The 40th Chinese Physics Olympiad

Final — Theoretical Examination (Problems)

2023

This file contains the seven problems of the theoretical examination of the final round. The source document is the official "reference solutions" booklet (October 2023 trial version), in which each problem statement is immediately followed by its official solution; the solutions have been separated into a companion file. All figures are cropped from the original document; where a figure carries Chinese labels a short glossary is given below it. Total mark: 320 points.

Problem 1 (40 points)

After the rain the sky clears, and a large number of small water droplets are suspended in the air. If sunlight shines from behind the observer at a low angle, the observer may see a rainbow (Fig. 1a). Sometimes two rainbows appear at the same time. The inner one is called the *primary bow* (order-1 bow); it comes from one reflection of the sunlight after the light has entered a droplet. The fainter bow outside the primary bow is called the *secondary bow* (order-2 bow); it comes from two reflections of the sunlight inside a droplet. Light can also undergo three or more reflections inside a droplet and produce bows of the corresponding orders. In nature only the order-1 and order-2 bows can usually be observed; bows of higher order can be produced in the laboratory.

Fig. 1a A rainbow in nature

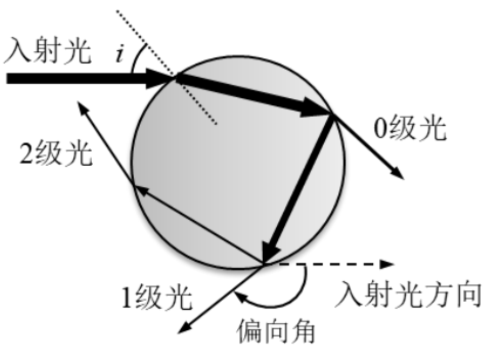

Fig. 1b Refraction, reflection and exit of one ray at a droplet

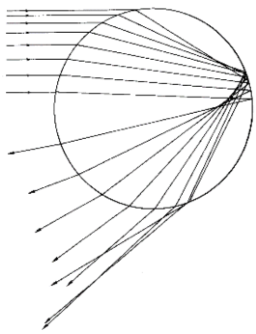

Fig. 1c Order-1 light after parallel light strikes a droplet

Labels: 入射光 = *incident light;* 0 级光 = *order-0 light;* 1 级光 = *order-1 light;* 2 级光 = *order-2 light;* 入射光方向 = *direction of the incident light;* 偏向角 = *deviation angle.*

Consider a spherical water droplet whose diameter is of order $10^2\,\mu\mathrm{m}$. As in Fig. 1b, one incident ray together with the centre of the sphere defines a plane; this plane cuts the surface of the spherical droplet in a circle (a great circle), and the ray that enters the droplet and the rays that leave it after refractions and reflections all lie in this plane. The ray that leaves the droplet after k ($k = 0, 1, 2, \dots$) reflections inside the droplet is called *order-k light*. The angle through which the outgoing light is turned relative to the direction of the incident light is called the *deviation angle* (see Fig. 1b).

(1) The angle of incidence of the ray on the outer surface of the droplet is i; the refractive indices of air and of water are 1 and n. Find the expression for the deviation angle θ_k of the order-k light.

(2) When a monochromatic parallel beam illuminates the droplet, different positions of incidence correspond to different angles of incidence (see Fig. 1c). As the angle of incidence i varies, the deviation angle θ_k of the order-k light has a minimum $\theta_{k\mathrm{m}}$. Find the expressions for $\theta_{k\mathrm{m}}$ and for the corresponding angle of incidence i_k.

(3) Because of dispersion, light of different wavelengths has a different refractive index n in water. For a definite angle of incidence i, find the relation between the rate of change $\dfrac{\mathrm{d}\theta_k}{\mathrm{d}n}$ of the deviation angle θ_k of the order-k light with respect to the refractive index n and the angle of incidence i; and find the expression for $\left.\dfrac{\mathrm{d}\theta_k}{\mathrm{d}n}\right|_{i=i_k}$, i.e. when i takes the value i_k of part (2).

(4) Figure 1c is a sketch of the order-1 light leaving the droplet when a monochromatic parallel beam illuminates it. Near the minimum deviation angle $\theta_{1\mathrm{m}}$ of the order-1 light the outgoing rays are more concentrated, i.e. the intensity is larger; the order-1 bow appears here, and the corresponding minimum deviation angle $\theta_{1\mathrm{m}}$ is the deviation angle of the order-1 bow. Similarly $\theta_{k\mathrm{m}}$ is the deviation angle of the order-k bow. For white light incident in parallel, take the refractive indices of red and of violet light in water to be $n_{\mathrm{red}} = 1.329$ and $n_{\mathrm{violet}} = 1.344$. Calculate separately the deviation angles $\theta_{k\mathrm{m}}^{\mathrm{red}}$ and $\theta_{k\mathrm{m}}^{\mathrm{violet}}$ of the $k = 1$ and $k = 2$ bows and the angular widths δ_k of the bows $\left(\delta_k = \left|\theta_{k\mathrm{m}}^{\mathrm{violet}} - \theta_{k\mathrm{m}}^{\mathrm{red}}\right|\right)$, and state in each case the order of the colours from the inside to the outside. Give the angles in degrees (°), accurate to 0.01°.

(5) Looking at a rainbow through a polarizer one finds that the light of the rainbow is polarized. Let the light incident on a droplet be monochromatic natural light. In the outgoing light let the intensity of the s polarization (polarization direction perpendicular to the plane of incidence) be I_{s} and the intensity of the p polarization (polarization direction parallel to the plane of incidence) be I_{p}. The degree of polarization of the light is defined as

$$P = \left|\frac{I_{\mathrm{s}} - I_{\mathrm{p}}}{I_{\mathrm{s}} + I_{\mathrm{p}}}\right|.$$

When the incident light is red light ($n_{\mathrm{red}} = 1.329$), calculate separately the degree of polarization P_k of the light leaving in the $k = 1$ and $k = 2$ bows.

Hint. When light goes from a medium of refractive index n_1 into a medium of refractive index n_2, with angle of incidence φ_1 and angle of refraction φ_2, then for the s polarization the ratios of the electric-field amplitudes of the reflected and of the refracted light to that of the incident light are

$$r_{\mathrm{s}} = \left|\frac{n_1\cos\varphi_1 - n_2\cos\varphi_2}{n_1\cos\varphi_1 + n_2\cos\varphi_2}\right|, \qquad t_{\mathrm{s}} = \frac{2n_1\cos\varphi_1}{n_1\cos\varphi_1 + n_2\cos\varphi_2},$$

and for the p polarization the corresponding ratios are

$$r_{\mathrm{p}} = \left|\frac{n_2\cos\varphi_1 - n_1\cos\varphi_2}{n_2\cos\varphi_1 + n_1\cos\varphi_2}\right|, \qquad t_{\mathrm{p}} = \frac{2n_1\cos\varphi_1}{n_2\cos\varphi_1 + n_1\cos\varphi_2}.$$

(6) Under certain conditions extra coloured fringes can be observed right next to the edge of a bow; this phenomenon is called *supernumerary bows*, as shown in Fig. 1d. Explain qualitatively the cause of the supernumerary bows.

Fig. 1d

Labels: 附属虹→ = *supernumerary bows.*

Problem 2 (50 points)

As in Fig. 2a, a uniform rigid thin rod of length l and mass m can turn about a horizontal axis O through one of its ends. Consider only the motion of the rod in the vertical plane perpendicular to the axis O. Use the angle θ between the rod and the vertically downward direction ($-\pi < \theta \le \pi$, counterclockwise positive) as the coordinate describing the position of the rod. Neglect air resistance and friction at the axis. The magnitude of the acceleration due to gravity is g.

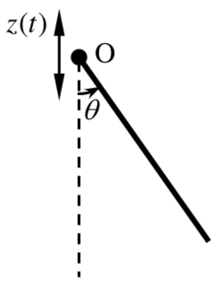

Fig. 2a

The axis O is known to perform a small-amplitude high-frequency simple harmonic oscillation in the vertical direction, with equation of motion $z(t) = A\cos\omega t$, amplitude $A \ll l$ and angular frequency $\omega \gg \sqrt{g/l}$. Discuss the motion of the rod in the translating reference frame that moves together with the axis O.

(1) Write down the dynamical equation satisfied by $\theta(t)$.

(2) $\theta(t)$ can be written as the sum of $\varphi(t)$ and $\delta(t)$, where $\varphi(t)$ denotes the smooth (slow) motion and $\delta(t)$ denotes a small-amplitude high-frequency simple harmonic oscillation of

angular frequency ω. Since $\omega \gg \sqrt{g/l}$ (i.e. the characteristic frequencies of $\varphi(t)$ and of $\delta(t)$ differ by several orders of magnitude), $\varphi(t)$ may be regarded as unchanged within one period of the variation of $\delta(t)$; the effect of the high-frequency motion on the smooth motion can be represented by its average over one period of the high-frequency motion. Under the above conditions and approximations, derive the dynamical equation satisfied by $\varphi(t)$.

(3) From the dynamical equation satisfied by $\varphi(t)$ one sees that the smooth motion of the rod is equivalent to motion of the rod in a conservative field. Find the corresponding effective potential energy $V_{\text{eff}}(\varphi)$ (take the potential energy to be zero at $\varphi = 0$).

(4) From the effective potential energy $V_{\text{eff}}(\varphi)$, determine the equilibrium positions φ_0 of the rod and discuss the stability of each equilibrium position (do not consider the case in which the parameters take critical values); find the frequency of small oscillations of the rod near each stable equilibrium position.

(5) Initially the rod is directly below the axis O and its initial angular velocity is $\dot{\varphi}(0) = \omega_0$ ($\omega_0 > 0$). For the rod to be able to move to the position directly above the axis O, ω_0 must be larger than a critical value ω_c; find ω_c. When $\omega_0 < \omega_c$, find the maximum angle $\varphi_{\max}$ that the rod can reach.

Problem 3 (40 points)

Rutherford's α-particle scattering experiment revealed the nuclear structure of the atom. Particle-scattering experiments can be used to determine the kind, the concentration and the depth distribution of the target atoms of a material. A sketch of a typical experimental setup is shown in Fig. 3a: a beam of α particles is incident on the material target to be measured (for example a gold foil), and the number of α particles scattered in different directions θ is measured.

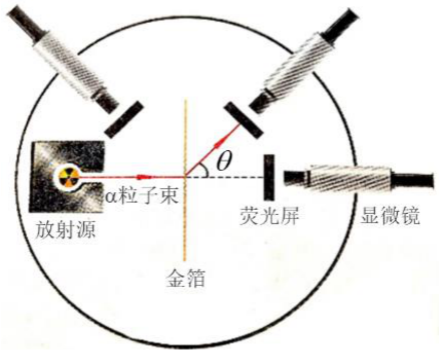

Fig. 3a

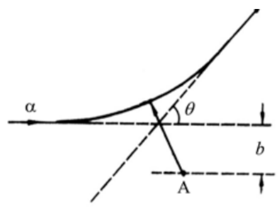

Fig. 3b

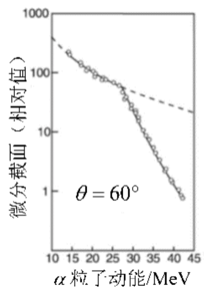

Fig. 3c

Labels: Fig. 3a: 放射源 *= radioactive source;* α 粒子束 *=* α*-particle beam;* 金箔 *= gold foil;* 荧光屏 *= fluorescent screen;* 显微镜 *= microscope. Fig. 3c:* 微分截面（相对值） *= differential cross-section (relative value);* α 粒子动能/MeV *= kinetic energy of the* α *particle / MeV.*

(1) α particles can be obtained from the decay of radioactive elements. A $^{210}_{84}\text{Po}$ (polonium) nucleus at rest decays into Pb (lead), emitting at the same time an α particle of kinetic energy 5.31 MeV. Write down the reaction equation of this decay process and calculate the total kinetic energy of the final-state particles of the decay (in MeV, to two significant figures).

(2) As in Fig. 3b, an α particle of mass m, charge $2e$ ($e > 0$) and kinetic energy E is incident from far away along a certain straight line; the distance from this line to the target nucleus A is b (the impact parameter). After this α particle has been scattered by the nucleus A

of charge number Z, the angle between its direction of motion far away afterwards and the incident direction far away is θ (the scattering angle). The target nucleus A may be regarded as always at rest. Find the relation $b(\theta)$ between b and θ.

(3) In an α-particle scattering experiment what is actually incident is a beam of α particles, of beam intensity I (the number of particles incident per unit time per unit cross-sectional area). The angular distribution of the scattered particles is axially symmetric about the line through A parallel to the direction of incidence of the α particles far away. Within the annular solid-angle element $\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta$ centred on the target nucleus A, the number of particles leaving per unit time $\dfrac{\mathrm{d}N}{\mathrm{d}t}$ is proportional to $I\,\mathrm{d}\Omega$,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \sigma(\theta)\, I \,\mathrm{d}\Omega,$$

where $\sigma(\theta)$ has the dimensions of area and is called the *differential scattering cross-section*. Find the expression for $\sigma(\theta)$ (it must not contain the parameter b).

(4) In experiments, α particles of different kinetic energies obtained with an accelerator are scattered by gold nuclei. The relation between the differential scattering cross-section (relative value) measured at the scattering angle $\theta = 60°$ and the kinetic energy of the incident α particles is shown in Fig. 3c; the experimental data show a kink at α-particle kinetic energy 25 MeV. Explain the reason for this kink and calculate the minimum distance of the α particle from the gold nucleus in that case. It is given that $\dfrac{e^2}{4\pi\varepsilon_0} = 1.44\,\mathrm{fm\,MeV}$ and that the nuclear charge number of gold is 79. The gold nucleus may be regarded as always at rest.

(5) In the α-particle scattering experiment shown in Fig. 3a, because of the blocking by the α-particle source or by the detector (fluorescent screen and microscope), a measurement at scattering angles near 180° cannot be made. Propose a scheme that makes the measurement at scattering angle 180° possible, and draw a sketch of the experimental scheme.

Problem 4 (50 points)

When the distance between two atoms is small, their interaction is a strong repulsion; when the distance is large, their interaction is a weak attraction. A large number of atoms can combine into a crystal through this interaction. As the temperature tends to 0 K the atoms are arranged in a periodic space lattice and are in mechanical equilibrium; at temperatures above 0 K the atoms perform small-amplitude oscillations about their equilibrium positions. Consider a cubic crystal made of identical atoms of mass M. Use the following model to discuss how the atoms combine into a crystal and how the vibrations of the atoms in the crystal affect its thermal properties.

(1) The interaction potential energy $V(R)$ between two atoms a distance R apart can be approximately written as the Lennard-Jones potential

$$V(R) = \varepsilon\left[\left(\frac{r_0}{R}\right)^{12} - \left(\frac{r_0}{R}\right)^{6}\right],$$

where $\varepsilon > 0$ characterizes the strength of the interaction and r_0 characterizes the range. Assume that at equilibrium the atoms are arranged at the vertices of a cubic lattice, i.e. the equilibrium positions of the atoms are

$$\boldsymbol{R}_{l_x l_y l_z} = (l_x a,\ l_y a,\ l_z a), \qquad l_x, l_y, l_z = \cdots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \cdots$$

where a is the lattice constant. For simplicity, assume that the form of the Lennard-Jones potential remains valid right up to the melting of the crystal and that the parameters ε and r_0 may be regarded as unchanged. Derive the expression for the lattice constant a. The result may contain the following constants:

$$A_n = \sum_{l_x,l_y,l_z} (l_x^2 + l_y^2 + l_z^2)^{-n/2},$$

where n is an arbitrary positive integer and $\sum_{l_x,l_y,l_z} \cdots$ denotes the sum over all l_x, l_y, l_z that are not all zero.

(2) Calculate the contributions of the nearest-neighbour, the next-nearest-neighbour and the next-next-nearest-neighbour atoms to the value of A_6.

(3) An exact treatment of the small oscillations of the atoms of a crystal about their equilibrium positions $\boldsymbol{R}_{l_x l_y l_z}$ is very complicated. For simplicity, Einstein assumed that the motions of the atoms do not interfere with one another, i.e. that when the vibration of any one atom is considered, all the other atoms are taken to be at rest at their own equilibrium positions. In this model, find the angular frequency ω_{E} of the small oscillations of each atom of the above crystal (the result may contain a and A_n).

In what follows assume that the Einstein model (including the above result for the angular frequency ω_{E}) remains applicable for large amplitudes:

(4) The distribution of the atoms over vibrational states is the Boltzmann distribution: at absolute temperature T the probability that the magnitude of the momentum of an atom lies in $[p, p + \mathrm{d}p]$ and that the magnitude of its displacement from the equilibrium position lies in $[u, u + \mathrm{d}u]$ is proportional to

$$\mathrm{e}^{-\frac{E}{k_{\mathrm{B}}T}}\, p^2\, \mathrm{d}p\, u^2\, \mathrm{d}u,$$

where E is the energy of the atom when the magnitude of its momentum is p and the magnitude of its displacement is u, and k_{B} is Boltzmann's constant. Avogadro's number is N_{A}. Calculate the molar heat capacity at constant volume of the above crystal (the necessary derivation must be given).

(5) The amplitude of the atomic vibration increases as the temperature rises. According to the Lindemann criterion, the crystal melts when the average value of the distance of deviation from the equilibrium position exceeds $C_{\mathrm{L}} a$, where C_{L} is a constant (of order 0.1). Derive the expression for the melting point of the crystal (the result may contain a and ω_{E}).

Hint.

$$\int_0^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}, \quad \int_0^{+\infty} \mathrm{e}^{-x^2} x^2 \mathrm{d}x = \frac{\sqrt{\pi}}{4}, \quad \int_0^{+\infty} \mathrm{e}^{-x^2} x^3 \mathrm{d}x = \frac{1}{2}, \quad \int_0^{+\infty} \mathrm{e}^{-x^2} x^4 \mathrm{d}x = \frac{3\sqrt{\pi}}{8};$$

$$\int_0^{+\infty}\int_0^{+\infty} f(x)g(y)\,\mathrm{d}x\mathrm{d}y = \int_0^{+\infty} f(x)\mathrm{d}x \cdot \int_0^{+\infty} g(y)\mathrm{d}y.$$

Problem 5 (40 points)

A black hole is a singular celestial object predicted by general relativity; its macroscopic properties are completely determined by the mass M, the angular momentum J and the charge Q. For a spherically symmetric simple black hole (a Schwarzschild black hole) with $J = 0$, $Q = 0$ and mass M one can define a sphere centred on the centre of the black hole and of radius r (called the *horizon*). According to classical theory, no matter inside the horizon can escape from the black hole.

(1) According to Newtonian mechanics, if the escape velocity of a particle at the surface of a spherically symmetric celestial body of mass M is exactly equal to the speed of light c in vacuum, then this body is a black hole. Derive the expression for the radius r of this body. The mass of the Sun is about $2 \times 10^{30}\,\mathrm{kg}$, the gravitational constant is $G = 6.67 \times 10^{-11}\,\mathrm{m^3/(kg\,s^2)}$ and the speed of light in vacuum is $c = 3 \times 10^8\,\mathrm{m/s}$; to what radius must the Sun at least shrink in order to become a black hole?

(2) The r obtained above from Newtonian mechanics happens to agree with the horizon radius given by general relativity for a black hole of the same mass. According to the theory of Bekenstein and Hawking, the entropy S of a black hole is proportional to the area A of its horizon,

$$S = \frac{k_\mathrm{B} c^3}{4G\hbar} A,$$

where $\hbar$ is the reduced Planck constant and k_B is Boltzmann's constant. When two simple black holes each of mass M collapse into one simple black hole of mass M' ($M' < 2M$), find the change of the entropy of the black-hole system (the result should not contain A).

(3) For a simple black hole of absolute temperature T and mass M, its internal energy U and its entropy S satisfy the fundamental thermodynamic relation $\mathrm{d}U = T\,\mathrm{d}S$. According to relativity, the internal energy of the black hole is given by Einstein's mass–energy relation $U = Mc^2$. Derive from this the relation between the black-hole temperature T and its mass M, and calculate the heat capacity of this black hole.

(4) Suppose a reversible heat engine works between two simple black holes whose initial masses are M_{10} and M_{20}, with $M_{20} > M_{10}$. Find the total work done by this engine on the outside over the whole process from the beginning to the end.

(5) According to quantum mechanics, Hawking proposed that the surface of a black hole (i.e. the black-hole horizon) can radiate electromagnetic waves outwards, and that this radiation is equivalent to black-body radiation at the same temperature as the black hole; it is called *Hawking radiation*. Find the relation $P(M)$ between the Hawking radiation power P and the black-hole mass M. The Stefan–Boltzmann constant is $\sigma = \dfrac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2}$.

(6) Because of Hawking radiation the energy of a black hole decreases, so its mass becomes smaller with time. Let the initial mass of the black hole be M_0. Ignoring all other factors, find the time needed for this black hole to disappear finally through Hawking radiation.

Problem 6 (50 points)

In the making of glass, coloured glass can be produced by doping it with molten metals (such as gold, silver or copper). Driven by the electric field of the light, the electrons in these metal particles resonantly absorb a particular frequency component of the incident white light, so that the light near this frequency is converted into heat and the light transmitted through the glass appears coloured. This resonance is called *plasmon resonance*.

(1) Consider a uniform dielectric sphere of radius R and relative permittivity ε_r in vacuum, placed in a uniform external electric field $\boldsymbol{E}_0$. Find the electric field strength $\boldsymbol{E}_\mathrm{in}$ inside the sphere and the electric dipole moment $\boldsymbol{p}$ of the sphere in the steady state. It is known that a uniform dielectric sphere in a uniform external field is uniformly polarized; the permittivity of vacuum is ε_0.

(2) Consider a uniform metallic conductor whose conduction-electron number density is n, electron mass m and electron charge $-e$ ($e > 0$). When the electrons drift with directed

velocity $\boldsymbol{v}$ under an external field, the average effect of the scattering by the lattice is equivalent to a drag force $\boldsymbol{f} = -\gamma m\boldsymbol{v}$, where γ is the drag coefficient. Neglect the interaction between the electrons.

(i) If a uniform constant electric field $\boldsymbol{E}$ exists in this metallic conductor, find the current density $\boldsymbol{j}$ in the metal in the steady state and the conductivity σ_0 of this metal.

(ii) If a uniform alternating electric field $\boldsymbol{E}(t) = \boldsymbol{E}_\mathrm{m}\cos\omega t$ of amplitude $\boldsymbol{E}_\mathrm{m}$ and angular frequency ω exists in the metallic conductor, the electrons will move and form a current. If in the steady state the current density in the metal is written as $\boldsymbol{j}(t) = \boldsymbol{j}_\mathrm{m}\cos(\omega t + \varphi_j)$, put $\sigma = \dfrac{j_\mathrm{m}}{E_\mathrm{m}}$; in analogy with the complex representation of alternating current, introduce the complex conductivity $\tilde{\sigma} = \sigma\mathrm{e}^{\mathrm{i}\varphi_j}$. In complex form the relation between the complex current density in the metal $\tilde{\boldsymbol{j}} = \boldsymbol{j}_\mathrm{m}\mathrm{e}^{\mathrm{i}(\omega t+\varphi_j)}$ and the complex electric field strength $\tilde{\boldsymbol{E}} = \boldsymbol{E}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t}$ can be written as $\tilde{\boldsymbol{j}} = \tilde{\sigma}\tilde{\boldsymbol{E}}$. Find the expression for the complex conductivity $\tilde{\sigma} = \mathrm{Re}\,\tilde{\sigma} + \mathrm{i}\,\mathrm{Im}\,\tilde{\sigma}$ of this metal (express it in terms of ω, γ, ε_0 and the parameter $\omega_\mathrm{p} = \sqrt{\dfrac{ne^2}{\varepsilon_0 m}}$).

(iii) In the above uniform alternating field the electrons move together synchronously; in analogy with a dielectric one can define the polarization in the metallic conductor. In complex form the relation between the complex polarization $\tilde{\boldsymbol{P}}$ and the complex electric field strength $\tilde{\boldsymbol{E}}$ in the metal can be written as $\tilde{\boldsymbol{P}} = \tilde{\chi}\varepsilon_0\tilde{\boldsymbol{E}}$, where $\tilde{\chi}$ is the complex susceptibility. Furthermore one can introduce the complex electric displacement $\tilde{\boldsymbol{D}} = \varepsilon_0\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{P}} = \tilde{\varepsilon}_\mathrm{r}\varepsilon_0\tilde{\boldsymbol{E}}$ and the complex relative permittivity $\tilde{\varepsilon}_\mathrm{r}$. Find the expression for the complex relative permittivity $\tilde{\varepsilon}_\mathrm{r}(\omega) = \mathrm{Re}\,\tilde{\varepsilon}_\mathrm{r}(\omega) + \mathrm{i}\,\mathrm{Im}\,\tilde{\varepsilon}_\mathrm{r}(\omega)$ of this metal (express it in terms of ω, γ and the parameter $\omega_\mathrm{p} = \sqrt{\dfrac{ne^2}{\varepsilon_0 m}}$).

(3) Spherical nanoparticles of radius R made of the above metallic conductor are placed in a plane harmonic electromagnetic wave of angular frequency ω ($\omega < \omega_\mathrm{p}$) whose electric-field amplitude has magnitude E_0 and whose wavelength is much larger than R. Consider only the polarization of the metal particle in the electric field; in complex form this can be treated in analogy with the polarization of the dielectric sphere of part (1). It is given that $\gamma \ll \omega_\mathrm{p}$; neglect electromagnetic radiation.

(i) Find the magnitude $E_\mathrm{in}(\omega)$ of the electric-field amplitude inside the metal nanoparticle in the steady state; when $\omega = \omega_\mathrm{r}$, E_in reaches its maximum and plasmon resonance occurs. Find the resonance angular frequency ω_r (for simplicity one may take $\gamma = 0$ when finding ω_r).

(ii) Find the magnitude $E_\mathrm{r} = E_\mathrm{in}(\omega_\mathrm{r})$ of the electric-field amplitude inside the particle at resonance, and the average heating power P_r inside the particle at resonance. (Since $\gamma \ll \omega_\mathrm{p}$, drop the higher-order small quantities in γ.)

Problem 7 (50 points)

Suppose that the magnetic field of a magnetic monopole (a point magnetic charge) g satisfies the so-called Coulomb law for magnetic charge,

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{g}{r^3}\boldsymbol{r},$$

where μ_0 is the permeability of vacuum and $\boldsymbol{r}$ is the position vector of the field point relative to the position of the magnetic charge. Conventional electromagnetic theory holds that magnetic monopoles do not exist, and up to now no magnetic monopole has been found experimentally. In 1931 Dirac proposed a model in which a string is constructed out of infinitely many (magnetic-charge) magnetic dipoles joined head to tail (the *Dirac string*); this string extends from one end point to infinity, as shown in the left part of Fig. 7a. If every small magnetic dipole is made equivalent to a small current loop, then the whole Dirac string is equivalent to a semi-infinite, extremely thin current-carrying solenoid of the same cross-section, uniformly wound, as shown in the right part of Fig. 7a. This equivalence means that the elementary magnetic dipole moment $g\,\mathrm{d}\boldsymbol{l}$ on the Dirac string equals the elementary magnetic moment $\mathrm{d}\boldsymbol{m}$ of the corresponding line element $\mathrm{d}\boldsymbol{l}$ on the axis of the solenoid, i.e. $\mathrm{d}\boldsymbol{m} = g\,\mathrm{d}\boldsymbol{l}$ (see Fig. 7a). In this picture a "magnetic monopole" also effectively exists at the end point of the solenoid, and the magnetic monopole introduced in this way obeys the ordinary laws of electromagnetism. In what follows the vector potential is introduced in order to discuss the Dirac string and related questions.

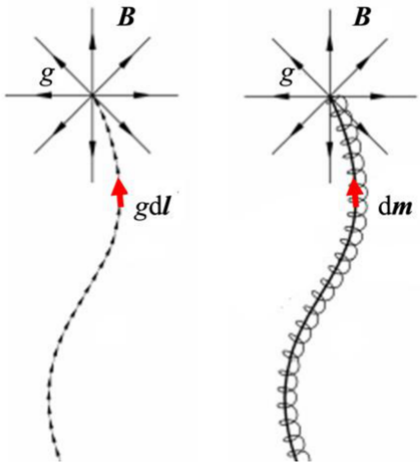

Fig. 7a

According to ordinary electromagnetic theory the flux of the magnetic induction $\boldsymbol{B}$ can be expressed as the circulation of the vector potential $\boldsymbol{A}$,

$$\iint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \oint_L \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l},$$

where L is the boundary loop of the surface S. The vector potential excited by a small current loop of magnetic moment $\boldsymbol{m}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3},$$

where $\boldsymbol{r}$ is the position vector of the field point relative to the position of the current loop.

(1) As in Fig. 7b, the magnetic charge at the end point of the extremely thin current-carrying solenoid equivalent to the Dirac string is g; the solenoid extends from the origin along the negative z axis to infinity.

(i) Derive the expression for the vector potential $\boldsymbol{A}$ of this solenoid at the position (r,θ,φ) $(0 \le \theta < \pi)$.

Hint: $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}}$ + an arbitrary constant.

(ii) For the directed circular loop L in the figure, which has the z axis as its axis and for which the spherical coordinates r, θ $(0 \le \theta < \pi)$ are fixed, first calculate the circulation Γ_L of the vector potential $\boldsymbol{A}$; then calculate, according to the Coulomb law for magnetic charge, the flux Φ_L of the field of the point magnetic charge g through the disc bounded by the ring L (the normal of the area element is along the positive

z direction and forms a right-handed screw relation with the sense of the loop; do not consider the case $\theta = \frac{\pi}{2}$); and compare Γ_L with Φ_L (if they differ, analyse the source of the difference).

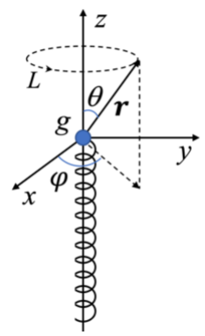

Fig. 7b

(2) In the extremely thin current-carrying solenoid that extends from the origin along the negative z axis and is equivalent to the Dirac string, the current $I(t) = I_0 \cos \omega t$ is a low-frequency alternating current. The cross-sectional area of the solenoid is S and the number of turns per unit length is n.

(i) Find the equivalent magnetic charge $g(t)$ at the origin.

(ii) A wire of resistance R_0 per unit length is bent into an equilateral triangular loop of side a. This triangle lies in the plane $z = -\frac{\sqrt{6}}{12}a$ and the solenoid passes through its centre, as shown in Fig. 7c. Find the average heating power of this triangular loop. Neglect radiation and the self-inductance of the triangular loop.
Hint: one may consider the solid angle subtended by the equilateral triangle at g.

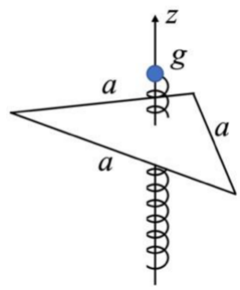

Fig. 7c

(3) Experiments on the Aharonov–Bohm effect (the A–B effect) prove that even in a region where the magnetic induction is zero, magnetic effects can appear because $\boldsymbol{A} \neq 0$. This reveals the physical meaning of the magnetic vector potential $\boldsymbol{A}$. The A–B effect can be verified with a free-electron double-slit interference experiment. In this experiment the distance between the double slit (whose slit width is very small) and the screen is D, and the separation of the slits is d ($d \ll D$). An infinitely long, extremely thin straight solenoid is placed perpendicular to the plane of the figure, between the paths of the electrons after they have passed the double slit; its number of turns per unit length is n and its cross-sectional area is S, as shown in Fig. 7d. The magnitude of the momentum of the free electrons emitted by the electron source is p and their charge is $-e$ ($e > 0$). If the current in the solenoid changes from 0 to I, find the distance through which the central bright fringe moves on the screen. It is known that the wave vector of an electron of momentum $\boldsymbol{p}$ in a vector-potential field $\boldsymbol{A}$ is $\boldsymbol{k} = \frac{1}{\hbar}(\boldsymbol{p} - e\boldsymbol{A})$, where $\hbar$ is the reduced Planck constant.

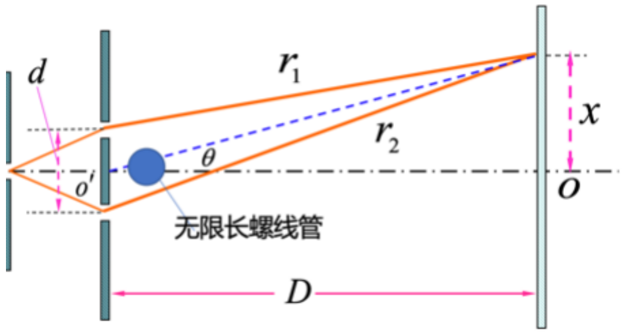

Fig. 7d

Labels: 无限长螺线管 = *infinitely long solenoid (the electron source is at the far left).*