2\pi}{\Lambda}\frac{z'+v_0t'}{\sqrt{1-\beta^2}}\right) \tag{4'}$$

(1 point)

$$B_{x'} = \frac{1}{\sqrt{1-\beta^2}}B_0\cos\left(\frac{2\pi}{\Lambda}\frac{z'+v_0t'}{\sqrt{1-\beta^2}}\right) \tag{5'}$$

(1 point)

(Note: if (4′) and (5′) are given directly, 8 points.)
In the frame $\mathrm{S}'(x',y',z')$, because the magnetic induction B_0 is not too large, the change of the electron velocity produced by the magnetic field is much smaller than v_0; so in the frame S' the electron moves near the origin O' ($x'=0, y'=0, z'=0$), and its displacement

away from O′ along the z' axis is much smaller than Λ. Therefore the electromagnetic field felt by the electron is approximately the electromagnetic field at O′ ($x'=0, y'=0, z'=0$):

$$E_{y'} = \frac{v_0 B_0}{\sqrt{1-\beta^2}} \cos\omega' t' \tag{6′}$$

(1 point)

$$B_{x'} = \frac{B_0}{\sqrt{1-\beta^2}} \cos\omega' t' \tag{7′}$$

(1 point)

where

$$\omega' = \frac{2\pi v_0}{\Lambda\sqrt{1-\beta^2}}$$

{Method 2(2)} Let the magnetic induction of the field along the x axis in the laboratory frame S(x, y, z) be B_x; then in the frame S′(x', y', z') the electromagnetic field felt by the electron (only the non-zero components) is

$$E_{y'} = \frac{v_0}{\sqrt{1-\beta^2}} B_x, \qquad B_{x'} = \frac{1}{\sqrt{1-\beta^2}} B_x$$

where $\beta = v_0/c$. The static magnetic field in the frame S(x, y, z) has the spatial period Λ; the spatial period observed in the frame S′(x', y', z') is

$$\Lambda' = \Lambda\sqrt{1-\beta^2} \tag{1″}$$

(4 points)

In the frame S′(x', y', z') the electric and magnetic fields measured at the origin O′ are alternating fields whose angular frequency is

$$\omega' = \frac{2\pi}{T'} = \frac{2\pi}{\frac{\Lambda'}{v_0}} = \frac{2\pi v_0}{\Lambda\sqrt{1-\beta^2}} \tag{2″}$$

(2 points)

In the frame S′(x', y', z'), because the magnetic induction B_0 is not too large, the change of the electron velocity produced by the magnetic field is much smaller than v_0; so in the frame S′ the electron moves near O′ and its displacement away from O′ along the z' axis is much smaller than Λ. Therefore the electromagnetic field felt by the electron is approximately the field at O′:

$$E_{y'} = \frac{v_0 B_0}{\sqrt{1-\beta^2}} \cos\omega' t' \tag{3″}$$

(2 points)

$$B_{x'} = \frac{B_0}{\sqrt{1-\beta^2}} \cos\omega' t' \tag{4″}$$

(2 points)

The initial position and the initial velocity of the electron in the frame S′ (the initial conditions) are respectively

$$(x', y', z') = (0, 0, 0), \qquad (v_{x'}(0), v_{y'}(0), v_{z'}(0)) = (0, 0, 0) \tag{8′}$$

(1 point)

In the frame S′ the speed of the electron is much smaller than the speed of light, so the equations of motion of the electron obey classical Newtonian dynamics:

$$m_e \frac{d^2 y'}{dt'^2} = -e\frac{v_0 B_0}{\sqrt{1-\beta^2}}\cos(\omega' t') - e\frac{B_0}{\sqrt{1-\beta^2}}\frac{dz'}{dt'}\cos(\omega' t') \tag{9'}$$

(2 points)

$$m_e \frac{d^2 z'}{dt'^2} = e\frac{B_0}{\sqrt{1-\beta^2}}\frac{dy'}{dt'}\cos(\omega' t') \tag{10'}$$

(2 points)

Because $dz'/dt' \ll v_0$, the second term on the right-hand side of (9′) may be neglected. Hence

$$m_e \frac{d^2 y'}{dt'^2} = -e\frac{v_0 B_0}{\sqrt{1-\beta^2}}\cos(\omega' t') \tag{11'}$$

(1 point)

Integrating (taking the initial conditions into account at the same time),

$$y' = \frac{e v_0 B_0}{\omega'^2 m_e \sqrt{1-\beta^2}}\left[\cos(\omega' t') - 1\right] \tag{12'}$$

(1 point)

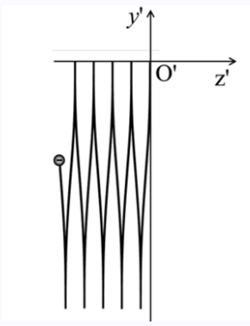
Substituting (12′) into (10′),

$$z' = \frac{e^2 v_0 B_0^2}{8\omega'^3 m_e^2 (1-\beta^2)}\left[\sin(2\omega' t') - 2\omega' t'\right] \tag{13'}$$

(1 point)

$$x' = 0$$

(If the initial position is not chosen at $z' = 0$, an additional constant term appears in (13′); no marks are deducted.)



Solution Figure 4a

That is, the electron performs a "Z"-shaped motion in the $y'z'$ plane and moves uniformly in a straight line along the z' axis, as shown in Solution Fig. 4a.

(2 points)
(14)

(2) The electromagnetic wave radiated along the z' axis comes from the electron oscillating along the y' axis; the frequency of the electromagnetic wave equals the oscillation frequency of the electron along the y' axis, so in the frame S′

$$\omega_0 = \omega' = \frac{2\pi v_0}{\Lambda\sqrt{1-\beta^2}} \tag{15}$$

(2 points)

By the Doppler effect, the frequency of the radiated electromagnetic wave measured in the laboratory frame S is

$$\omega = \frac{2\pi c}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_0 = \frac{2\pi v_0}{\Lambda\sqrt{1-\beta^2}}\sqrt{\frac{1+\beta}{1-\beta}} = \frac{2\pi v_0}{\Lambda(1-\beta)} \tag{16}$$

(5 points)

From (16) and the data of the problem,

$$v_0 = \frac{\Lambda}{\Lambda+\lambda}c \approx \frac{0.1}{0.1+4\times 10^{-8}}c = (1-4.00\times 10^{-7})c \tag{17}$$

(2 points)

[Translator's note: Λ and λ are here expressed in centimetres.]

The accelerating voltage of the electron accelerator satisfies

$$eU = \frac{m_e c^2}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}} - m_e c^2 \tag{18}$$

(5 points)

From (17), (18) and the data of the problem,

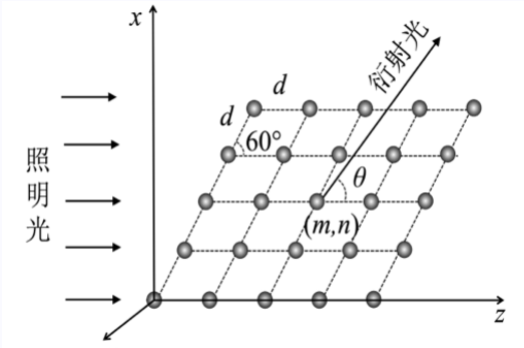
$$\begin{aligned} U &= \frac{m_e c^2}{e}\left[\frac{1}{\sqrt{1-\left(\frac{v_0}{c}\right)^2}} - 1\right] \\ &\approx \frac{m_e c^2}{e}\frac{1}{\sqrt{2\times 4.00\times 10^{-7}}} \\ &= \frac{9.11\times 10^{-31}(3.00\times 10^8)^2}{1.60\times 10^{-19}\sqrt{2\times 4.00\times 10^{-7}}}\ \mathrm{V} \\ &\approx 5.73\times 10^8\ \mathrm{V} \end{aligned} \tag{19}$$

(2 points)

(3) From Fig. 4b, the parallel light (the X laser) falls on every lattice point of the two-dimensional crystal plane; each lattice point may be regarded as a secondary wave source that sends out secondary waves in all directions, and far away in the xz plane one observes the diffracted light making an angle θ with the z direction. The diffraction of the X laser by the two-dimensional crystal is Fraunhofer diffraction. Let the incident light wave vector be $\vec{k}$ and the diffracted light wave vector be $\vec{k}'$, and let the Fraunhofer diffraction field of the lattice point $(0,0)$ at the coordinate origin be $\tilde{U}_0(\theta)$. The phase difference between the diffraction field of the lattice point $(0,0)$ and that of the lattice point (m,n) is

$$\Delta\varphi = \vec{k}'\cdot\vec{r} - \vec{k}\cdot\vec{r} \tag{20}$$

(2 points)



Solution Figure 4b

Labels: 照明光 *= illuminating light;* 衍射光 *= diffracted light.*

So the Fraunhofer diffraction field of the lattice point (m, n) is

$$\tilde{U}_{mn}(\theta) = \tilde{U}_0(\theta)\mathrm{e}^{-i\Delta\varphi} = \tilde{U}_0(\theta)\mathrm{e}^{-i(\vec{k}'-\vec{k})\cdot\vec{r}} \tag{21}$$

(2 points)

where $\vec{r}$ is the position vector of the lattice point (m, n): $\vec{r} = \left(\frac{\sqrt{3}}{2}md,\ 0,\ nd + \frac{1}{2}md\right)$, $\vec{k} = \frac{2\pi}{\lambda}(0, 0, 1)$, $\vec{k}' = \frac{2\pi}{\lambda}(\sin\theta,\ 0,\ \cos\theta)$. Hence from the above the diffraction field of the lattice point (m, n) is

$$\begin{aligned}\tilde{U}_{mn}(\theta) &= \tilde{U}_0(\theta)\exp\left\{-i\frac{2\pi}{\lambda}\left[\frac{\sqrt{3}}{2}md\sin\theta + \left(n + \frac{m}{2}\right)d(\cos\theta - 1)\right]\right\} \\ &= \tilde{U}_0(\theta)\exp\left\{-i\frac{2\pi}{\lambda}\left[md\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\right) + nd(\cos\theta - 1)\right]\right\} \\ &= \tilde{U}_0(\theta)\exp(-i2m\beta_1 - i2n\beta_2)\end{aligned} \tag{22}$$

(2 points)

where

$$\beta_1 = \frac{\pi}{\lambda}d\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\right), \qquad \beta_2 = \frac{\pi}{\lambda}d(\cos\theta - 1)$$

The total diffraction field of the two-dimensional crystal plane is the coherent superposition of the secondary waves emitted by all the lattice points, i.e.

$$\begin{aligned}\tilde{U}(\theta) &= \sum_{m=0}^{N_1-1}\sum_{n=0}^{N_2-1}\tilde{U}_{mn}(\theta) = \tilde{U}_0(\theta)\sum_{m=0}^{N_1-1}\exp(-i2m\beta_1)\sum_{n=0}^{N_2-1}\exp(-i2n\beta_2) \\ &= \tilde{U}_0(\theta)\mathrm{e}^{-i(N_1-1)\beta_1}\mathrm{e}^{-i(N_2-1)\beta_2}\left(\frac{\sin N_1\beta_1}{\sin\beta_1}\right)\left(\frac{\sin N_2\beta_2}{\sin\beta_2}\right)\end{aligned} \tag{23}$$

(4 points)

"Here the formula for the sum of a geometric series $\sum_{n=0}^{N-1}\mathrm{e}^{-i2n\beta} = \dfrac{1 - \mathrm{e}^{-i2N\beta}}{1 - \mathrm{e}^{-i2\beta}}$ has been used, together with $(1 - \mathrm{e}^{i\Phi}) = -2i\sin\frac{\Phi}{2} \times \mathrm{e}^{i\frac{\Phi}{2}}$."

From (23) the intensity distribution of the diffracted light is

$$I(\theta) = \left|\tilde{U}(\theta)\right|^2 = \left|\tilde{U}_0(\theta)\right|^2\left(\frac{\sin N_1\beta_1}{\sin\beta_1}\right)^2\left(\frac{\sin N_2\beta_2}{\sin\beta_2}\right)^2 \tag{24}$$

(2 points)

Hence the positions of the principal maxima:

$$\beta_1 = \frac{\pi}{\lambda} d\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\right) = k_1\pi, \qquad \beta_2 = \frac{\pi}{\lambda} d(\cos\theta - 1) = k_2\pi \tag{25}$$

(2 points)

that is,

$$d\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\right) = k_1\lambda \tag{26}$$

$$d(\cos\theta - 1) = k_2\lambda \tag{27}$$

where k_1, k_2 are integers. (Note: in (25) each formula 1 point; if (26) and (27) are given directly, each formula 1 point; if (25), (26) and (27) are all given, 2 points in total.)

[**Method 2.** With a plane wave incident, every lattice point on the two-dimensional crystal plane may be regarded as a secondary wave source that sends out secondary waves in all directions. The position of a principal maximum requires that the secondary waves emitted by all the lattice points in a row interfere constructively, and at the same time that the secondary waves emitted by all the lattice points in a column interfere constructively, as shown in Solution Fig. 5b. The positions of the principal maxima simultaneously satisfy

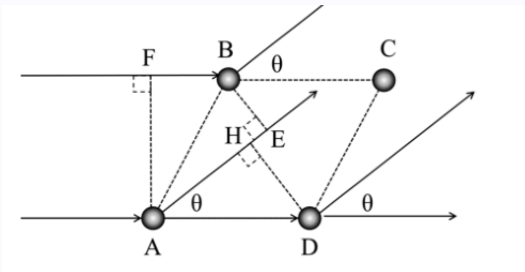
$$\Delta L_{\text{row}} = \overline{AE} - \overline{FB} = d\cos\left(\frac{\pi}{3} - \theta\right) - d\sin\left(\frac{\pi}{6}\right) = d\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\right) = k_1\lambda \tag{20$'$}$$

(7 points)

$$\Delta L_{\text{col}} = \overline{\text{AH}} - \overline{\text{AD}} = d(\cos\theta - 1) = k_2\lambda \tag{21$'$}$$

(7 points)

where k_1, k_2 are integers.]



Solution Figure 5b (as labelled in the original; it belongs to Problem 4).

From (26),

$$-1 \le \sin\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2} + k_1\frac{\lambda}{d} \le 1$$

Because $d = 2\lambda$, therefore

$$k_1 = +1,\ 0,\ -1,\ -2,\ -3 \tag{28}$$

(or 2 points)

From (27),

$$-1 \le \cos\theta = 1 + k_2\frac{\lambda}{d} \le 1$$

Because $d = 2\lambda$, therefore

$$k_2 = 0,\ -1,\ -2,\ -3,\ -4 \tag{29}$$

(or 2 points)

(Note: writing one of (28), (29) earns 2 points; the two formulas are not marked twice.)

Hence

$$\cos\theta = 1,\ \frac{1}{2},\ 0,\ -\frac{1}{2},\ -1$$

Substituting into (24) the values of $\cos\theta$ that correspond to the admissible integers k_2 found above,

[Translator's note: as printed; the equation into which the values are substituted is (26).]

$$k_1 = 2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\right) = \pm\sqrt{3 - 3\cos^2\theta} + \cos\theta - 1 \tag{30}$$

Requiring k_1 in the above expression to be an integer, we get:

$k_2 = 0$, $\cos\theta = 1$, giving $k_1 = 0$, $\theta = 0$: a zeroth-order principal maximum appears. (31)

$k_2 = -1$, $\cos\theta = \frac{1}{2}$, giving $k_1 = 1$, $\sin\theta = \frac{\sqrt{3}}{2}$, $\theta = 60°$; or $k_1 = -2$, $\sin\theta = -\frac{\sqrt{3}}{2}$, $\theta = -60°$; i.e. one principal maximum appears at each of $\theta = \pm 60°$, two principal maxima in all. *(4 points)* (32)

(Note: 2 points for each of the two results.)

$k_2 = -2$, $\cos\theta = 0$, giving $k_1 = \pm\sqrt{3} - 1$, not an integer; no principal maximum appears. (33)

$k_2 = -3$, $\cos\theta = -\frac{1}{2}$, giving $k_1 = 0$, $\sin\theta = \frac{\sqrt{3}}{2}$, $\theta = 120°$; or $k_1 = -3$, $\sin\theta = -\frac{\sqrt{3}}{2}$, $\theta = -120°$; i.e. one principal maximum appears at each of $\theta = \pm 120°$, two principal maxima in all. *(4 points)* (34)

(Note: 2 points for each of the two results.)

$k_2 = -4$, $\cos\theta = 1$, giving $k_1 = -2$, $\sin\theta = 0$, $\theta = 180°$: one principal maximum appears. (35)

[Translator's note: as printed; for $k_2 = -4$ one has $\cos\theta = -1$, which is what the quoted values of k_1 and θ correspond to.]

Conclusion: 6 principal maxima can be observed; they appear at $\theta = 0,\ \pm 60°,\ \pm 120°,\ 180°$. (Note: the principal maxima corresponding to $\theta = 0°$ or $180°$ cannot be observed, so they do not carry marks.)

Mark scheme (60 points in total). Part (1), 20 points. 【Method 1】: (1), (2), (5), (10), (11) 2 points each; (3), (4), (6), (7), (8), (9), (12), (13) 1 point each. 【Method 2】: (1′), (2′), (3′) 2 points each; (4′), (5′) 1 point each (note: if (4′) and (5′) are given directly, 8 points); (6′), (7′) 1 point each. {Method 2(2)}: (1″) 4 points, (2″) 2 points, (3″) 2 points, (4″) 2 points (Method 2 and Method 2(2) are not marked twice). (8′) 1 point, (9′) and (10′) 2 points each, (11′), (12′), (13′) 1 point each. (14) 2 points. Part (2), 16 points: (15) 2 points, (16) 5 points, (17) 2 points, (18) 5 points, (19) 2 points. Part (3), 24 points: (20) 2 points, (21) 2 points, (22) 2 points, (23) 4 points, (24) 2 points, (25) 2 points (note: in (25) each formula 1 point; if (26) and (27) are given directly, each formula 1 point; if (25), (26) and (27) are all given, 2 points in total). "Method 2: (20′) 7 points, (21′) 7 points; the two methods are not marked twice."(28) (or (29)) 2 points (note: writing one of (28), (29) earns 2 points; the two formulas are not marked twice), (32) 4 points, (34) 4 points.

Problem 5 (o and e rays in a uniaxial crystal; phase matching in BBO)

Solution

(1) *(10 points)*

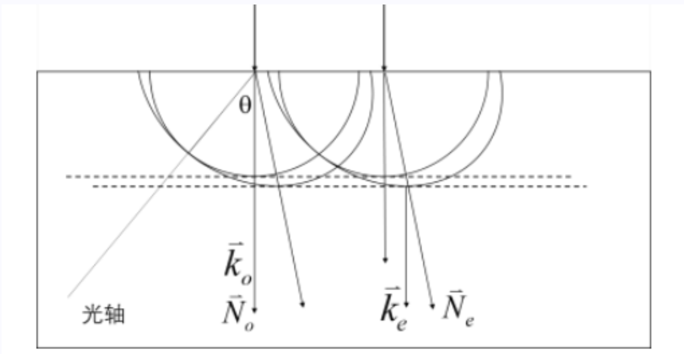
Instructions for the construction:

1) For the o light, draw circles centred on the two points of incidence; the common tangent plane (the envelope) of the two circles is parallel to the surface.

2) For the e light, draw ellipses whose short axis is along the optic axis and whose long axis is along the direction perpendicular to the optic axis. From the two ellipses obtain their common tangent plane (the envelope).

3) The ellipse of the e light is externally tangent, on the optic axis, to the circle of the o light.

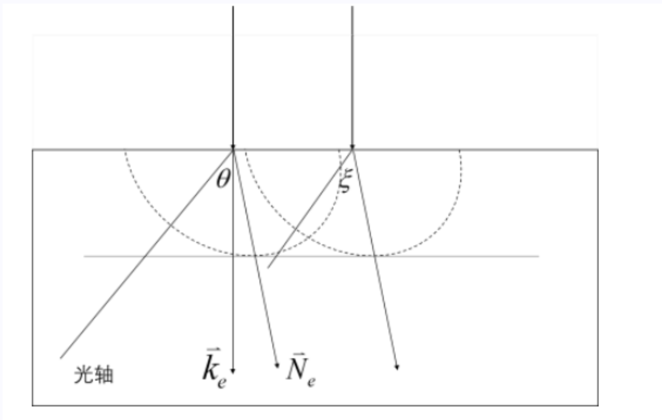
4) $\vec{k}_o$, $\vec{k}_e$, $\vec{N}_o$ all have the same direction; the direction of $\vec{N}_e$ is the direction of the line joining the point of incidence to the point at which the envelope touches the ellipse.



Solution Figure 5a
Labels: 光轴 = *optic axis.*

(2) *(9 points)*

From the figure drawn according to Huygens' principle in part (1), ξ is the angle between the propagation direction $\vec{N}_e$ of the e light and the optic axis, and θ is the angle between the normal $\vec{k}_e$ of the envelope on the ellipse of the e light and the optic axis (Solution Fig. 5b). Therefore this part asks for the relation between the angle that $\vec{N}_e$ makes with the y axis and the angle that $\vec{k}_e$ makes with the y axis, at a point of the ellipse.



Solution Figure 5b
Labels: 光轴 = *optic axis.*

At time t the equation of the ellipse of the equal-phase surface (the equal-time surface) of the e light of a point source is

$$\frac{v_y^2 t^2}{v_o^2 t^2} + \frac{v_x^2 t^2}{v_e^2 t^2} = 1 \tag{1}$$

where v_x and v_y are the two rectangular coordinate components of the wave speed of the e light. If in the above formula the factor t^2 is cancelled one obtains the velocity ellipse of the e light (in the three-dimensional case, the velocity ellipsoid).
On the ellipse (or velocity ellipse) expressed by (1), the geometry gives

$$\tan\xi = \frac{v_x}{v_y} \tag{2}$$

$$\tan\theta = -\frac{\mathrm{d}v_y}{\mathrm{d}v_x} \tag{3}$$

Taking the differential of the equation (1) of the ellipse,

$$2v_y \frac{\mathrm{d}v_y}{v_o^2} + 2v_x \frac{\mathrm{d}v_x}{v_e^2} = 0 \tag{4}$$

Simplifying,

$$\frac{v_x}{v_y} = -\frac{v_e^2}{v_o^2}\frac{\mathrm{d}v_y}{\mathrm{d}v_x} \tag{5}$$

That is,

$$\therefore\ \tan\xi = \frac{v_e^2}{v_o^2}\tan\theta = \frac{n_o^2}{n_e^2}\tan\theta \tag{6}$$

(3) *(4 points)*
For light of vacuum wavelength 800.0 nm,

$$n_\mathrm{o}(800) = n_{o,800} = 1.661 \tag{7}$$

$$n_\mathrm{e}(800) = n_{e,800} = 1.544 \tag{8}$$

For light of vacuum wavelength 400.0 nm,

$$n_\mathrm{o}(400) = n_{o,400} = 1.693 \tag{9}$$

$$n_\mathrm{e}(400) = n_{e,400} = 1.568 \tag{10}$$

(4) *(6 points)*
(4.1) Conservation of momentum requires that the momentum of the two photons before the conversion equal the momentum of the photon after the conversion, hence

$$2\vec{k}_1 = \vec{k}_2 \tag{11}$$

(4.2) In the frequency-doubling relation, to satisfy conservation of energy the vacuum wavelength of the fundamental photon must be twice the vacuum wavelength of the frequency-doubled photon. From the definition of the magnitude of the wave vector in a medium, $k = \dfrac{2\pi n}{\lambda}$, in order that momentum be conserved the refractive indices of the fundamental and the doubled light must be equal. This refractive index is the phase-velocity refractive index, and this requirement means that the fundamental light and the frequency-doubled light have the same phase velocity in the material.

From the refractive index values calculated in part (3), when the fundamental and the doubled light are both o light, or both e light, they cannot have the same refractive index because of dispersion. Such a scheme has to be excluded, and therefore one of the two beams must be o light and the other e light.
Take the 800 nm light to be o light: its refractive index is 1.661; the 400 nm light is then e light, whose refractive index lies between 1.544 and 1.693 and varies with the angle. Therefore an angle can be found at which the fundamental and the doubled light have the same refractive index; the scheme is feasible.

[Translator's note: as printed; the refractive index of the 400 nm *e light lies between* $n_{e,400} = 1.568$ *and* $n_{o,400} = 1.693$*, so "1.544" should read "1.568".]*

Take the 400 nm light to be o light: its refractive index is 1.693; the 800 nm light is then e light, whose refractive index lies between 1.544 and 1.661 and varies with the angle. Therefore no angle can be found at which the fundamental and the doubled light have the same refractive index; the scheme is not feasible.
Conclusion: this frequency-doubling scheme takes the 800 nm fundamental light to be o light and the 400 nm frequency-doubled light to be e light. (12)
(Remark: it is enough to write down the conclusion; the analysis is not required.)

(5) *(18 points)*
On the velocity ellipse of the e light of vacuum wavelength 400 nm, following the convention of part (2), take the propagation direction $\vec{N}_e$ of the e light to make the angle ξ with the optic axis, and denote the speed in that direction by $v(400,\xi)$. Considering the propagation of the e light beam inside the crystal, the projection of $v(400,\xi)$ on the wave-vector direction is the phase velocity $v_p(400)$, so that

$$v_p(400) = v(400,\xi)\cos(\xi-\theta) \tag{13}$$

According to the analysis in (4.2), this phase velocity should equal the phase velocity $v_o(800)$ of the o light of vacuum wavelength 800 nm, i.e.

$$v(400,\xi)\cos(\xi-\theta) = v_o(800) = \frac{c}{n_{o,800}} \tag{14}$$

For the e light of vacuum wavelength 400 nm, substituting $v_x = v(400,\xi)\sin\xi$, $v_y = v(400,\xi)\cos\xi$ into the ellipse equation (1) gives

$$\frac{v(400,\xi)^2\cos^2\xi}{v_o(400)^2} + \frac{v(400,\xi)^2\sin^2\xi}{v_e(400)^2} = 1 \tag{15}$$

whence

$$v(400,\xi)^2 = \frac{c^2}{n_{o,400}^2\cos^2\xi + n_{e,400}^2\sin^2\xi} \tag{15$'$}$$

Using trigonometric formulas,

$$\cos^2(\xi-\theta) = (\cos\xi\cos\theta+\sin\xi\sin\theta)^2 = \cos^2\xi\cos^2\theta(1+\tan\xi\tan\theta)^2 \tag{16}$$

Therefore, combining (14), (15′) and (16),

$$\frac{c^2}{n_{o,800}^2} = \frac{c^2}{n_{o,400}^2\cos^2\xi + n_{e,400}^2\sin^2\xi}\cos^2\xi\cos^2\theta(1+\tan\xi\tan\theta)^2 \tag{17}$$

Substituting (6) and simplifying,

$$\frac{1}{n_{o,800}^2} = \frac{\cos^2\theta(1+\tan\xi\tan\theta)^2}{n_{o,400}^2\left(1+\dfrac{n_{e,400}^2}{n_{o,400}^2}\tan^2\xi\right)}$$

$$= \frac{\cos^2\theta}{n_{o,400}^2}\left(1+\frac{n_{o,400}^2}{n_{e,400}^2}\tan^2\theta\right)$$

$$\therefore\ \frac{1}{n_{o,800}^2} = \frac{1}{n_{o,400}^2}\left(\cos^2\theta+\frac{n_{o,400}^2}{n_{e,400}^2}\left(1-\cos^2\theta\right)\right) \tag{18}$$

Substituting $n_{\mathrm{o}}(800) = 1.661$, $n_{\mathrm{o}}(400) = 1.693$ and $n_{\mathrm{e}}(400) = 1.568$ gives

$$\cos\theta = 0.8749$$

so that the angle θ is 29.0 degrees, or 0.506 radians. (19)

(6) *(3 points)*

From

$$\tan(\xi) = \frac{v_e^2}{v_o^2}\tan(\theta) = \frac{n_o^2}{n_e^2}\tan(\theta)$$

we obtain

$$\tan(\xi) = \frac{n_o^2}{n_e^2}\tan(\theta) = \frac{1.693^2}{1.568^2}\tan(29.0^\circ) = 0.646 \tag{20}$$

$$\xi = 32.9^\circ$$

so the angle between the wave vector $\vec{k}_e$ of the e light and its propagation direction $\vec{N}_e$ is

$$\alpha = \xi - \theta = 3.9^\circ \tag{21}$$

Mark scheme (50 points in total). Part (1), 10 points: 1) a correct construction, with the four vector directions obtained in the figure and correctly labelled, 2 points for each label; 2) the ellipse of the e light in the figure being externally tangent to the circle of the o light on the optic axis, 2 points. Part (2), 9 points: (1) 2 points, (2) 2 points, (3) 2 points, (6) 3 points. Part (3), 4 points: (7), (8), (9), (10) 1 point each. Part (4), 6 points: (11) 2 points; (12) the result 4 points — it is enough that the conclusion is correct, no analysis is required. Part (5), 18 points: (13) 4 points, (14) 4 points, (15) (or (15′)) 4 points, (18) 4 points, (19) 2 points. Part (6), 3 points: (20) 2 points, (21) 1 point.

Problem 6 (the van der Waals gas: heat capacity, entropy, Carnot cycle, compressibility)

Solution

(1) The physical meaning of b: because gas molecules have a size, it is the sum of the volumes that cannot be freely reached by 1 mol of gas molecules moving in the container. (1)

【**Remark:** one may set $a = 0$ and then compare $p(v-b) = RT$ with the ideal-gas equation $pv = RT$; one sees that $(v-b)$ is the volume of the region in which 1 mol of van

der Waals gas can move freely in a container of volume v, from which the physical meaning of b can be analysed.】
Considering the presence of one gas molecule of radius r, the volume that another gas molecule cannot freely reach is $V_0 = \frac{4}{3}\pi(2r)^3$. Since this inaccessible region is shared by a pair of molecules,

$$b = \frac{1}{2}N_A V_0 = \frac{16}{3}N_A \pi r^3 \tag{2}$$

(2) Because T and p are intensive quantities, the van der Waals equation for N moles of the gas of the problem occupying the volume V is

$$\left(p + \frac{N^2 a}{V^2}\right)\left(\frac{V}{N} - b\right) = RT. \tag{3}$$

That is,

$$\left(p + \frac{N^2 a}{V^2}\right)(V - Nb) = NRT.$$

(3) Rewriting the van der Waals equation for 1 mol of this gas as

$$p = \frac{RT}{v-b} - \frac{a}{v^2}, \tag{4}$$

we have

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{R}{v-b}, \qquad \left(\frac{\partial^2 p}{\partial T^2}\right)_v = 0 \tag{5}$$

and therefore

$$\left(\frac{\partial C_V}{\partial v}\right)_T = T\left(\frac{\partial^2 p}{\partial T^2}\right)_v = 0. \tag{6}$$

That is, the C_V of a van der Waals gas does not depend on v (it may depend only on T). Considering the limit $v \to \infty$ (or $v \gg b$, $vRT \gg a$), the van der Waals equation (4) returns to the form of the ideal-gas equation of state, so the C_V of a van der Waals gas is the same as the C_V of the corresponding ideal gas. For the non-relativistic monatomic gas of the problem, C_V is a constant and

$$C_V = \frac{3}{2}R \tag{7}$$

(4) From the relation $u = C_V T - \dfrac{a}{v}$ given in the problem we obtain

$$\mathrm{d}u = C_V \mathrm{d}T + \frac{a}{v^2}\mathrm{d}v \tag{8}$$

For a reversible quasi-static process, the first law of thermodynamics

$$\mathrm{d}u = đQ + đW$$

(where $đQ$ is the heat absorbed by the gas in a small reversible quasi-static process, and $đW = p(-\mathrm{d}v)$ is the work done on the gas by the surroundings in a small reversible quasi-static process) together with the second law of thermodynamics

$$\mathrm{d}s = \frac{đQ}{T}$$

gives the fundamental differential equation of thermodynamics for 1 mol of gas:

$$\mathrm{d}u = T\mathrm{d}s - p\mathrm{d}v \tag{9}$$

Therefore

$$ds = \frac{1}{T}du + \frac{p}{T}dv \tag{10}$$

Substituting (4) and (8) into (10),

$$ds = \frac{1}{T}\left(C_V dT + \frac{a}{v^2}dv\right) + \frac{1}{T}\left(\frac{RT}{v-b} - \frac{a}{v^2}\right)dv = C_V\frac{dT}{T} + \frac{R}{v-b}dv \tag{11}$$

Integrating,

$$s(T,v) = C_V \ln T + R\ln(v-b) + s_0 \tag{12}$$

where s_0 is an undetermined integration constant. Or

$$s(T,v) = C_V \ln\frac{T}{T_0} + R\ln\left(\frac{v-b}{v_0-b}\right) \qquad \text{(where } T_0,\ v_0 \text{ are constants)} \tag{12$'$}$$

A reversible adiabatic process is an isentropic process; from (12), for the adiabatic process of 1 mol of the above van der Waals gas,

$$C_V \ln T + R\ln(v-b) = \text{constant} \tag{13$'$}$$

or

$$T^{(C_V/R)}(v-b) = T^{3/2}(v-b) = \text{constant} \tag{13}$$

which is the required equation of the adiabatic process. (The constant in (13) may be written as $T_0^{(C_V/R)}(v_0-b)$.)

(5) The isothermal expansion is the heat-absorbing sub-process; the heat absorbed is

$$Q_1 = T_1\Delta s_1 \tag{14}$$

where Δs_1 is the entropy change of that process. From (12),

$$\Delta s_1 = [C_V\ln T_1 + R\ln(v_2-b) + s_0] - [C_V\ln T_1 + R\ln(v_1-b)+s_0] = R\ln\frac{v_2-b}{v_1-b} \tag{15}$$

Substituting (15) into (14),

$$Q_1 = T_1 R\ln\frac{v_2-b}{v_1-b} \tag{16}$$

The isothermal compression is the heat-releasing sub-process; the heat released is

$$Q_2 = T_2\,|\Delta s_2| \tag{17}$$

where Δs_2 (< 0) is the entropy change of that process. From (12),

$$\Delta s_2 = [C_V\ln T_2 + R\ln(v_4-b) + s_0] - [C_V\ln T_1 + R\ln(v_3-b)+s_0] = R\ln\frac{v_4-b}{v_3-b} \tag{18}$$

[Translator's note: as printed; the second bracket should contain $C_V \ln T_2$, since process III is isothermal at T_2 — as the stated result already assumes.]

Substituting (18) into (17),

$$Q_2 = T_2 R\ln\frac{v_3-b}{v_4-b} \tag{19}$$

So the cycle efficiency is

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2\ln\dfrac{v_3-b}{v_4-b}}{T_1\ln\dfrac{v_2-b}{v_1-b}} \tag{20}$$

For the two adiabatic processes, (13) gives

$$T_1^{(C_V/R)}(v_2 - b) = T_2^{(C_V/R)}(v_3 - b), \qquad T_1^{(C_V/R)}(v_1 - b) = T_2^{(C_V/R)}(v_4 - b) \tag{21}$$

Therefore

$$\frac{v_2 - b}{v_1 - b} = \frac{v_3 - b}{v_4 - b} \tag{22}$$

Substituting (22) into (20),

$$\eta = 1 - \frac{T_2}{T_1} \tag{23}$$

which is the required η as a function of T_1 and T_2.

(6) From the van der Waals equation of state of 1 mol of gas, $\left(p + \frac{a}{v^2}\right)(v - b) = RT$, we get

$$R\mathrm{d}T = (v-b)\mathrm{d}p + \left(p + \frac{a}{v^2}\right)\mathrm{d}v - (v-b)\frac{2a}{v^3}\mathrm{d}v = (v-b)\mathrm{d}p + \left[\frac{RT}{v-b} - (v-b)\frac{2a}{v^3}\right]\mathrm{d}v \tag{24}$$

At constant temperature $\mathrm{d}T = 0$, so (24) gives

$$\left(\frac{\partial v}{\partial p}\right)_T = \left[\frac{2a}{v^3} - \frac{RT}{(v-b)^2}\right]^{-1} = \frac{v^3(v-b)^2}{2a(v-b)^2 - RTv^3} \tag{25}$$

For the van der Waals model we obtain

$$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T = \frac{v^2(v-b)^2}{RTv^3 - 2a(v-b)^2} \tag{26}$$

Rewriting the above,

$$\kappa_T^{-1} = \frac{RTv}{(v-b)^2} - \frac{2a}{v^2}$$

and substituting the van der Waals equation

$$\frac{RT}{v-b} = p + \frac{a}{v^2}$$

we get

$$\kappa_T^{-1} = \frac{v}{v-b}\left[p + \frac{a}{v^2} - \frac{2a(v-b)}{v^3}\right]$$

From the result of part (1) we know that $v > b$; combining this with the given condition $a \ll pv^2$, we obtain

$$\kappa_T^{-1} > \frac{v}{v-b}\left(p + \frac{a}{v^2} - \frac{2a}{v^2}\right) = \frac{v}{v-b}\left(p - \frac{a}{v^2}\right) > 0 \tag{27}$$

Therefore for a van der Waals gas the range of κ_T is

$$\kappa_T > 0 \tag{28}$$

Its intuitive physical meaning is: under isothermal compression the volume of the substance decreases as the pressure increases. (29)

We know that the van der Waals gas is a stable state of matter; generalizing from everyday experience, all stable material systems have $\kappa_T > 0$.

Mark scheme (50 points in total). Part (1), 6 points: (1) and (2) 3 points each. Part (2), 2 points: (3) 2 points. Part (3), 8 points: (5) 3 points, (6) 1 point, (7) 4 points (the argument or the equation must be given; writing the result directly earns 2 points). Part (4), 10 points: (8) 2 points, (9) 2 points, (11) 2 points, (12) 2 points, (13) (or (13′)) 2 points. Part (5), 12 points: (14), (16), (17), (19), (22) 2 points each; (20) and (23) 1 point each. Part (6), 12 points: (25) 4 points, (26) 2 points, (28) 4 points, (29) 2 points.