The 40th Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2023 — official reference solutions

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*Translated from the official booklet "Reference solutions of the theoretical examination of the 40th National Physics Competition for Secondary School Students (final round)", dated October 2023 and marked **trial version** (试用版) by the source. Equation numbers* (1), (2), ... *within each problem follow the circled numbers of the Chinese original; primed variants* (5′), (5″), ... *belong to the alternative methods, which the original marks with* 【解法一】, 【解法二】, ... *and which are rendered here as "Method 1", "Method 2", ... Passages that the original encloses in the brackets* 【】 *are set here as bracketed remarks. The per-question marking schemes printed at the end of each problem are translated in full. Obvious misprints of the source are transcribed as printed, with a translator's note giving the correct form.*

Problem 1 (rainbows: deviation angle, colour order, polarization)

Solution

(1) The angle of incidence is i; let the angle of refraction be r. The law of refraction gives

$$\sin i = n \sin r \tag{1}$$

Each refraction turns the ray through the angle $(i-r)$ and each reflection turns it through the angle $(\pi - 2r)$, so the deviation angle of the order-k outgoing light is

$$\theta_k = 2(i-r) + k(\pi - 2r) = k\pi + 2i - 2(k+1)r \tag{2}$$

Substituting (1) gives

$$\theta_k = k\pi + 2i - 2(k+1)\arcsin\left(\frac{1}{n}\sin i\right) \tag{3}$$

(2) From (2), for θ_k to take a minimum we require

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}i} = 2 - 2(k+1)\frac{\mathrm{d}r}{\mathrm{d}i} = 0 \tag{4}$$

From (1),

$$\frac{\mathrm{d}r}{\mathrm{d}i} = \frac{\cos i}{n\cos r} = \frac{\cos i}{\sqrt{n^2 - \sin^2 i}} \tag{5}$$

Substituting into (4),

$$2 - 2(k+1)\frac{\cos i}{\sqrt{n^2 - \sin^2 i}} = 0 \tag{6}$$

[Or: using the differentiation formula $\dfrac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$, directly from (3),

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}i} = 2 - 2(k+1)\frac{\cos i/n}{\sqrt{1-(\sin i/n)^2}} = 2 - 2(k+1)\frac{\cos i}{\sqrt{n^2 - \sin^2 i}} = 0, \tag{6′}$$

] Solving,

$$i_k = \arccos\sqrt{\frac{n^2-1}{k(k+2)}} = \arcsin\sqrt{1-\frac{n^2-1}{k(k+2)}} \tag{7}$$

Substituting into (3),

$$\theta_{k\mathrm{m}} = k\pi + 2\arccos\sqrt{\frac{n^2-1}{k(k+2)}} - 2(k+1)\arcsin\left(\frac{1}{n}\sqrt{1-\frac{n^2-1}{k(k+2)}}\right) \tag{8}$$

(3) From (2), the rate of change of θ_k with respect to the refractive index n is

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}n} = -2(k+1)\frac{\mathrm{d}r}{\mathrm{d}n} \tag{9}$$

From (1),

$$\frac{\mathrm{d}r}{\mathrm{d}n} = -\frac{\sin r}{n\cos r} = -\frac{\sin i}{n\sqrt{n^2-\sin^2 i}} \tag{10}$$

Substituting into (9),

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}n} = \frac{2(k+1)}{n}\frac{\sin i}{\sqrt{n^2-\sin^2 i}} \tag{11}$$

[Or: using $\dfrac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$, directly from (3),

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}n} = -2(k+1)\frac{(-1/n^2)\sin i}{\sqrt{1-(\sin i/n)^2}} = \frac{2(k+1)}{n}\frac{\sin i}{\sqrt{n^2-\sin^2 i}} \tag{11$'$}$$

] When $i = i_k$, substituting (7),

$$\begin{aligned}\left.\frac{\mathrm{d}\theta_k}{\mathrm{d}n}\right|_{i=i_k} &= \frac{2(k+1)}{n}\sqrt{\frac{1-\dfrac{n^2-1}{k(k+2)}}{n^2-1+\dfrac{n^2-1}{k(k+2)}}} \\ &= \frac{2(k+1)}{n}\sqrt{\frac{k(k+2)+1-n^2}{[k(k+2)+1](n^2-1)}} = \frac{2}{n}\sqrt{\frac{(k+1)^2-n^2}{n^2-1}}\end{aligned} \tag{12}$$

(4) For the order $k=1$ (the primary bow), (8) gives

$$\theta_{1\mathrm{m}}^{\mathrm{red}} = 137.34^\circ, \qquad \theta_{1\mathrm{m}}^{\mathrm{violet}} = 139.50^\circ \tag{13}$$

The angular width is

$$\delta_1 = \left|\theta_{1\mathrm{m}}^{\mathrm{violet}} - \theta_{1\mathrm{m}}^{\mathrm{red}}\right| = 2.16^\circ \tag{14}$$

The angles subtended at the observer by the red bow and by the violet bow are $\pi - \theta_{1\mathrm{m}}^{\mathrm{red}} = 42.66^\circ$ and $\pi - \theta_{1\mathrm{m}}^{\mathrm{violet}} = 40.50^\circ$, so

$$\text{violet is on the inside, red is on the outside} \tag{15}$$

For the order $k=2$ (the secondary bow), (8) gives

$$\theta_{2\mathrm{m}}^{\mathrm{red}} = 229.84^\circ, \qquad \theta_{2\mathrm{m}}^{\mathrm{violet}} = 233.73^\circ \tag{16}$$

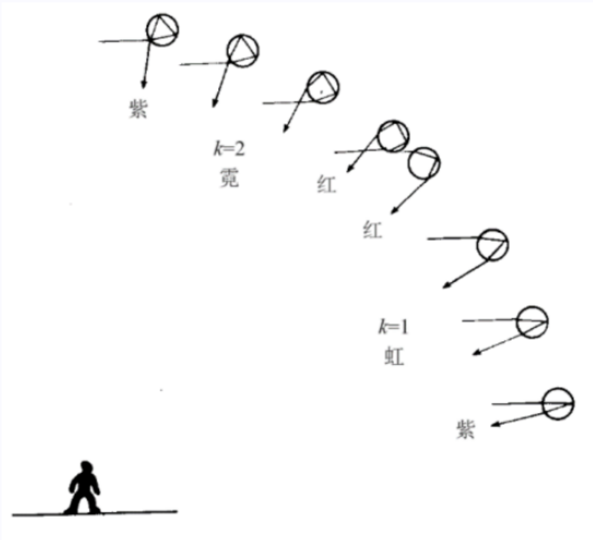
The angular width is

$$\delta_2 = \left|\theta_{2\mathrm{m}}^{\mathrm{violet}} - \theta_{2\mathrm{m}}^{\mathrm{red}}\right| = 3.89° \tag{17}$$

The angles subtended at the observer by the red secondary bow and by the violet secondary bow are $\theta_{2\mathrm{m}}^{\mathrm{red}} - \pi = 49.84°$ and $\theta_{2\mathrm{m}}^{\mathrm{violet}} - \pi = 53.73°$, so

$$\text{red is on the inside, violet is on the outside} \tag{18}$$

Appendix: the sketch of the primary and the secondary bow is as shown.



Labels: 红 = red; 紫 = violet; 虹 = primary bow; 霓 = secondary bow.

(5) Let the electric-field amplitudes of the s and of the p polarization of the incident light both be E_0. For the order-k bow the angle of incidence is i_k and the angle of refraction is r_k; after the refraction from air into water, the k reflections inside the water and the refraction from water into air, Fresnel's formulas give for the electric-field amplitudes of the s and of the p polarization

$$E_{\mathrm{s}k} = \frac{2\cos i_k}{\cos i_k + n\cos r_k}\left|\frac{n\cos r_k - \cos i_k}{n\cos r_k + \cos i_k}\right|^k \frac{2n\cos r_k}{n\cos r_k + \cos i_k}E_0 \tag{19}$$

$$E_{\mathrm{p}k} = \frac{2\cos i_k}{n\cos i_k + \cos r_k}\left|\frac{\cos r_k - n\cos i_k}{\cos r_k + n\cos i_k}\right|^k \frac{2n\cos r_k}{\cos r_k + n\cos i_k}E_0 \tag{20}$$

For the order $k = 1$ (the primary bow), $i_1 = 59.64°$, $r_1 = 40.49°$; substituting into (18) and (19)[a] gives

$$E_{1\mathrm{s}} = 0.2963E_0, \qquad E_{1\mathrm{p}} = 0.0618E_0 \tag{21}$$

Since the intensity is proportional to the square of the electric-field amplitude,

$$\frac{I_{\mathrm{s}1}}{I_{\mathrm{p}1}} = \frac{E_{\mathrm{s}1}^2}{E_{\mathrm{p}1}^2} = \frac{0.2963^2E_0^2}{0.0618^2E_0^2} = \frac{0.08779}{0.00382} \tag{22}$$

The degree of polarization is

$$P_1 = \frac{I_{\mathrm{s}1} - I_{\mathrm{p}1}}{I_{\mathrm{s}1} + I_{\mathrm{p}1}} = 91.65\% \tag{23}$$

For the order $k = 2$ (the secondary bow), $i_2 = 71.97°$, $r_2 = 45.68°$; substituting into (19) and (20) gives

$$E_{2\mathrm{s}} = 0.1875E_0, \qquad E_{2\mathrm{p}} = 0.0625E_0, \tag{24}$$

Since the intensity is proportional to the square of the electric-field amplitude,

$$\frac{I_{\mathrm{s}2}}{I_{\mathrm{p}2}} = \frac{E_{\mathrm{s}2}^2}{E_{\mathrm{p}2}^2} = \frac{0.1875^2E_0^2}{0.0625^2E_0^2} = \frac{0.03516}{0.00391} \tag{25}$$

The degree of polarization is

$$P_2 = \frac{I_{\mathrm{s}2} - I_{\mathrm{p}2}}{I_{\mathrm{s}2} + I_{\mathrm{p}2}} = 79.99\% \tag{26}$$

(6) Near the primary bow the wavefront of the outgoing light is different from that of the incident light: it is no longer a plane wave but has become distorted. This extra optical path difference makes the outgoing rays interfere, and thus produces the "supernumerary bows".

Marking scheme.
This problem carries 40 points.
Part (1) 3 points: (1)(2)(3) 1 point each;
Part (2) 6 points: (4)(5) 1 point each, (7)(8) 2 points each;
Part (3) 6 points: (9)(10) 1 point each, (11)(12) 2 points each;
Part (4) 10 points: (13)(16) 2 points each, (14)(17) 1 point each, (16)(18) 2 points each[b];
Part (5) 12 points: (19)(20)(21)(23)(24)(26) 2 points each;
Part (6) 3 points.

[a] *Translator's note:* The source prints "(18)(19)"; the equations actually used are (19) and (20).
[b] *Translator's note:* As printed, equation (16) appears twice in the part-(4) marking scheme. From the total of 10 points the last item should read "(15)(18) 2 points each".

Problem 2 (rod on a rapidly oscillating pivot: effective potential)

Solution

(1) In the translating non-inertial frame of the axis O, introduce the translational inertial force (taking vertically upward as positive)

$$F_{\mathrm{in}} = -m\ddot{z} = m\omega^2 A\cos\omega t \tag{1}$$

This translational inertial force acts effectively at the centre of mass of the uniform rod; the rotation theorem for the rod gives

$$-(mg - F_{\mathrm{in}})\frac{l}{2}\sin\theta = \frac{1}{3}ml^2\ddot{\theta} \tag{2}$$

Substituting (1) gives the dynamical equation for $\theta(t)$,

$$-(mg - m\omega^2 A\cos\omega t)\frac{l}{2}\sin\theta = \frac{1}{3}ml^2\ddot{\theta} \tag{3}$$

(2) Substituting $\theta = \varphi + \delta$ ($\delta \ll 1$) into (3) and expanding to first order in δ,

$$-(mg - m\omega^2 A\cos\omega t)\frac{l}{2}(\sin\varphi + \cos\varphi\cdot\delta) = \frac{1}{3}ml^2(\ddot{\varphi} + \ddot{\delta}) \tag{4}$$

that is,

$$-mg\frac{l}{2}\sin\varphi - mg\frac{l}{2}\cos\varphi\cdot\delta + m\omega^2 A\cos\omega t\,\frac{l}{2}\sin\varphi + m\omega^2 A\cos\omega t\,\frac{l}{2}\cos\varphi\cdot\delta = \frac{1}{3}ml^2(\ddot{\varphi}+\ddot{\delta})$$

Method 1. Since $\delta(t)$ is a small high-frequency oscillation of angular frequency $\omega \gg \sqrt{g/l}$, the time average of a physical quantity x over one high-frequency period $T = \dfrac{2\pi}{\omega}$ is

$$\langle x\rangle = \frac{1}{T}\int_t^{t+T} x(t')\mathrm{d}t'$$

Taking the time average of (4) over one period of $\delta(t)$, with φ and $\ddot{\varphi}$ regarded as unchanged,

$$\begin{aligned} &-mg\frac{l}{2}\sin\varphi - mg\frac{l}{2}\cos\varphi\langle\delta\rangle + m\omega^2 A\frac{l}{2}\sin\varphi\langle\cos\omega t\rangle \\ &+ m\omega^2 A\frac{l}{2}\cos\varphi\langle\delta\cdot\cos\omega t\rangle = \frac{1}{3}ml^2(\ddot{\varphi}+\langle\ddot{\delta}\rangle) \end{aligned}$$

Taking into account

$$\langle\delta\rangle = 0,\quad \langle\cos\omega t\rangle = 0,\quad \langle\delta\cdot\cos\omega t\rangle \text{ to be determined},\quad \langle\ddot{\delta}\rangle = 0$$

we obtain

$$-mg\frac{l}{2}\sin\varphi + m\omega^2 A\frac{l}{2}\cos\varphi\langle\delta\cdot\cos\omega t\rangle = \frac{1}{3}ml^2\ddot{\varphi} \tag{5}$$

In order to determine $\langle\delta\cdot\cos\omega t\rangle$, multiply both sides of (4) by $\cos\omega t$:

$$\begin{aligned} &-mg\frac{l}{2}\sin\varphi\cos\omega t - mg\frac{l}{2}\cos\varphi\,\delta\cos\omega t + m\omega^2 A\frac{l}{2}\sin\varphi\cos^2\omega t \\ &+ m\omega^2 A\frac{l}{2}\cos\varphi\,\delta\cos^2\omega t = \frac{1}{3}ml^2(\ddot{\varphi}\cos\omega t + \ddot{\delta}\cos\omega t) \end{aligned} \tag{6}$$

Taking into account

$$\ddot{\delta} = -\omega^2\delta \tag{7}$$

substituting into (6), and then taking the time average of (6) over one period of $\delta(t)$ with φ, $\ddot{\varphi}$ regarded as unchanged,

$$\begin{aligned} &-mg\frac{l}{2}\sin\varphi\langle\cos\omega t\rangle - mg\frac{l}{2}\cos\varphi\langle\delta\cdot\cos\omega t\rangle + m\omega^2 A\frac{l}{2}\sin\varphi\langle\cos^2\omega t\rangle \\ &+ m\omega^2 A\frac{l}{2}\cos\varphi\langle\delta\cdot\cos^2\omega t\rangle = \frac{1}{3}ml^2\left(\ddot{\varphi}\langle\cos\omega t\rangle - \omega^2\langle\delta\cdot\cos\omega t\rangle\right) \end{aligned} \tag{8}$$

Taking into account

$$\langle\cos\omega t\rangle = 0,\quad \langle\cos^2\omega t\rangle = \frac{1}{2},\quad \langle\delta\cdot\cos^2\omega t\rangle = 0$$

we obtain

$$-mg\frac{l}{2}\cos\varphi\langle\delta\cdot\cos\omega t\rangle + m\omega^2 A\frac{l}{2}\sin\varphi\cdot\frac{1}{2} = -\frac{1}{3}ml^2\omega^2\langle\delta\cdot\cos\omega t\rangle \tag{9}$$

Solving,

$$\langle\delta\cdot\cos\omega t\rangle = \frac{-\dfrac{3}{4}\omega^2 A\sin\varphi}{\omega^2 l - \dfrac{3}{2}g\cos\varphi} \approx -\frac{3}{4}\frac{A}{l}\sin\varphi \qquad (\text{since } \omega^2 \gg g/l) \tag{10}$$

Substituting into (5) gives the dynamical equation for $\varphi(t)$,

$$-mg\frac{l}{2}\sin\varphi - \frac{3}{8}m\omega^2A^2\sin\varphi\cos\varphi = \frac{1}{3}ml^2\ddot{\varphi} \tag{11}$$

Method 2. Since $\delta(t)$ is a small high-frequency oscillation of angular frequency $\omega \gg \sqrt{g/l}$, put

$$\delta(t) = \delta_0\cos\omega t, \qquad \ddot{\delta} = -\omega^2\delta_0\cos\omega t \tag{5$'$}$$

Substituting into (4),

$$\begin{aligned}&-mg\frac{l}{2}\sin\varphi - mg\frac{l}{2}\cos\varphi\,\delta_0\cos\omega t + m\omega^2A\cos\omega t\,\frac{l}{2}\sin\varphi\\&+m\omega^2A\frac{l}{2}\cos\varphi\,\delta_0\cos^2\omega t = \frac{1}{3}ml^2(\ddot{\varphi}+\ddot{\delta})\end{aligned} \tag{6$'$}$$

Taking the time average of (6′) over one period of $\delta(t)$, with φ and $\ddot{\varphi}$ regarded as unchanged and using $\langle\cos\omega t\rangle = 0$, $\langle\cos^2\omega t\rangle = \frac{1}{2}$, we obtain

$$-mg\frac{l}{2}\sin\varphi + m\omega^2A\frac{l}{2}\cos\varphi\frac{\delta_0}{2} = \frac{1}{3}ml^2\ddot{\varphi} \tag{7$'$}$$

To determine δ_0, substitute (5′) into (4) and multiply both sides by $\cos\omega t$:

$$\begin{aligned}&-mg\frac{l}{2}\sin\varphi\cos\omega t - mg\frac{l}{2}\cos\varphi\,\delta_0\cos^2\omega t + m\omega^2A\frac{l}{2}\sin\varphi\cos^2\omega t\\&+m\omega^2A\frac{l}{2}\cos\varphi\,\delta_0\cos^3\omega t = \frac{1}{3}ml^2(\ddot{\varphi}\cos\omega t - \omega^2\delta_0\cos^2\omega t)\end{aligned} \tag{8$'$}$$

Taking the time average of (8′) over one period of $\delta(t)$, with φ, $\ddot{\varphi}$ regarded as unchanged and using $\langle\cos\omega t\rangle = \langle\cos^3\omega t\rangle = 0$, $\langle\cos^2\omega t\rangle = \frac{1}{2}$, we obtain

$$-mg\frac{l}{2}\cos\varphi\frac{\delta_0}{2} + m\omega^2A\frac{l}{2}\sin\varphi\frac{1}{2} = \frac{1}{3}ml^2\left(-\omega^2\frac{\delta_0}{2}\right) \tag{9$'$}$$

Solving,

$$\delta_0 = \frac{-\frac{3}{2}\omega^2A\sin\varphi}{\omega^2l - \frac{3}{2}g\cos\varphi} \approx -\frac{3}{2}\frac{A}{l}\sin\varphi \qquad (\text{since } \omega^2 \gg g/l) \tag{10$'$}$$

Substituting (10′) into (7′) gives the dynamical equation for $\varphi(t)$,

$$-mg\frac{l}{2}\sin\varphi - \frac{3}{8}m\omega^2A^2\sin\varphi\cos\varphi = \frac{1}{3}ml^2\ddot{\varphi} \tag{11$'$}$$

Method 3. Since $\delta(t)$ is a small high-frequency oscillation of angular frequency $\omega \gg \sqrt{g/l}$, $\delta \propto A$ is a small quantity while $\ddot{\delta} \propto \omega^2A$ is not. On the left-hand side of the last equation, the first term has nothing to do with the high-frequency micro-oscillation, the second and the fourth terms both contain δ and are small, and the third term is the term that determines δ; that is,

$$m\omega^2A\frac{l}{2}\sin\varphi\cos\omega t = \frac{1}{3}ml^2\ddot{\delta} \tag{5$''$}$$

Since $\delta(t)$ is a steady micro-oscillation, the expression for $\delta(t)$ is

$$\delta(t) = -\frac{3}{2}\frac{A}{l}\sin\varphi\cos\omega t \tag{6$''$}$$

Substituting $\delta(t)$ back into the original equation (i.e. into (4)),

$$\begin{aligned} &-mg\frac{l}{2}\sin\varphi - mg\frac{l}{2}\cos\varphi\cdot\delta + m\omega^2 A\cos\omega t\,\frac{l}{2}\cos\varphi\cdot\delta \\ &-\frac{3}{4}m\omega^2A^2\sin\varphi\cos\varphi\cdot\cos^2\omega t = \frac{1}{3}ml^2(\ddot{\varphi}+\ddot{\delta}) \end{aligned} \tag{7''}$$

Taking the time average of (7″) over one period of $\delta(t)$, with φ, $\ddot{\varphi}$ regarded as unchanged and using $\langle\delta\rangle = 0$, $\langle\ddot{\delta}\rangle = 0$, $\langle\cos^2\omega t\rangle = \frac{1}{2}$, we obtain the dynamical equation for $\varphi(t)$,

$$-mg\frac{l}{2}\sin\varphi - \frac{3}{8}m\omega^2A^2\sin\varphi\cos\varphi = \frac{1}{3}ml^2\ddot{\varphi} \tag{8''}$$

(3) The left-hand side of (11) may be regarded as a conservative torque; its relation to the potential energy is

$$M(\varphi) = -mg\frac{l}{2}\sin\varphi - \frac{3}{8}m\omega^2A^2\sin\varphi\cos\varphi = -\frac{\mathrm{d}V_{\text{eff}}(\varphi)}{\mathrm{d}\varphi} \tag{12}$$

Combining with $V_{\text{eff}}(\varphi = 0) = 0$ we obtain

$$V_{\text{eff}}(\varphi) = \int_{\varphi}^{0} M(\varphi)\mathrm{d}\varphi = \frac{1}{2}mgl(1-\cos\varphi) + \frac{3}{16}m\omega^2A^2(1-\cos^2\varphi) \tag{13}$$

(4) At an equilibrium position φ_0 the function $V_{\text{eff}}(\varphi)$ has an extremum; the condition to be satisfied is

$$V'_{\text{eff}}(\varphi_0) = -M(\varphi_0) = \left(\frac{1}{2}mgl + \frac{3}{8}m\omega^2A^2\cos\varphi_0\right)\sin\varphi_0 = 0 \tag{14}$$

Solving,

$$\sin\varphi_0 = 0 \quad \text{or} \quad \cos\varphi_0 = -\frac{4gl}{3\omega^2A^2} \qquad \left(\text{requires } \omega^2 > \frac{4gl}{3A^2}\right) \tag{15}$$

The equilibrium positions are respectively

$$\varphi_0 = 0 \tag{16-1}$$

$$\varphi_0 = \pi \tag{16-2}$$

$$\varphi_0 = \pm\arccos\left(-\frac{4gl}{3\omega^2A^2}\right) = \pm\left(\pi - \arccos\frac{4gl}{3\omega^2A^2}\right) \qquad \left(\text{requires } \omega^2 > \frac{4gl}{3A^2}\right) \tag{16-3}$$

When the rod deviates from the equilibrium position φ_0 by a small angle φ' ($\varphi' \ll \varphi_0$), the energy of the rod can be written as

$$E = \frac{1}{2}\frac{1}{3}ml^2\dot{\varphi}^2 + V_{\text{eff}}(\varphi = \varphi_0 + \varphi') = \frac{1}{2}\frac{1}{3}ml^2\dot{\varphi}'^2 + V_{\text{eff}}(\varphi_0) + \frac{1}{2}V''_{\text{eff}}(\varphi_0)\varphi'^2 \tag{17}$$

where

$$V''_{\text{eff}}(\varphi_0) = \frac{1}{2}mgl\cos\varphi_0 + \frac{3}{8}m\omega^2A^2(2\cos^2\varphi_0 - 1) \tag{18}$$

At a stable equilibrium position $V_{\text{eff}}(\varphi)$ must have a minimum, $V''_{\text{eff}}(\varphi_0) > 0$; at an unstable equilibrium position $V_{\text{eff}}(\varphi)$ must have a maximum, $V''_{\text{eff}}(\varphi_0) < 0$.

(i) For $\varphi_0 = 0$: from (18),

$$V''_{\text{eff}}(\varphi_0 = 0) = \frac{1}{2}mgl + \frac{3}{8}m\omega^2A^2 > 0 \tag{19}$$

so

$$\varphi_0 = 0 \text{ is a } \textbf{stable} \text{ equilibrium position} \tag{20}$$

From (17), the frequency of the smooth small oscillation of the rod after a small disturbance is

$$f = \frac{1}{2\pi}\sqrt{\frac{V''_{\text{eff}}(\varphi_0 = 0)}{ml^2/3}} = \frac{1}{2\pi}\sqrt{\frac{3}{2}\left(\frac{g}{l} + \frac{3}{4}\frac{\omega^2 A^2}{l^2}\right)} \tag{21}$$

(ii) For $\varphi_0 = \pi$: from (18),

$$V''_{\text{eff}}(\varphi_0 = \pi) = -\frac{1}{2}mgl + \frac{3}{8}m\omega^2 A^2 \begin{cases} < 0 & \omega^2 < \dfrac{4gl}{3A^2} \\ > 0 & \omega^2 > \dfrac{4gl}{3A^2} \end{cases} \tag{22}$$

so

$$\text{when } \omega^2 < \frac{4gl}{3A^2},\ \varphi_0 = \pi \text{ is an } \textbf{unstable} \text{ equilibrium position} \tag{23-1}$$

$$\text{when } \omega^2 > \frac{4gl}{3A^2},\ \varphi_0 = \pi \text{ is a } \textbf{stable} \text{ equilibrium position} \tag{23-2}$$

(One sees that the rod, originally in unstable equilibrium directly above the axis, can now hold stable equilibrium at this position because the axis oscillates at high frequency; i.e. the stability of the equilibrium has changed.)

When $\omega^2 > \dfrac{4gl}{3A^2}$, from (17) the frequency of the smooth small oscillation of the rod after a small disturbance is

$$f = \frac{1}{2\pi}\sqrt{\frac{V''_{\text{eff}}(\varphi_0 = \pi)}{ml^2/3}} = \frac{1}{2\pi}\sqrt{\frac{3}{2}\left(\frac{3}{4}\frac{\omega^2 A^2}{l^2} - \frac{g}{l}\right)} \tag{24}$$

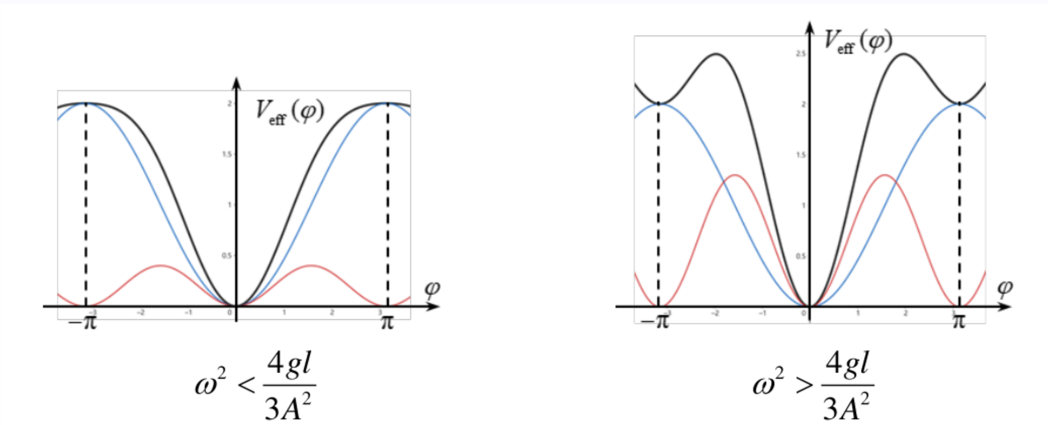
(iii) For $\varphi_0 = \pm\arccos\left(-\dfrac{4gl}{3\omega^2 A^2}\right) = \pm\left(\pi - \arccos\dfrac{4gl}{3\omega^2 A^2}\right)$ (requires $\omega^2 > \dfrac{4gl}{3A^2}$): from (18),

$$V''_{\text{eff}}(\varphi_0) = \frac{1}{2}mgl\cos\varphi_0 + \frac{3}{8}m\omega^2 A^2(2\cos^2\varphi_0 - 1) = \frac{2mg^2l^2}{3\omega^2 A^2} - \frac{3}{8}m\omega^2 A^2 < 0 \tag{25}$$

so

$$\varphi_0 = \pm\arccos\left(-\frac{4gl}{3\omega^2 A^2}\right) = \pm\left(\pi - \arccos\frac{4gl}{3\omega^2 A^2}\right) \text{ are } \textbf{unstable} \text{ equilibrium positions} \tag{26}$$

(5) The potential-energy curves $V_{\text{eff}}(\varphi)$ for $\omega^2 < \dfrac{4gl}{3A^2}$ and for $\omega^2 > \dfrac{4gl}{3A^2}$ are shown respectively in the figures below.



(i) When $\omega^2 < \dfrac{4gl}{3A^2}$: $\varphi = 0$ is a minimum of $V_{\text{eff}}(\varphi)$, $\varphi = \pi$ is a maximum of $V_{\text{eff}}(\varphi)$, and $V_{\text{eff}}(\varphi)$ increases monotonically in $0 \le \varphi \le \pi$; so for the rod to be able to move to $\varphi = \pi$ the initial angular velocity ω_0 must satisfy

$$\frac{1}{2}\frac{1}{3}ml^2\omega_0^2 > V_{\text{eff}}(\varphi = \pi) = mgl \tag{27}$$

Solving,

$$\omega_0 > \sqrt{\frac{6g}{l}} = \omega_{\text{c}} \tag{28}$$

(ii) When $\omega^2 > \dfrac{4gl}{3A^2}$: $\varphi = 0$ and $\varphi = \pi$ are minima of $V_{\text{eff}}(\varphi)$, while $\varphi = \pi - \arccos\dfrac{4gl}{3\omega^2A^2}$ is a maximum of $V_{\text{eff}}(\varphi)$; this is the potential barrier within $0 \le \varphi \le \pi$, so for the rod to be able to move to $\varphi = \pi$ it must cross this barrier, and the initial angular velocity ω_0 must satisfy

$$\frac{1}{2}\frac{1}{3}ml^2\omega_0^2 > V_{\text{eff}}\left(\varphi = \pi - \arccos\frac{4gl}{3\omega^2A^2}\right) = \frac{1}{2}mgl + \frac{3}{16}m\omega^2A^2 + \frac{mg^2l^2}{3\omega^2A^2} \tag{29}$$

Solving,

$$\omega_0 > \sqrt{\frac{3g}{l} + \frac{9\omega^2A^2}{8l^2} + \frac{2g^2}{\omega^2A^2}} = \omega_{\text{c}} \tag{30}$$

In summary, the critical angular velocity is

$$\omega_{\text{c}} = \begin{cases} \sqrt{\dfrac{6g}{l}} & \omega^2 < \dfrac{4gl}{3A^2} \\ \sqrt{\dfrac{3g}{l} + \dfrac{9\omega^2A^2}{8l^2} + \dfrac{2g^2}{\omega^2A^2}} & \omega^2 > \dfrac{4gl}{3A^2} \end{cases}$$

When $\omega_0 < \omega_{\text{c}}$, the maximum angle $\varphi_{\max}$ that the rod can reach satisfies

$$\frac{1}{2}\frac{1}{3}ml^2\omega_0^2 = V_{\text{eff}}(\varphi_{\max}) = \frac{1}{2}mgl(1 - \cos\varphi_{\max}) + \frac{3}{16}m\omega^2A^2(1 - \cos^2\varphi_{\max}) \tag{31}$$

that is,

$$\cos^2\varphi_{\max} + \frac{8gl}{3\omega^2A^2}\cos\varphi_{\max} + \frac{8\omega_0^2l^2}{9\omega^2A^2} - \frac{8gl}{3\omega^2A^2} - 1 = 0 \tag{32}$$

Solving,

$$\cos\varphi_{\max} = \frac{1}{2}\left[-\frac{8gl}{3\omega^2A^2} \pm \sqrt{\left(\frac{8gl}{3\omega^2A^2}\right)^2 - 4\left(\frac{8\omega_0^2l^2}{9\omega^2A^2} - \frac{8gl}{3\omega^2A^2} - 1\right)}\right]$$

Note that here the smaller solution for φ should be taken, so the "+" sign should be taken:

$$\varphi_{\max} = \arccos\frac{1}{2}\left[-\frac{8gl}{3\omega^2A^2} + \sqrt{\left(\frac{8gl}{3\omega^2A^2}\right)^2 - 4\left(\frac{8\omega_0^2l^2}{9\omega^2A^2} - \frac{8gl}{3\omega^2A^2} - 1\right)}\right] \tag{33}$$

Marking scheme.
This problem carries 50 points.
Part (1) 3 points: (1) 1 point, (3) 2 points;
Part (2) 12 points: (4) 1 point; **Method 1** (5)(6)(9) 2 points each, (10) 1 point, (11) 4

points; **Method 2** $(7')(8')(9')$ 2 points each, $(10')$ 1 point, $(11')$ 4 points; **Method 3** $(5'')$ 6 points, $(6'')$ 1 point, $(8'')$ 4 points;
Part (3) 4 points: (12)(13) 2 points each;
Part (4) 18 points: (14) 1 point, (16-1), (16-2), (16-3) 1 point each, (19) 2 points, (20) 1 point, (21)(22) 2 points each, (23-1), (23-2) 1 point each, (24)(25) 2 points each, (26) 1 point;
Part (5) 13 points: (27)(28) 1 point each, (29) 4 points, (30)(31) 2 points each, (33) 3 points.

Problem 3 (Rutherford scattering: impact parameter and cross-section)

Solution

(1) The reaction equation for the α decay of ${}^{210}_{84}\mathrm{Po}$ is

$$ {}^{210}_{84}\mathrm{Po} \longrightarrow {}^{206}_{82}\mathrm{Pb} + {}^{4}_{2}\mathrm{He} \tag{1} $$

Since the parent nucleus is at rest before the decay, let the velocities of the α particle and of the daughter nucleus ${}^{206}_{82}\mathrm{Pb}$ after the decay be v and V. Conservation of momentum gives

$$ 0 = M_{\mathrm{Pb}} V + m_\alpha v \tag{2} $$

where m and M are the masses of the α particle and of the ${}^{206}_{82}\mathrm{Pb}$ nucleus. The total kinetic energy of the final-state particles of the decay is

$$ E_{\mathrm{d}} = \frac{1}{2} M_{\mathrm{Pb}} V^2 + \frac{1}{2} m_\alpha v^2 \tag{3} $$

Solving (2) and (3),

$$ E_{\mathrm{d}} = E_\alpha \left(1 + \frac{m_\alpha}{M_{\mathrm{Pb}}}\right) \approx \frac{A}{A-4} E_\alpha \tag{4} $$

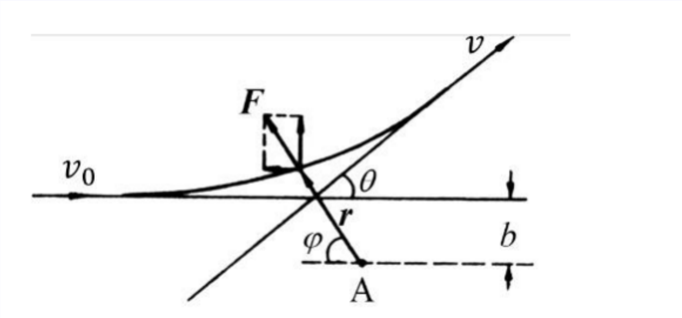
where A is the mass number of the Po nucleus and E_α is the kinetic energy of the α particle,

$$ E_\alpha = \frac{1}{2} m_\alpha v^2 = 5.31\,\mathrm{MeV} $$

From (4) and the data given in the problem,

$$ E_{\mathrm{d}} = 5.41\,\mathrm{MeV} \tag{5} $$

(2)
Method 1. As in solution figure 3a, the nucleus A, of mass M and positive charge Ze (e being the elementary charge), is at the origin, while the α particle, of mass m, energy E and charge $+2e$, is incident with velocity v_0 and impact parameter b; because of the Coulomb repulsion $\boldsymbol{F}$ of the nucleus, the α particle changes its direction and leaves with the deflection angle θ. During the scattering, since the mass of an electron is much smaller than the mass of the α particle, the interaction between the atomic electrons and the α particle is neglected; and since the mass of the target atom is usually much larger than that of the α particle, the nucleus is approximately taken to be at rest.



Solution figure 3a

The scattering of the α particle by the nucleus is an elastic process, so after the scattering the speed of the α particle at infinity is still v_0. Therefore the velocity of the α particle at infinity after the scattering, in the perpendicular direction, is

$$v_{\perp\infty} = v_0 \sin\theta \tag{6}$$

During the scattering, the force on the α particle in the perpendicular direction depends on the distance between the α particle and the nucleus; it is

$$F_{\perp} = F\sin\varphi = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{r^2}\sin\varphi \tag{7}$$

Since the change of momentum equals the impulse,

$$\mathrm{d}v_{\perp} = \frac{F_{\perp}}{m}\mathrm{d}t = \frac{2Ze^2}{4\pi\varepsilon_0}\frac{\sin\varphi}{mr^2}\mathrm{d}t \tag{8}$$

The Coulomb force is a central force, so the angular momentum is conserved during the scattering of the α particle:

$$L = mr^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} = mv_0 b \tag{9}$$

From (8) and (9),

$$\mathrm{d}v_{\perp} = \frac{2Ze^2}{4\pi\varepsilon_0 m v_0 b}\sin\varphi\,\mathrm{d}\varphi \tag{10}$$

When the α particle comes in, its velocity in the perpendicular direction is zero; considering its asymptotic behaviour at infinity, at incidence $\varphi = 0$, and when it has been scattered out to infinity $\varphi = \pi - \theta$. Therefore, integrating (10),

$$v_{\perp\infty} = \int_0^{\pi-\theta}\frac{2Ze^2}{4\pi\varepsilon_0 m v_0 b}\sin\varphi\,\mathrm{d}\varphi = \frac{2Ze^2}{4\pi\varepsilon_0 m v_0 b}(1+\cos\theta) \tag{11}$$

From (6) and (11),

$$b = \frac{D}{2}\cot\frac{\theta}{2} = \frac{Ze^2}{4\pi\varepsilon_0 E}\cot\frac{\theta}{2} \tag{12}$$

where $D = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{\frac{1}{2}mv_0^2} = \frac{2Ze^2}{4\pi\varepsilon_0 E}$.

Method 2. Conservation of energy and of angular momentum give

$$E = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(2e)}{r} + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) = \frac{1}{2}mv_0^2 \tag{6$'$}$$

$$L = mr^2\dot{\varphi} = mv_0 b \tag{7$'$}$$

where r is the distance from the α particle to the nucleus A (the origin). Also

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\dot{\varphi}$$

From (6′) and (7′),

$$\frac{2mE}{L^2} = \frac{4Ze^2m}{4\pi\varepsilon_0 L^2}\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 \tag{8′}$$

First one sees that taking the coefficient function $\frac{1}{r^4}$ of the derivative term on the right-hand side of (8′) inside the derivative, i.e.

$$\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \left(-\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r}\right)^2 = \left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r}\right)^2$$

suggests introducing the new variable

$$\rho = \frac{1}{r}$$

Equation (8′) then becomes

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\varphi^2} + \rho = C \tag{9′}$$

where $C = \frac{-2Ze^2}{4\pi\varepsilon_0 m v_0^2 b^2}$. Solving the differential equation (10′)[a] gives

$$\frac{1}{r} = C(1+\cos\varphi) + \frac{1}{b}\sin\varphi \tag{10′}$$

In the case in which the α particle is scattered through the angle θ, when the α particle goes out to infinity

$$\frac{1}{r} \to 0, \qquad \varphi = \pi - \theta$$

Substituting into (10′), one finds that the scattering angle θ satisfies

$$\cot\frac{\theta}{2} = -\frac{1}{Cb}, \tag{11′}$$

Putting $C = \frac{-2Ze^2}{4\pi\varepsilon_0 m v_0^2 b^2}$ into (11′),

$$b = \frac{Ze^2}{4\pi\varepsilon_0 E}\cot\frac{\theta}{2} \tag{12′}$$

Method 3. Take A in Fig. 3b as the pole and the direction of the pericentre as the direction of the polar axis, and set up polar coordinates (r, ϕ); then the orbit equation of the α particle is

$$r = \frac{-p}{1 - q\cos\phi} \tag{6″}$$

where

$$p = \left(\frac{2Ze^2}{4\pi\varepsilon_0}\right)^{-1}\frac{L^2}{m}$$

$$q = \sqrt{1 + \left(\frac{2Ze^2}{4\pi\varepsilon_0}\right)^{-2}\frac{2EL^2}{m}} \tag{7″}$$

and the orbital angular momentum is

$$L = mv_0 b = \sqrt{2mE}\, b \tag{8''}$$

Since $r > 0$, from (6$''$)

$$\cos\phi > \frac{1}{q} = \cos\phi_0$$

so $\phi \in (-\phi_0, \phi_0)$, and the corresponding scattering angle is

$$\theta = \pi - 2\phi_0 \tag{9''}$$

so

$$\cot\frac{\theta}{2} = \tan\phi_0 = \sqrt{q^2 - 1} \tag{10''}$$

Substituting (7$''$) into (10$''$),

$$\cot\frac{\theta}{2} = \frac{4\pi\varepsilon_0 E b}{Ze^2} \tag{11''}$$

that is,

$$b = \frac{Ze^2}{4\pi\varepsilon_0 E}\cot\frac{\theta}{2} \tag{12''}$$

Method 4 (using the Runge–Lenz vector). Let the velocity of the α particle be $\boldsymbol{v}$ and its angular momentum $\boldsymbol{L}$, and put the parameter $\beta = \dfrac{2Ze^2}{4\pi\varepsilon_0}$; then

$$\boldsymbol{F} = \frac{\beta}{r^2}\hat{\boldsymbol{r}} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} \tag{6'''}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{r}} = \frac{\boldsymbol{L}}{mr^2} \times \hat{\boldsymbol{r}} \tag{7'''}$$

Define the Runge–Lenz vector of the α particle as

$$\boldsymbol{B} = m\boldsymbol{v} \times \boldsymbol{L} + m\beta\hat{\boldsymbol{r}} \tag{8'''}$$

Using (6$'''$) and (7$'''$),

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} \times \boldsymbol{L} + m\beta\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \frac{\beta}{r^2}\hat{\boldsymbol{r}} \times \boldsymbol{L} + m\beta\frac{\boldsymbol{L}}{mr^2} \times \hat{\boldsymbol{r}} = 0 \tag{9'''}$$

so the Runge–Lenz vector is conserved.
Setting the positive directions of the x, y, z axes as in the figure, we have $\boldsymbol{L} = -L\hat{\boldsymbol{z}}$, and the initial velocity of the α particle is $\boldsymbol{v}_0 = v_0\hat{\boldsymbol{x}}$. Using (8$'''$) to compute the initial and the final Runge–Lenz vectors,

$$\boldsymbol{B}_{\text{initial}} = mv_0 L\hat{\boldsymbol{y}} - m\beta\hat{\boldsymbol{x}} \tag{10'''}$$

$$\boldsymbol{B}_{\text{final}} = mv_0 L\left(-\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}\right) + m\beta\left(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}}\right) \tag{11'''}$$

where

$$L = mv_0 b = \sqrt{2mE}\, b \tag{12'''}$$

Using the conservation of the Runge–Lenz vector,

$$mv_0 L = mv_0 L\cos\theta - m\beta\sin\theta$$

so

$$\cot\frac{\theta}{2} = \frac{\sin\theta}{1-\cos\theta} = \frac{v_0 L}{\beta} = \frac{4\pi\varepsilon_0 E b}{Ze^2} \tag{13'''}$$

that is,

$$b = \frac{Ze^2}{4\pi\varepsilon_0 E}\cot\frac{\theta}{2} \tag{14'''}$$

Method 5. During the scattering the α particle experiences the Coulomb repulsion of the nucleus, and its trajectory is a hyperbola. Let the semi-major axis be a and the distance from the α particle to the nucleus be c. Then

$$E = k\frac{2Ze^2}{2a} = \frac{Ze^2}{4\pi\varepsilon_0 a} \tag{6''''}$$

where $a = k\dfrac{2Ze^2}{2E} = \dfrac{Ze^2}{4\pi\varepsilon_0 E}$. The scattering angle θ satisfies

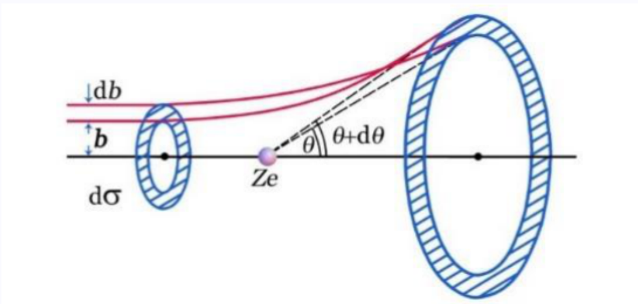
$$\tan\left(\frac{\pi-\theta}{2}\right) = \frac{b}{a} = \frac{Eb}{kZe^2} \tag{7''''}$$

$$\theta = 2\,\mathrm{arccot}\left(\frac{Eb}{kZe^2}\right) = 2\,\mathrm{arccot}\left(\frac{4\pi\varepsilon_0 Eb}{Ze^2}\right) \tag{8''''}$$

so

$$b = \frac{Ze^2}{4\pi\varepsilon_0 E}\cot\frac{\theta}{2} \tag{9''''}$$

(3) From (12) one sees that there is a one-to-one correspondence between b and θ. Consider the α particles whose impact parameter lies between b and $b+\mathrm{d}b$; after the scattering they leave at angles between θ and $\theta+\mathrm{d}\theta$. Therefore all the α particles that pass through the annular area $\mathrm{d}\sigma$ of inner radius b and outer radius $b+\mathrm{d}b$ must be scattered into a hollow cone whose angle lies between θ and $\theta+\mathrm{d}\theta$, as shown in solution figure 3b.



Solution figure 3b

The annular area $\mathrm{d}\sigma$ is

$$\mathrm{d}\sigma = 2\pi b\,|\mathrm{d}b| = \frac{\pi D^2}{4}\frac{\cos\dfrac{\theta}{2}}{\sin^3\dfrac{\theta}{2}}\,\mathrm{d}\theta \tag{13}$$

where (12) has been used in the derivation. The solid-angle element of the hollow cone whose angle lies between θ and $\theta+\mathrm{d}\theta$ is

$$\mathrm{d}\Omega = \frac{2\pi r\sin\theta\, r\,\mathrm{d}\theta}{r^2} = 2\pi\sin\theta\,\mathrm{d}\theta \tag{14}$$

The probability that one α particle is scattered by one nucleus into unit solid angle in the direction θ is called the differential scattering cross-section; it is

$$\sigma(\theta) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{D^2}{16}\frac{1}{\sin^4\dfrac{\theta}{2}} = \left(\frac{2Ze^2}{4\pi\varepsilon_0 E}\right)^2\frac{1}{16\sin^4\dfrac{\theta}{2}} \tag{15}$$

From (15) one sees that in a fixed scattering direction (when the scattering angle is $\theta = 60°$) the differential scattering cross-section is inversely proportional to the square of the kinetic energy E of the α particle, i.e. it decreases as E^2 increases.

(4) The measured α-particle scattering cross-section shows a kink at the α-particle energy 25 MeV, i.e. the cross-section suddenly drops sharply. This shows that the α particle approaches the region of the nucleus, that the strong nuclear force comes into play, and that this makes the relation between the scattering cross-section and the α-particle kinetic energy E change abruptly. (16)

Method 1. From (12) one also sees that, for a fixed angle θ, the larger E is the smaller b is. When b becomes small enough the α particle enters the nucleus and the scattering relation changes abruptly (the strong nuclear force comes into play). At the kink point the α particle just grazes the surface of the nucleus, and the closest distance r is approximately the radius of the nucleus. Let the closest distance of the α particle from the nucleus be $r_{\min}$, at which point its speed is v'; conservation of angular momentum and of energy give

$$mv_0 b = mv' r_{\min} \tag{17}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv'^2 + \frac{2Ze^2}{4\pi\varepsilon_0 r_{\min}} \tag{18}$$

$$r_{\min} = \frac{D}{2}\left(1 + \frac{1}{\sin\dfrac{\theta}{2}}\right) = \frac{Ze^2}{4\pi\varepsilon_0 E}\left(1 + \frac{1}{\sin\dfrac{\theta}{2}}\right) \tag{19}$$

From (19) and the data given in the problem ($E = 25\,\mathrm{MeV}$, $\theta = 60°$, $Z = 79$),

$$r_{\min} \approx 1.365 \times 10^{-14}\,\mathrm{m} = 13.65\,\mathrm{fm} \tag{20}$$

Method 2. During the scattering the α particle experiences the Coulomb repulsion of the nucleus, and its trajectory is a hyperbola. Let the semi-major axis be a and the distance from the α particle to the nucleus be c. Then

$$a = k\frac{2Ze^2}{2E} = \frac{Ze^2}{4\pi\varepsilon_0 E} \tag{17'}$$

$$c = k\frac{a}{\cos\left(\dfrac{\pi-\theta}{2}\right)} = 2a \tag{18'}$$

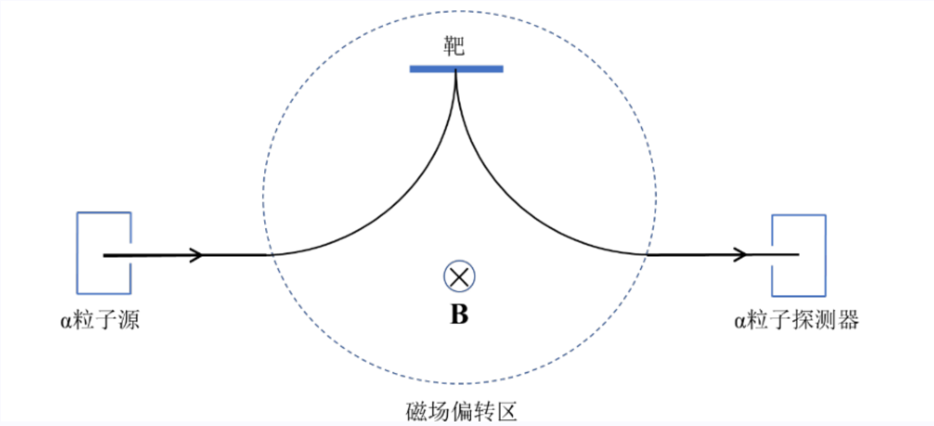
(or $b = a\tan\left(\dfrac{\pi-\theta}{2}\right) = \sqrt{3}\,a$, $c = \sqrt{b^2+a^2} = 2a$)

$$r_{\min} = c + a \tag{19'}$$

$$r_{\min} = 3a = 13.65\,\mathrm{fm} \tag{20'}$$

(5) In the usual Rutherford α-particle scattering experiment, because of the blocking by the α-particle source or by the detector, a measurement at the scattering angle 180° cannot be made. Since the α particle carries positive charge, a static magnetic field can be added on the flight path of the α particle in order to change its direction of motion. (If a student writes: add an alternating electric field, that is also counted as correct.) (21)

The experimental scheme is shown in solution figure 3c: a beam of α particles emitted from the α-particle source is deflected through 90° by a static magnetic field onto the target particles; the particles scattered at 180°, after leaving, are deflected through 90° by the static magnetic field and enter the α-particle detector, where they are measured. (22)



Solution figure 3c

Labels: α 粒子源 = α-particle source; 靶 = target; 磁场偏转区 = magnetic deflection region; α 粒子探测器 = α-particle detector.

Marking scheme.
This problem carries 40 points.
Part (1) 8 points: (1)(4) 3 points each, (5) 2 points;
Part (2) 12 points: **Method 1** (7)(8)(9)(10)(11)(12) 2 points each; **Method 2** (6′)(7′)(8′)(10′)(11′)(12′) 2 points each; **Method 3** (6″)(7″) 3 points each, (8″)(10″)(12″) 2 points each; **Method 4** (7‴)(9‴)(10‴)(11‴)(12‴)(14‴) 2 points each; **Method 5** (6⁗)(7⁗) 4 points each, (8⁗)(9⁗) 2 points each;
Part (3) 5 points: (13) 3 points, (15) 2 points;
Part (4) 10 points: **Method 1** (16)(17)(18)(19)(20) 2 points each; **Method 2** (16)(17′)(18′)(19′)(20′) 2 points each;
Part (5) 5 points: (21) 2 points, (22) 3 points.

[a] *Translator's note:* As printed; the differential equation being solved is (9′).

Problem 4 (Lennard-Jones crystal and the Einstein model)

Solution

(1) Since these atoms are all identical it is enough to consider the atom at the origin. The interaction potential energy of the atom at the origin with the other atoms is

$$\begin{aligned} V_0 &= \sum_{l_x,l_y,l_z} \varepsilon\left[\left(\frac{r_0}{R_{l_xl_yl_z}}\right)^{12} - \left(\frac{r_0}{R_{l_xl_yl_z}}\right)^{6}\right] \\ &= \varepsilon \sum_{l_x,l_y,l_z} \frac{r_0^{12}}{(l_x^2a^2+l_y^2a^2+l_z^2a^2)^6} - \varepsilon \sum_{l_x,l_y,l_z} \frac{r_0^{6}}{(l_x^2a^2+l_y^2a^2+l_z^2a^2)^3} \\ &= \varepsilon\left(\frac{r_0}{a}\right)^{12} \sum_{l_x,l_y,l_z} \frac{1}{(l_x^2+l_y^2+l_z^2)^6} - \varepsilon\left(\frac{r_0}{a}\right)^{6} \sum_{l_x,l_y,l_z} \frac{1}{(l_x^2+l_y^2+l_z^2)^3} \\ &= \varepsilon\left[\left(\frac{r_0}{a}\right)^{12} A_{12} - \left(\frac{r_0}{a}\right)^{6} A_6\right] \end{aligned} \tag{1}$$

After equilibrium is reached the potential energy is a minimum; therefore

$$0 = \frac{\mathrm{d}V_0}{\mathrm{d}a} = \varepsilon\left[-12A_{12}\frac{r_0^{12}}{a^{13}} + 6A_6\frac{r_0^6}{a^7}\right] \tag{2}$$

so that

$$-12A_{12}\frac{r_0^{12}}{a^{13}} + 6A_6\frac{r_0^6}{a^7} = 0$$

from which the lattice constant a is found to be

$$a = \left(\frac{2A_{12}}{A_6}\right)^{1/6} r_0 \tag{3}$$

(2) The atom at the origin has altogether 6 nearest neighbours, whose equilibrium positions are

$$(l_x, l_y, l_z) = (-1,0,0),\ (+1,0,0),\ (0,-1,0),\ (0,+1,0),\ (0,0,-1),\ (0,0,+1)$$

By the definition of A_n their contribution to the constant A_6 is

$$A_6' = 1+1+1+1+1+1 = 6 \tag{4}$$

Similarly there are 12 next-nearest-neighbour atoms, whose equilibrium positions are

$$\begin{aligned}(l_x, l_y, l_z) = {} & (-1,-1,0),\ (-1,+1,0),\ (+1,-1,0),\ (+1,+1,0),\\ & (-1,0,-1),\ (-1,0,+1),\ (+1,0,-1),\ (+1,0,+1),\\ & (0,-1,-1),\ (0,-1,+1),\ (0,+1,-1),\ (0,+1,+1)\end{aligned}$$

Their contribution to the constant A_6 is

$$A_6'' = 12 \times 2^{-3} = \frac{3}{2} \tag{5}$$

The atom at the origin has altogether 8 next-next-nearest-neighbour atoms, whose equilibrium positions are

$$\begin{aligned}(l_x, l_y, l_z) = {} & (-1,-1,-1),\ (-1,-1,+1),\ (-1,+1,-1),\ (-1,+1,+1),\ (+1,-1,-1),\\ & (+1,-1,+1),\ (+1,+1,-1),\ (+1,+1,+1)\end{aligned}$$

Their contribution to the constant A_6 is

$$A_6''' = 8 \times 3^{-3} = \frac{8}{27} \tag{6}$$

It can be seen that the other atoms, far from this atom, contribute little to the constant A_6.

(3) Since all the atoms are identical, we need only consider the atom whose equilibrium position is the origin. According to Einstein's assumption the other atoms remain at $\boldsymbol{R}_{l_x l_y l_z}$; therefore, when the atom at the origin is displaced from its equilibrium position by $\boldsymbol{u}$, its interaction potential energy with the other atoms is

$$V = \sum_{l_x, l_y, l_z} \varepsilon\left[\left(\frac{r_0}{|\boldsymbol{R}_{l_x l_y l_z} - \boldsymbol{u}|}\right)^{12} - \left(\frac{r_0}{|\boldsymbol{R}_{l_x l_y l_z} - \boldsymbol{u}|}\right)^{6}\right] \tag{7}$$

When u is small the above expression can be expanded in a series in the small quantity u. Using the following formula,

$$\begin{aligned}
|\boldsymbol{R}-\boldsymbol{u}|^{-n} &= \left|R^2-2\boldsymbol{R}\cdot\boldsymbol{u}+u^2\right|^{-\frac{n}{2}} = R^{-n}\left(1-2\frac{\boldsymbol{R}\cdot\boldsymbol{u}}{R^2}+\frac{u^2}{R^2}\right)^{-\frac{n}{2}} \\
&= R^{-n}\left[1-\frac{n}{2}\left(-2\frac{\boldsymbol{R}\cdot\boldsymbol{u}}{R^2}+\frac{u^2}{R^2}\right)+\frac{1}{2}\frac{-n}{2}\left(\frac{-n}{2}-1\right)\left(-2\frac{\boldsymbol{R}\cdot\boldsymbol{u}}{R^2}+\frac{u^2}{R^2}\right)^2+\cdots\right] \\
&= R^{-n}\left[1+n\frac{\boldsymbol{R}\cdot\boldsymbol{u}}{R^2}-\frac{n}{2}\frac{u^2}{R^2}+\frac{n(n+2)}{2}\left(\frac{\boldsymbol{R}\cdot\boldsymbol{u}}{R^2}\right)^2+\cdots\right]
\end{aligned} \tag{8}$$

and keeping terms up to u^2 we obtain

$$\begin{aligned}
\sum_{l_x,l_y,l_z}\left|\boldsymbol{R}_{l_xl_yl_z}-\boldsymbol{u}\right|^{-n} &= \sum R_{l_xl_yl_z}^{-n}+n\sum\frac{\boldsymbol{R}_{l_xl_yl_z}\cdot\boldsymbol{u}}{R_{l_xl_yl_z}^{n+2}} \\
&\quad -\frac{n}{2}\sum\frac{u^2}{R_{l_xl_yl_z}^{n+2}}+\frac{n(n+2)}{2}\sum\frac{(\boldsymbol{R}_{l_xl_yl_z}\cdot\boldsymbol{u})^2}{R_{l_xl_yl_z}^{n+4}}
\end{aligned} \tag{9}$$

Using the symmetry that (l_x,l_y,l_z) and $(-l_x,-l_y,-l_z)$ occur in pairs,

$$\sum\frac{l_x}{(l_x^2+l_y^2+l_z^2)^{n/2}}=\sum\frac{l_y}{(l_x^2+l_y^2+l_z^2)^{n/2}}=\sum\frac{l_z}{(l_x^2+l_y^2+l_z^2)^{n/2}}=0 \tag{10}$$

$$\sum\frac{l_xl_y}{(l_x^2+l_y^2+l_z^2)^{n/2}}=\sum\frac{l_yl_z}{(l_x^2+l_y^2+l_z^2)^{n/2}}=\sum\frac{l_zl_x}{(l_x^2+l_y^2+l_z^2)^{n/2}}=0 \tag{11}$$

Similarly, using the symmetry $l_x\leftrightarrow l_y\leftrightarrow l_z$,

$$\begin{aligned}
\sum\frac{l_x^2}{(l_x^2+l_y^2+l_z^2)^{n/2}} &= \sum\frac{l_y^2}{(l_x^2+l_y^2+l_z^2)^{n/2}}=\sum\frac{l_z^2}{(l_x^2+l_y^2+l_z^2)^{n/2}} \\
&= \frac{1}{3}\sum\frac{l_x^2+l_y^2+l_z^2}{(l_x^2+l_y^2+l_z^2)^{n/2}} \\
&= \frac{1}{3}\sum(l_x^2+l_y^2+l_z^2)^{-\frac{n-2}{2}}=\frac{1}{3}A_{n-2}
\end{aligned} \tag{12}$$

Hence

$$\sum\frac{\boldsymbol{R}_{l_xl_yl_z}\cdot\boldsymbol{u}}{R_{l_xl_yl_z}^{n+2}}=\frac{1}{a^{n+1}}\sum\frac{u_xl_x+u_yl_y+u_zl_z}{(l_x^2+l_y^2+l_z^2)^{(n+2)/2}}=0 \tag{13}$$

$$\begin{aligned}
\sum\frac{(\boldsymbol{R}_{l_xl_yl_z}\cdot\boldsymbol{u})^2}{R_{l_xl_yl_z}^{n+4}} &= \frac{1}{a^{n+2}}\sum\frac{u_x^2l_x^2+u_y^2l_y^2+u_z^2l_z^2+2u_xu_yl_xl_y+2u_yu_zl_yl_z+2u_zu_xl_zl_x}{(l_x^2+l_y^2+l_z^2)^{(n+4)/2}} \\
&= \frac{1}{a^{n+2}}\sum\frac{u_x^2l_x^2+u_y^2l_y^2+u_z^2l_z^2}{(l_x^2+l_y^2+l_z^2)^{(n+4)/2}} \\
&= \frac{u^2}{3a^{n+2}}\sum(l_x^2+l_y^2+l_z^2)^{-\frac{n+2}{2}}=\frac{u^2}{3a^{n+2}}A_{n+2}
\end{aligned} \tag{14}$$

Using these results we obtain

$$\begin{aligned}
\sum_{l_x,l_y,l_z}\left|\boldsymbol{R}_{l_xl_yl_z}-\boldsymbol{u}\right|^{-n} &= A_na^{-n}-\frac{n}{2}A_{n+2}a^{-(n+2)}u^2+\frac{n(n+2)}{6}A_{n+2}a^{-(n+2)}u^2 \\
&= A_na^{-n}+\frac{n(n-1)}{6}A_{n+2}a^{-(n+2)}u^2
\end{aligned} \tag{15}$$

So the energy of the system when the atom at the origin is displaced from its equilibrium position by $\boldsymbol{u}$ is

$$\begin{aligned} V &= \varepsilon \sum_{l_x,l_y,l_z} \left(r_0^{12} \left| \boldsymbol{R}_{l_x l_y l_z} - \boldsymbol{u} \right|^{-12} - r_0^6 \left| \boldsymbol{R}_{l_x l_y l_z} - \boldsymbol{u} \right|^{-6} \right) \\ &= \varepsilon r_0^{12} \left(A_{12} a^{-12} + \frac{12 \times 11}{6} A_{14} a^{-14} u^2 \right) - \varepsilon r_0^6 \left(A_6 a^{-6} + \frac{6 \times 5}{6} A_8 a^{-8} u^2 \right) \\ &= \varepsilon \left[A_{12} \left(\frac{r_0}{a} \right)^{12} - A_6 \left(\frac{r_0}{a} \right)^6 \right] + \varepsilon \left(22 A_{14} \frac{r_0^{12}}{a^{14}} - 5 A_8 \frac{r_0^6}{a^8} \right) u^2 \\ &= V_0 + \frac{1}{2} M \omega_{\mathrm{E}}^2 u^2 \end{aligned} \tag{16}$$

Here $V_0 = \varepsilon \left[A_{12} \left(\frac{r_0}{a} \right)^{12} - A_6 \left(\frac{r_0}{a} \right)^6 \right]$, from which the Einstein frequency is obtained as

$$\omega_{\mathrm{E}} = \sqrt{\frac{2\varepsilon}{M} \left(22 A_{14} \frac{r_0^{12}}{a^{14}} - 5 A_8 \frac{r_0^6}{a^8} \right)} \tag{17}$$

(4) The energy E of an atom whose momentum has magnitude p and whose displacement has magnitude u is

$$E(p,u) = V_0 + \frac{p^2}{2M} + \frac{1}{2} M \omega_{\mathrm{E}}^2 u^2 \tag{18}$$

The average energy of each atom is

$$\begin{aligned} \bar{E} &= \frac{\int_0^{+\infty} \int_0^{+\infty} E \, \mathrm{e}^{-\frac{E}{k_{\mathrm{B}} T}} p^2 \mathrm{d}p \, u^2 \mathrm{d}u}{\int_0^{+\infty} \int_0^{+\infty} \mathrm{e}^{-\frac{E}{k_{\mathrm{B}} T}} p^2 \mathrm{d}p \, u^2 \mathrm{d}u} \\ &= V_0 + \frac{1}{2M} \frac{\int_0^{+\infty} \mathrm{e}^{-\frac{p^2}{2M k_{\mathrm{B}} T}} p^4 \mathrm{d}p}{\int_0^{+\infty} \mathrm{e}^{-\frac{p^2}{2M k_{\mathrm{B}} T}} p^2 \mathrm{d}p} + \frac{1}{2} M \omega_{\mathrm{E}}^2 \frac{\int_0^{+\infty} \mathrm{e}^{-\frac{M \omega_{\mathrm{E}}^2 u^2}{2 k_{\mathrm{B}} T}} u^4 \mathrm{d}u}{\int_0^{+\infty} \mathrm{e}^{-\frac{M \omega_{\mathrm{E}}^2 u^2}{2 k_{\mathrm{B}} T}} u^2 \mathrm{d}u} \\ &= V_0 + 2 k_{\mathrm{B}} T \frac{\int_0^{+\infty} \mathrm{e}^{-x^2} x^4 \mathrm{d}x}{\int_0^{+\infty} \mathrm{e}^{-x^2} x^2 \mathrm{d}x} \end{aligned} \tag{19}$$

Since

$$\frac{\mathrm{d}\, \mathrm{e}^{-x^2}}{\mathrm{d}x} = -2x \, \mathrm{e}^{-x^2}$$

integration by parts gives

$$\int_0^{\infty} \mathrm{e}^{-x^2} x^2 \mathrm{d}x = -\frac{1}{2} \int_0^{\infty} x \, \mathrm{d}\, \mathrm{e}^{-x^2} = \frac{1}{2} \int_0^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{4} \tag{20}$$

$$\int_0^{\infty} \mathrm{e}^{-x^2} x^4 \mathrm{d}x = -\frac{1}{2} \int_0^{\infty} x^3 \, \mathrm{d}\, \mathrm{e}^{-x^2} = \frac{3}{2} \int_0^{\infty} x^2 \mathrm{e}^{-x^2} \mathrm{d}x = \frac{3}{2} \frac{\sqrt{\pi}}{4} = \frac{3\sqrt{\pi}}{8} \tag{21}$$

Substituting these two integrals into the expression for the average energy, the average energy of each atom is

$$\bar{E} = V_0 + 3 k_{\mathrm{B}} T \tag{22}$$

One mole of the crystal contains N_A (Avogadro's number) atoms. Therefore, in the Einstein model, the internal energy of one mole of the crystal is

$$U = N_\mathrm{A}\bar{E} = N_\mathrm{A}V_0 + 3N_\mathrm{A}k_\mathrm{B}T \tag{23}$$

so that the molar heat capacity at constant volume is

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3N_\mathrm{A}k_\mathrm{B} \tag{24}$$

(5) The average value of the distance of deviation of an atom from its equilibrium position is

$$\bar{u} = \frac{\displaystyle\int_0^{+\infty}\int_0^{+\infty} u\,\mathrm{e}^{-\frac{E}{k_\mathrm{B}T}}p^2\mathrm{d}p\,u^2\mathrm{d}u}{\displaystyle\int_0^{+\infty}\int_0^{+\infty} \mathrm{e}^{-\frac{E}{k_\mathrm{B}T}}p^2\mathrm{d}p\,u^2\mathrm{d}u} = \frac{\displaystyle\int_0^{+\infty}\mathrm{e}^{-\frac{M\omega_\mathrm{E}^2u^2}{2k_\mathrm{B}T}}u^3\mathrm{d}u}{\displaystyle\int_0^{+\infty}\mathrm{e}^{-\frac{M\omega_\mathrm{E}^2u^2}{2k_\mathrm{B}T}}u^2\mathrm{d}u} = \sqrt{\frac{2k_\mathrm{B}T}{M\omega_\mathrm{E}^2}}\frac{\displaystyle\int_0^{+\infty}\mathrm{e}^{-x^2}x^3\mathrm{d}x}{\displaystyle\int_0^{+\infty}\mathrm{e}^{-x^2}x^2\mathrm{d}x} \tag{25}$$

Using integration by parts and the change of variable $x^2 = t$,

$$\int_0^{+\infty}\mathrm{e}^{-x^2}x^3\mathrm{d}x = -\frac{1}{2}\int_0^{+\infty}x^2\,\mathrm{d}\,\mathrm{e}^{-x^2} = -\frac{1}{2}\int_0^{+\infty}t\,\mathrm{d}\,\mathrm{e}^{-t} = \frac{1}{2}\int_0^{+\infty}\mathrm{e}^{-t}\mathrm{d}t = \frac{1}{2} \tag{26}$$

we thus obtain

$$\bar{u} = \sqrt{\frac{2k_\mathrm{B}T}{M\omega_\mathrm{E}^2}}\frac{\frac{1}{2}}{\frac{\sqrt{\pi}}{4}} = 2\sqrt{\frac{2k_\mathrm{B}T}{\pi M\omega_\mathrm{E}^2}} \tag{27}$$

When the temperature equals the melting point, $T = T_\mathrm{M}$, the Lindemann criterion gives

$$\bar{u}_\mathrm{M} = C_\mathrm{L}a$$

that is,

$$\sqrt{\frac{8k_\mathrm{B}T_\mathrm{M}}{\pi M\omega_\mathrm{E}^2}} = C_\mathrm{L}a$$

Therefore

$$T_\mathrm{M} = \frac{\pi Ma^2\omega_\mathrm{E}^2C_\mathrm{L}^2}{8k_\mathrm{B}} \tag{28}$$

Marking scheme.
This problem carries 50 points.
Part (1) 8 points: (1) 4 points, (2)(3) 2 points each;
Part (2) 6 points: (4)(5)(6) 2 points each;
Part (3) 17 points: (7) 4 points, (8) 3 points, (9) 2 points, (15) 2 points, (16) 4 points, (17) 2 points;
Part (4) 11 points: (18) 2 points, (19) 5 points, (22)(23) 1 point each, (24) 2 points;
Part (5) 8 points: (25) 3 points, (27) 2 points, (28) 3 points.

Problem 5 (black-hole thermodynamics and Hawking radiation)

Solution

(1) Taking the potential energy to be zero at infinity, the gravitational potential energy between the celestial body and a test particle of mass m is

$$V(d) = -\frac{GMm}{d} \tag{1}$$

where d is the distance from the test particle to the centre of the black hole. The radius of the body is r; conservation of mechanical energy gives for the escape velocity v_e the equation

$$\frac{1}{2}mv_e^2 - \frac{GMm}{r} = 0 \tag{2}$$

so the magnitude of the escape velocity is

$$v_e = \sqrt{\frac{2GM}{r}} \tag{3}$$

When the escape velocity is $v_e = c$, the radius of the body satisfies

$$r = \frac{2GM}{c^2} \tag{4}$$

To make the Sun a black hole, substituting the mass of the Sun $M = 2 \times 10^{30}$ kg into (4), the radius of the Sun must shrink at least to

$$r = \frac{2 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{9 \times 10^{16}} \approx 2964\,\mathrm{m} \approx 3\,\mathrm{km} \tag{5}$$

(2) From the statement of the problem the relation between the entropy of the black hole and its mass is

$$S = \frac{k_B c^3}{4G\hbar}A = \frac{k_B c^3}{4G\hbar}4\pi r^2 = \frac{\pi k_B c^3}{G\hbar}\left(\frac{2GM}{c^2}\right)^2 = \frac{4\pi k_B G M^2}{c\hbar} \tag{6}$$

so that when two spherical black holes each of mass M collapse into one black hole of mass M', the change of the entropy of the system is

$$\Delta S = S_2 - S_1 = \frac{4\pi k_B G}{c\hbar}\left[M'^2 - 2M^2\right] \tag{7}$$

(3) Using the thermodynamic formula

$$\frac{1}{T} = \frac{\mathrm{d}S}{\mathrm{d}U}$$

together with Einstein's relation

$$U = Mc^2$$

we obtain

$$\frac{1}{T} = \frac{\mathrm{d}S}{\mathrm{d}U} = \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}M}\left(\frac{4\pi k_B G M^2}{c\hbar}\right) = \frac{8\pi k_B G M}{c^3\hbar} \tag{8}$$

so the temperature of the black hole is

$$T = \frac{c^3\hbar}{8\pi k_B G M} \tag{9}$$

One sees that the temperature of the black hole is inversely proportional to its mass: the smaller the mass, the higher the temperature. From the definition of heat capacity, the heat capacity of the black hole is

$$C = T\frac{\mathrm{d}S}{\mathrm{d}T} = T\frac{\mathrm{d}S}{\mathrm{d}M}\frac{\mathrm{d}M}{\mathrm{d}T} = T\frac{8\pi k_\mathrm{B}GM}{c\hbar}\frac{-c^3\hbar}{8\pi k_\mathrm{B}GT^2} = -\frac{c^2M}{T} = -\frac{8\pi k_\mathrm{B}GM^2}{c\hbar} \tag{10}$$

One sees that the heat capacity of a black hole is always negative, which means that a black hole is thermodynamically unstable.

(4) Method 1. During the working of the heat engine let the masses of the two black holes be M_1 and M_2. From the statement of the problem the engine cycle is a reversible process, so after a number of complete cycles the entropy of the system does not change. From (6), the constancy of the entropy gives

$$M_1^2 + M_2^2 = M_{10}^2 + M_{20}^2 \tag{11}$$

Since $M_{20} > M_{10}$, (9) shows that the black hole of small mass is the high-temperature reservoir and the black hole of large mass is the low-temperature reservoir. During the engine cycle the engine absorbs heat from the high-temperature reservoir (the small black hole) and gives out heat to the low-temperature reservoir (the large black hole), so the mass of the small black hole keeps decreasing until it finally disappears and the engine cycle stops, i.e.

$$M_1 = 0 \tag{12}$$

Substituting into the previous equation,

$$M_2 = \sqrt{M_{10}^2 + M_{20}^2} \tag{13}$$

From energy conservation, the total work finally done by the engine on the outside is

$$W = \left(M_{10} + M_{20} - \sqrt{M_{10}^2 + M_{20}^2}\right)c^2 \tag{14}$$

Method 2. The black hole of small mass is the high-temperature reservoir and the black hole of large mass is the low-temperature reservoir. The engine absorbs heat $\mathrm{d}Q_1$ from the high-temperature reservoir (the small black hole) and gives out heat $\mathrm{d}Q_2$ to the low-temperature reservoir (the large black hole), so the mass of the small black hole keeps decreasing and finally disappears. Carnot's theorem gives

$$\frac{\mathrm{d}Q_2}{\mathrm{d}Q_1} = \frac{T_2}{T_1} \tag{15}$$

From Einstein's mass–energy relation and the first law of thermodynamics,

$$\mathrm{d}Q_1 = -\mathrm{d}(M_1c^2), \qquad \mathrm{d}Q_2 = \mathrm{d}(M_2c^2) \tag{16}$$

Furthermore the black-hole temperature is inversely proportional to its mass,

$$T \propto \frac{1}{M},$$

so from the two relations above,

$$\frac{\mathrm{d}M_2}{\mathrm{d}M_1} = -\frac{T_2}{T_1} = -\frac{M_1}{M_2} \tag{17}$$

Therefore

$$M_1 \mathrm{d}M_1 + M_2 \mathrm{d}M_2 = 0 \tag{18}$$

Integrating both sides of the above equation,

$$M_1^2 + M_2^2 = M_{10}^2 + M_{20}^2 \tag{19}$$

So during the engine cycle the mass of the large black hole keeps increasing and satisfies

$$M_2 = \sqrt{M_{10}^2 + M_{20}^2 - M_1^2} \tag{20}$$

Finally the mass of the small black hole is used up and the engine cycle stops, i.e. the final state is

$$M_1 = 0 \tag{21}$$

The efficiency of a reversible heat engine depends only on the temperatures of the high- and the low-temperature reservoirs,

$$\eta = \frac{\mathrm{d}W}{\mathrm{d}Q_1} = 1 - \frac{T_2}{T_1} \tag{22}$$

so the total work finally done by the engine on the outside is

$$\begin{aligned} W &= \int \left(1 - \frac{T_2}{T_1}\right) \mathrm{d}Q_1 = -\int_{M_{10}}^{0} \left(1 - \frac{M_1}{M_2}\right) c^2 \mathrm{d}M_1 \\ &= -\int_{M_{10}}^{0} \left(1 - \frac{M_1}{\sqrt{M_{10}^2 + M_{20}^2 - M_1^2}}\right) c^2 \mathrm{d}M_1 \\ &= \left(M_{10} + M_{20} - \sqrt{M_{10}^2 + M_{20}^2}\right) c^2 \end{aligned} \tag{23}$$

(5) By the Stefan–Boltzmann law, the energy radiated outwards by the black hole per unit time (i.e. the Hawking radiation power) is

$$\begin{aligned} P &= \sigma T^4 A = 4\pi\sigma r^2 T^4 = 4\pi\sigma \left(\frac{2GM}{c^2}\right)^2 \left(\frac{c^3\hbar}{8\pi k_\mathrm{B} GM}\right)^4 \\ &= \sigma \frac{c^8\hbar^4}{256\pi^3 k_\mathrm{B}^4 G^2 M^2} = \frac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2} \frac{c^8\hbar^4}{256\pi^3 k_\mathrm{B}^4 G^2 M^2} \\ &= \frac{c^6\hbar}{15360\pi G^2 M^2} \end{aligned} \tag{24}$$

(6) Because of Hawking radiation, the decrease of the black-hole energy equals the radiated energy, i.e.

$$-c^2 \frac{\mathrm{d}M}{\mathrm{d}t} = P = \frac{c^6\hbar}{15360\pi G^2 M^2} \tag{25}$$

so

$$-M^2 \mathrm{d}M = \frac{c^4\hbar}{15360\pi G^2} \mathrm{d}t \tag{26}$$

Integrating both sides,

$$-\int_{M_0}^{0} M^2 \mathrm{d}M = \int_0^{t_f} \frac{c^4\hbar}{15360\pi G^2} \mathrm{d}t \tag{27}$$

so the time needed for the black hole to disappear through Hawking radiation is

$$t_f = \frac{5120\pi G^2 M_0^3}{c^4 \hbar} \tag{28}$$

Marking scheme.
This problem carries 40 points.
Part (1) 5 points: (1)(2)(3)(4)(5) 1 point each;
Part (2) 4 points: (6) 3 points, (7) 1 point;
Part (3) 8 points: (8) 3 points, (9) 1 point, (10) 4 points;
Part (4) 10 points: **Method 1** (11) 3 points, (12)(13) 2 points each, (14) 3 points; **Method 2** (15)(16)(17)(18)(19)(20)(21)(22) 1 point each, (23) 2 points;
Part (5) 5 points: (24) 5 points;
Part (6) 8 points: (25) 3 points, (26) 1 point, (27)(28) 2 points each.

Problem 6 (plasmon resonance of a metal nanosphere)

Solution

(1) A uniform dielectric sphere in a uniform external field is uniformly polarized. Let the polarization be $\boldsymbol{P}$; the surface density of the polarization charge on the dielectric sphere is distributed as

$$\sigma'(\theta) = \boldsymbol{P} \cdot \boldsymbol{n} = P\cos\theta \tag{1}$$

This is a cosine distribution of charge on a spherical surface, and the depolarizing field produced inside the sphere by the polarization charge is uniform:

$$\boldsymbol{E}' = -\frac{\boldsymbol{P}}{3\varepsilon_0} \tag{2}$$

The total electric field strength inside the sphere is

$$\boldsymbol{E}_{\text{in}} = \boldsymbol{E}_0 + \boldsymbol{E}' \tag{3}$$

The polarization law of the dielectric is

$$\boldsymbol{P} = (\varepsilon_{\text{r}} - 1)\varepsilon_0 \boldsymbol{E}_{\text{in}} \tag{4}$$

Solving (2), (3) and (4) simultaneously,

$$\boldsymbol{P} = \frac{\varepsilon_{\text{r}} - 1}{\varepsilon_{\text{r}} + 2} 3\varepsilon_0 \boldsymbol{E}_0 \tag{5}$$

Substituting (5) and (2) into (3),

$$\boldsymbol{E}_{\text{in}} = \frac{3}{\varepsilon_{\text{r}} + 2} \boldsymbol{E}_0 \tag{6}$$

The electric dipole moment of the dielectric sphere is

$$\boldsymbol{p} = \boldsymbol{P} \cdot \frac{4}{3}\pi R^3 = \frac{\varepsilon_{\text{r}} - 1}{\varepsilon_{\text{r}} + 2} 4\pi\varepsilon_0 R^3 \boldsymbol{E}_0 \tag{7}$$

(2)(i) The dynamical equation of an electron in the constant electric field $\boldsymbol{E}$ is

$$-e\boldsymbol{E} - \gamma m \boldsymbol{v} = m\dot{\boldsymbol{v}} \tag{8}$$

In the steady state $\dot{\boldsymbol{v}} = 0$; substituting into (8), the drift velocity is

$$\boldsymbol{v} = -\frac{e}{\gamma m}\boldsymbol{E} \tag{9}$$

The current density inside the conductor is

$$\boldsymbol{j} = n(-e)\boldsymbol{v} = \frac{ne^2}{\gamma m}\boldsymbol{E} = \sigma_0\boldsymbol{E} \tag{10}$$

so the conductivity of the conductor is

$$\sigma_0 = \frac{ne^2}{\gamma m} \tag{11}$$

(ii) The dynamical equation of an electron in the alternating electric field $\boldsymbol{E}(t) = \boldsymbol{E}_\mathrm{m}\cos\omega t$ is

$$-e\boldsymbol{E}_\mathrm{m}\cos\omega t - \gamma m\boldsymbol{v} = m\dot{\boldsymbol{v}} \tag{12}$$

which can be written in complex form as

$$-e\boldsymbol{E}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t} - \gamma m\tilde{\boldsymbol{v}} = m\dot{\tilde{\boldsymbol{v}}} \tag{13}$$

In the steady state, let the steady solution for the electron velocity be

$$\tilde{\boldsymbol{v}}(t) = \tilde{\boldsymbol{v}}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t} \tag{14}$$

Then $\dot{\tilde{\boldsymbol{v}}} = \mathrm{i}\omega\tilde{\boldsymbol{v}}$; substituting into (13),

$$-e\boldsymbol{E}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t} - \gamma m\tilde{\boldsymbol{v}} = \mathrm{i}\omega m\tilde{\boldsymbol{v}} \tag{15}$$

Solving,

$$\tilde{\boldsymbol{v}} = \frac{-e/m}{\gamma + \mathrm{i}\omega}\boldsymbol{E}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t} = \frac{-e/m}{\gamma + \mathrm{i}\omega}\tilde{\boldsymbol{E}} \tag{16}$$

The current density is

$$\tilde{\boldsymbol{j}} = n(-e)\tilde{\boldsymbol{v}} = \frac{ne^2/m}{\gamma + \mathrm{i}\omega}\tilde{\boldsymbol{E}} = \tilde{\sigma}\tilde{\boldsymbol{E}} \tag{17}$$

The complex conductivity is

$$\tilde{\sigma} = \frac{ne^2/m}{\gamma + \mathrm{i}\omega} = \frac{\varepsilon_0\omega_\mathrm{p}^2}{\gamma + \mathrm{i}\omega} = \frac{\varepsilon_0\omega_\mathrm{p}^2\gamma}{\gamma^2 + \omega^2} - \mathrm{i}\,\frac{\varepsilon_0\omega_\mathrm{p}^2\omega}{\gamma^2 + \omega^2} \tag{18}$$

[Note: when $\omega = 0$, $\tilde{\sigma} = \dfrac{ne^2}{m\gamma} = \sigma_0$ reduces to the static result.]

(iii) From the velocity $\tilde{\boldsymbol{v}}$ of the electron motion one obtains the position

$$\tilde{\boldsymbol{x}} = \frac{\tilde{\boldsymbol{v}}}{\mathrm{i}\omega} = \frac{e/m}{\omega^2 - \mathrm{i}\gamma\omega}\boldsymbol{E}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t} = \frac{e/m}{\omega^2 - \mathrm{i}\gamma\omega}\tilde{\boldsymbol{E}} \tag{19}$$

[Or: the dynamical equation of an electron in the alternating field $\boldsymbol{E}(t) = \boldsymbol{E}_\mathrm{m}\cos\omega t$ is

$$-e\boldsymbol{E}_\mathrm{m}\cos\omega t - \gamma m\dot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}}$$

which can be written in complex form as

$$-e\boldsymbol{E}_\mathrm{m}\mathrm{e}^{\mathrm{i}\omega t} - \gamma m\dot{\tilde{\boldsymbol{x}}} = m\ddot{\tilde{\boldsymbol{x}}}$$

In the steady state, let the steady solution for the electron position be $\tilde{\boldsymbol{x}}(t)=\tilde{\boldsymbol{x}}_{\mathrm{m}}\mathrm{e}^{\mathrm{i}\omega t}$; then $\dot{\tilde{\boldsymbol{x}}}=\mathrm{i}\omega\tilde{\boldsymbol{x}}$, $\ddot{\tilde{\boldsymbol{x}}}=-\omega^2\tilde{\boldsymbol{x}}$, and substituting,

$$-e\boldsymbol{E}_{\mathrm{m}}\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{i}\omega\gamma m\tilde{\boldsymbol{x}}=-m\omega^2\tilde{\boldsymbol{x}}$$

Solving,

$$\tilde{\boldsymbol{x}}=\frac{e/m}{\omega^2-\mathrm{i}\gamma\omega}\boldsymbol{E}_{\mathrm{m}}\mathrm{e}^{\mathrm{i}\omega t}=\frac{e/m}{\omega^2-\mathrm{i}\gamma\omega}\tilde{\boldsymbol{E}}$$

] The polarization is

$$\tilde{\boldsymbol{P}}=n(-e)\tilde{\boldsymbol{x}}=-\frac{ne^2/m}{\omega^2-\mathrm{i}\gamma\omega}\tilde{\boldsymbol{E}}=\tilde{\chi}\varepsilon_0\tilde{\boldsymbol{E}} \tag{20}$$

The complex susceptibility is

$$\tilde{\chi}=-\frac{ne^2/m\varepsilon_0}{\omega^2-\mathrm{i}\gamma\omega}=-\frac{\omega_{\mathrm{p}}^2}{\omega^2-\mathrm{i}\gamma\omega}=-\frac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}-\mathrm{i}\,\frac{\gamma}{\omega}\frac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2} \tag{21}$$

The complex relative permittivity is

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega)=1+\tilde{\chi}=1-\frac{\omega_{\mathrm{p}}^2}{\omega^2-\mathrm{i}\gamma\omega}=\left(1-\frac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}\right)+\mathrm{i}\left(-\frac{\gamma}{\omega}\frac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}\right) \tag{22}$$

[Note: the complex conductivity $\tilde{\sigma}$ and the complex relative permittivity $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$ satisfy $\tilde{\varepsilon}_{\mathrm{r}}(\omega)=1+\dfrac{\tilde{\sigma}}{\mathrm{i}\omega\varepsilon_0}$.]

(3)(i) In complex form, the polarization of the metal sphere in the alternating electric field can be treated in analogy with the polarization of the dielectric sphere in the electrostatic field; from (6) the complex amplitude of the field inside the sphere is

$$\tilde{E}_{\mathrm{in}}(\omega)=\frac{3}{2+\tilde{\varepsilon}_{\mathrm{r}}(\omega)}E_0 \tag{23}$$

Substituting (22), the magnitude of the field amplitude is

$$E_{\mathrm{in}}(\omega)=\left|\tilde{E}_{\mathrm{in}}(\omega)\right|=\frac{3}{|2+\tilde{\varepsilon}_{\mathrm{r}}(\omega)|}E_0=\frac{3}{\sqrt{\left(3-\dfrac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}\right)^2+\left(\dfrac{\gamma}{\omega}\dfrac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}\right)^2}}E_0 \tag{24}$$

Near the resonance region take $\gamma\approx 0$:

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega)=1-\frac{\omega_{\mathrm{p}}^2}{\omega^2} \tag{25}$$

The field amplitude is then

$$E_{\mathrm{in}}(\omega)\approx\frac{3}{3-\omega_{\mathrm{p}}^2/\omega^2}E_0 \tag{26}$$

When $3-\dfrac{\omega_{\mathrm{p}}^2}{\omega^2}\to 0$, E_{in} is a maximum: resonance occurs, with resonance angular frequency

$$\omega_{\mathrm{r}}=\frac{\omega_{\mathrm{p}}}{\sqrt{3}}\left(=\sqrt{\frac{ne^2}{3\varepsilon_0 m}}\right) \tag{27}$$

[Note: from $\dfrac{\mathrm{d}E_{\mathrm{in}}(\omega)}{\mathrm{d}\omega}=0$, i.e. from

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\left(3-\frac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}\right)^2+\left(\frac{\gamma}{\omega}\frac{\omega_{\mathrm{p}}^2}{\omega^2+\gamma^2}\right)^2\right]=0,$$

one gets

$$\frac{12\omega_{\mathrm{p}}^2\omega^4-4\omega_{\mathrm{p}}^4\omega^2-2\omega_{\mathrm{p}}^4\gamma^2}{\omega^3(\gamma^2+\omega^2)^2}=0$$

and, solving,

$$\omega_{\mathrm{r}}=\sqrt{\frac{\omega_{\mathrm{p}}^2+\sqrt{\omega_{\mathrm{p}}^2(\omega_{\mathrm{p}}^2+6\gamma^2)}}{6}}\approx\frac{\omega_{\mathrm{p}}}{\sqrt{3}}$$

in agreement with the previous result.]

(ii) Taking into account $\gamma\ll\omega_{\mathrm{p}}$, when $\omega=\omega_{\mathrm{r}}=\dfrac{\omega_{\mathrm{p}}}{\sqrt{3}}$,

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega_{\mathrm{r}})\approx-2-\mathrm{i}\,3\sqrt{3}\frac{\gamma}{\omega_{\mathrm{p}}}\tag{28}$$

The complex field amplitude at resonance is

$$\tilde{E}_{\mathrm{in}}(\omega_{\mathrm{r}})=\frac{3}{\tilde{\varepsilon}_{\mathrm{r}}(\omega)+2}E_0\approx\mathrm{i}\,\frac{\omega_{\mathrm{p}}}{\sqrt{3}\gamma}E_0\tag{29}$$

The magnitude of the field amplitude is

$$E_{\mathrm{r}}=\frac{\omega_{\mathrm{p}}}{\sqrt{3}\gamma}E_0\gg E_0\tag{30}$$

From (17) and (18), the complex amplitude of the current density inside the sphere is

$$\tilde{j}=\tilde{\sigma}E_{\mathrm{r}}=\frac{\varepsilon_0\omega_{\mathrm{p}}^2}{\gamma+\mathrm{i}\omega}E_{\mathrm{r}}=\frac{\varepsilon_0\omega_{\mathrm{p}}^2}{\gamma^2+\omega^2}(\gamma-\mathrm{i}\omega)E_{\mathrm{r}}=j\,\mathrm{e}^{\mathrm{i}\varphi_j}\tag{31}$$

where

$$j=\frac{\varepsilon_0\omega_{\mathrm{p}}^2}{\sqrt{\gamma^2+\omega^2}}E_{\mathrm{r}},\qquad\cos\varphi_j=\frac{\gamma}{\sqrt{\gamma^2+\omega^2}}\tag{32}$$

The average heat-power density is

$$\bar{p}_{\mathrm{r}}=\overline{j(t)E_{\mathrm{r}}(t)}=\overline{j\cos(\omega t+\varphi_j)\cdot E_{\mathrm{r}}\cos\omega t}=\frac{1}{2}jE_{\mathrm{r}}\cos\varphi_j\tag{33}$$

Substituting (32) together with $\omega=\omega_{\mathrm{r}}=\dfrac{\omega_{\mathrm{p}}}{\sqrt{3}}$ and $E_{\mathrm{r}}=\dfrac{\omega_{\mathrm{p}}}{\sqrt{3}\gamma}E_0$, and taking into account $\gamma\ll\omega_{\mathrm{p}}$,

$$\bar{p}_{\mathrm{r}}=\frac{1}{2}\frac{\varepsilon_0\omega_{\mathrm{p}}^2\gamma}{\gamma^2+\omega^2}E_{\mathrm{r}}^2\approx\frac{\varepsilon_0\omega_{\mathrm{p}}^2E_0^2}{2\gamma}\tag{34}$$

The average heating power is

$$\bar{P}_{\mathrm{r}}=\bar{p}_{\mathrm{r}}\cdot\frac{4}{3}\pi R^3=\frac{2\pi\varepsilon_0\omega_{\mathrm{p}}^2E_0^2R^3}{3\gamma}\tag{35}$$

Marking scheme.
This problem carries 50 points.
Part (1) 10 points: (1)(3)(4)(5) 1 point each, (2)(6)(7) 2 points each;
Part (2) 20 points:
(i) 4 points: (8)(9)(10)(11) 1 point each;
(ii) 8 points: (12) 2 points, (16) 4 points, (18) 2 points;
(iii) 8 points: (19) 4 points, (20)(21) 1 point each, (22) 2 points;
Part (3) 20 points:
(i) 6 points: (23) 2 points, (24)(26) 1 point each, (27) 2 points;
(ii) 14 points: (29)(30)(31)(32)(33)(34)(35) 2 points each.

Problem 7 (Dirac string, magnetic monopole and the A–B effect)

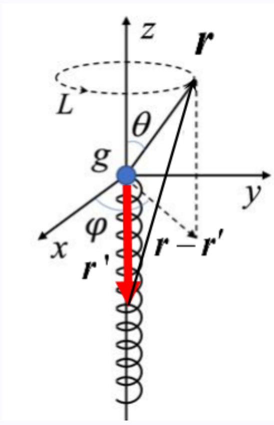
Solution

(1)(i) As in solution figure 7a, the condition given in the problem gives $\mathrm{d}\boldsymbol{m} = -g\,\mathrm{d}\boldsymbol{r}'$, where $\boldsymbol{r}' = z\boldsymbol{e}_z$ ($z \in (0, -\infty]$).
The vector potential produced by the Dirac string at the point $\boldsymbol{r}$ in space can be regarded as the superposition of the vector potentials produced at $\boldsymbol{r}$ by infinitely many small magnetic moments $\mathrm{d}\boldsymbol{m}$ joined tail (S pole) to head (N pole), as in solution figure 7a; thus

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \mathrm{d}\boldsymbol{m} \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \frac{\mu_0 g}{4\pi}\int \mathrm{d}\boldsymbol{r}' \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{1}$$

where

$$\mathrm{d}\boldsymbol{r}' \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \frac{-r\sin\theta}{(r^2+z^2-2rz\cos\theta)^{3/2}}\,\mathrm{d}z\,\boldsymbol{e}_\varphi \tag{2}$$



Solution figure 7a

Substituting (2) into (1) and simplifying,

$$\boldsymbol{A} = \frac{\mu_0 g r}{4\pi}\sin\theta\,\boldsymbol{e}_\varphi \int_0^{-\infty} \frac{-\mathrm{d}z'}{\sqrt{\left[(z'-r\cos\theta)^2 + r^2\sin^2\theta\right]^3}} \tag{3}$$

Using the indefinite integral formula given in the problem,

$$\int \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + \text{an arbitrary constant}$$

and integrating (3),

$$\begin{aligned}\boldsymbol{A} &= \frac{\mu_0 g r}{4\pi}\sin\theta\,\boldsymbol{e}_\varphi\,\frac{z'-r\cos\theta}{r^2\sin^2\theta\sqrt{(z'-r\cos\theta)^2+r^2\sin^2\theta}}\Bigg|_{z'=-\infty}^{z'=0} \\ &= \frac{\mu_0 g r}{4\pi}\sin\theta\,\boldsymbol{e}_\varphi\left[\frac{-r\cos\theta}{r^2\sin^2\theta\sqrt{(r\cos\theta)^2+r^2\sin^2\theta}}-\frac{-1}{r^2\sin^2\theta}\right] \\ &= \frac{\mu_0 g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\boldsymbol{e}_\varphi\end{aligned} \tag{4}$$

Equivalent forms are

$$\boldsymbol{A} = \frac{\mu_0 g}{4\pi r}\frac{\sin\theta}{1+\cos\theta}\boldsymbol{e}_\varphi = \frac{\mu_0 g}{4\pi r}\tan\frac{\theta}{2}\boldsymbol{e}_\varphi = \frac{\mu_0 g}{4\pi r}\frac{\boldsymbol{e}_z\times\boldsymbol{e}_r}{1+\boldsymbol{e}_z\cdot\boldsymbol{e}_r}$$

Remark: the above expression for $\boldsymbol{A}$ is undefined at $\theta=\pi$.

(ii)
1. Calculation of the circulation Γ_L. On the directed circular loop L in solution figure 7a take a line element

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}l\,\boldsymbol{e}_\varphi = r\sin\theta\,\mathrm{d}\varphi\,\boldsymbol{e}_\varphi \tag{5}$$

Using (4) and (5),

$$\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \frac{\mu_0 g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\times r\sin\theta\,\mathrm{d}\varphi = \frac{\mu_0 g}{4\pi}(1-\cos\theta)\,\mathrm{d}\varphi \tag{6}$$

so the circulation of $\boldsymbol{A}$ is

$$\Gamma_L = \oint_L \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \frac{\mu_0 g}{4\pi}(1-\cos\theta)\int_0^{2\pi}\mathrm{d}\varphi = \frac{\mu_0 g}{2}(1-\cos\theta) \tag{7}$$

2. Calculation of the flux Φ_L.
Method 1. Divide the disc into annuli of radius ρ ($\rho < r\sin\theta$); the area element is

$$\mathrm{d}\boldsymbol{S} = 2\pi\rho\,\mathrm{d}\rho\,\boldsymbol{e}_z$$

By the Coulomb law for magnetic charge, the elementary flux of the field of the magnetic charge g through the area element $\mathrm{d}\boldsymbol{S}$ is

$$\mathrm{d}\Phi = 2\pi\rho\,\mathrm{d}\rho\times\frac{\mu_0}{4\pi}\frac{g}{|\boldsymbol{\xi}|^3}\boldsymbol{\xi}\cdot\boldsymbol{e}_z = \frac{\mu_0 g}{2}\frac{\rho\,\mathrm{d}\rho}{(\rho^2+r^2\cos^2\theta)^{3/2}}r\sin\theta \tag{8}$$

where the vector $\boldsymbol{\xi}$ is the position vector of a point of the area element relative to the position of the magnetic charge, so that $|\boldsymbol{\xi}| = \sqrt{\rho^2+r^2\cos^2\theta}$. When $\theta\neq\frac{\pi}{2}$, the flux is

$$\begin{aligned}\Phi_L &= \frac{\mu_0 g}{2}r\cos\theta\int_0^{r\sin\theta}\frac{\rho\,\mathrm{d}\rho}{(\rho^2+r^2\cos^2\theta)^{3/2}} \\ &= -\frac{\mu_0 g}{2}r\cos\theta\,\frac{1}{\sqrt{\rho^2+r^2\cos^2\theta}}\Bigg|_0^{r\sin\theta} \\ &= \frac{\mu_0 g}{2}\left(\frac{\cos\theta}{|\cos\theta|}-\cos\theta\right)\end{aligned} \tag{9}$$

Hence

$$\Phi_L = \begin{cases} \dfrac{\mu_0 g}{2}(1-\cos\theta) & \text{when } \theta < \dfrac{\pi}{2} \\ -\dfrac{\mu_0 g}{2}(1+\cos\theta) & \text{when } \theta > \dfrac{\pi}{2} \end{cases} \tag{10}$$

Method 2. The solid angle subtended at the position of the magnetic charge by a surface which is bounded by the directed loop L and crosses the positive z axis (it may be part of a spherical surface) is

$$\Omega(\theta) = 2\pi \int_0^\theta \sin\theta' \, \mathrm{d}\theta' = 2\pi\,(1-\cos\theta) \tag{8$'$}$$

When $\theta \neq \frac{\pi}{2}$, by the "Gauss theorem" corresponding to the Coulomb law for magnetic charge and by the spherical symmetry of the field excited by the magnetic charge, the flux of the field of the magnetic charge g through the disc is

$$\Phi_L = \mu_0 g \frac{\Omega_L}{4\pi} \tag{9$'$}$$

where the solid angle subtended by the disc at the position of the magnetic charge is

$$\Omega_L = \begin{cases} \Omega(\theta) & \text{when } \theta < \dfrac{\pi}{2} \\ \Omega(\theta) - 4\pi & \text{when } \theta > \dfrac{\pi}{2} \end{cases}$$

Substituting (8′) into the above and then into (9′) gives

$$\Phi_L = \begin{cases} \dfrac{\mu_0 g}{2}(1-\cos\theta) & \text{when } \theta < \dfrac{\pi}{2} \\ -\dfrac{\mu_0 g}{2}(1+\cos\theta) & \text{when } \theta > \dfrac{\pi}{2} \end{cases} \tag{10$'$}$$

3. Comparison of Γ_L and Φ_L. Comparing (7) and (10),

$$\text{when } \theta < \frac{\pi}{2}, \quad \Gamma_L = \Phi_L$$

$$\text{when } \theta > \frac{\pi}{2}, \quad \Gamma_L \neq \Phi_L \tag{11}$$

Computing the difference of the two when $\theta > \frac{\pi}{2}$,

$$\Gamma_L - \Phi_L = \mu_0 g \tag{12}$$

The difference in (12) corresponds to the magnetic flux contributed by the interior of the solenoid that the disc is crossed by,

$$\Phi_0 = \mu_0 \frac{\mathrm{d}m}{\mathrm{d}l} = \mu_0 g \tag{13}$$

where $\mathrm{d}l$ is a line element of the solenoid along its axis and $\mathrm{d}m$ is the elementary magnetic moment on $\mathrm{d}l$ (along the $\boldsymbol{e}_z$ direction).

Remarks:

1) Let the number of turns per unit length of the solenoid be n, its cross-sectional area be S and its current be I; then

$$\mathrm{d}m = S\,\mathrm{d}I = nIS\,\mathrm{d}l$$

and the magnetic field inside the solenoid is $B = \mu_0 nI$, so the magnetic flux inside the solenoid is

$$\Phi_0 = \mu_0 \frac{\mathrm{d}m}{\mathrm{d}l}$$

2) The $\boldsymbol{A}$ computed in (4) is the magnetic vector potential of the semi-infinite solenoid obtained from ordinary electromagnetic theory, so Γ_L corresponds to the magnetic flux, through the plane enclosed by the ring, of the field of the semi-infinite solenoid; whereas the field excited by the equivalent magnetic charge according to the Coulomb law for magnetic charge applies only outside the solenoid, and the flux it contributes inside the (extremely thin) solenoid is zero. Hence when $\theta > \frac{\pi}{2}$,

$$\Gamma_L = \Phi_L + \Phi_0$$

(2)(i) From (13) (or from $g = \frac{\mathrm{d}m}{\mathrm{d}l}$),

$$g = \frac{\Phi_0}{\mu_0} = nIS = nSI_0 \cos\omega t \tag{14}$$

(ii) One can see that the position of g is exactly the centre of the regular tetrahedron whose base is the equilateral triangle.
Take the distance from g to a vertex of the triangle as radius and g as centre and construct a sphere, i.e. the circumscribed sphere of the regular tetrahedron. The magnetic flux through this spherical surface is the magnetic flux through the regular triangular pyramid. If the flux along the positive z direction is taken as positive, then by symmetry the flux through this triangular face is

$$\phi'_g = -\frac{1}{4}\mu_0 g = -\frac{1}{4}\mu_0 nSI_0 \cos\omega t \tag{15}$$

Because the Dirac string passes through the triangular plane, the flux through the triangular plane contains not only the flux ϕ'_g of the magnetic charge g but also the flux ϕ'_L of the solenoid part, i.e.

$$\phi' = \phi'_L + \phi'_g = \mu_0 g - \frac{1}{4}\mu_0 g = \frac{3}{4}\mu_0 nSI_0 \cos\omega t \tag{16}$$

The induced electromotive force is

$$\varepsilon = -\frac{\mathrm{d}\phi'}{\mathrm{d}t} = \frac{3}{4}\mu_0 nSI_0\omega \sin\omega t \tag{17}$$

The induced current is

$$I = \frac{\varepsilon}{3aR_0} = \frac{\mu_0 nSI_0\omega}{4aR_0}\sin\omega t \tag{18}$$

Using (17) and (18), the average heat power over one period is

$$\begin{aligned} W_R &= \frac{1}{T}\int_0^T I\varepsilon \,\mathrm{d}t = \frac{1}{T}\int_0^T \left(\frac{\mu_0 nSI_0\omega}{4\rho a}\sin\omega t\right)\left(\frac{3}{4}\mu_0 nSI_0\omega\sin\omega t\right)\mathrm{d}t \\ &= \frac{3\mu_0^2 n^2 S^2 I_0^2\omega^2}{16aR_0T}\int_0^T \sin^2\omega t\,\mathrm{d}t = \frac{3\mu_0^2 n^2 S^2 I_0^2\omega^2}{16aR_0T}\frac{T}{2} = \frac{3\mu_0^2 n^2 S^2 I_0^2\omega^2}{32aR_0} \end{aligned} \tag{19}$$

[Translator's note: In the first line of (19) the source prints the denominator "$4\rho a$"; from (18) it should be $4aR_0$, as the subsequent lines correctly use.]

(3) The wave vector of an electron of momentum $\boldsymbol{p}$ in the vector-potential field $\boldsymbol{A}$ is

$$\boldsymbol{k} = \frac{1}{\hbar}(\boldsymbol{p} - e\boldsymbol{A}) = \boldsymbol{k}_0 - \frac{e}{\hbar}\boldsymbol{A}$$

The phase difference of the two electron wave functions arriving at a point on the screen along the paths C_1 and C_2 is

$$\begin{aligned}\Delta\phi_1 &= \int_{C_1} \boldsymbol{k}\cdot \mathrm{d}\boldsymbol{l} - \int_{C_2} \boldsymbol{k}\cdot \mathrm{d}\boldsymbol{l} = \left(\int_{C_1} \boldsymbol{k}_0\cdot \mathrm{d}\boldsymbol{l} - \int_{C_2} \boldsymbol{k}_0\cdot \mathrm{d}\boldsymbol{l}\right) - \frac{e}{\hbar}\left(\int_{C_1} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} + \int_{-C_2} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}\right) \\ &= \Delta\phi_0 - \frac{e}{\hbar}\oint_C \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}\end{aligned} \tag{20}$$

where C is the loop enclosed by C_1 and $-C_2$.
The change of the phase difference caused by $\boldsymbol{A}$ after the solenoid carries current is

$$\Delta\phi = \Delta\phi_1 - \Delta\phi_0 = -\frac{e}{\hbar}\oint_C \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} = -\frac{e}{\hbar}\Phi = -\frac{e}{\hbar}\mu_0 nIS \tag{21}$$

When no current flows, the two beams of electron waves formed by the double slit satisfy at the positions of the bright fringes on the screen the condition of constructive interference

$$\Delta\phi_0 = k_0 d\sin\theta = 2l\pi \qquad l = 0, \pm 1, \pm 2, \dots \tag{22}$$

Let the distance between the position of the l-th order bright fringe on the screen and the central axis of the double slit be x; then

$$x = D\tan\theta \approx D\sin\theta = \frac{2l\pi D}{k_0 d} \tag{23}$$

When the current I flows, the condition of constructive interference becomes

$$\Delta\phi_1 = k_0 d\sin\theta - \frac{e}{\hbar}\Phi = 2l\pi \tag{24}$$

The position of the l-th order bright fringe becomes

$$x' = \frac{2l\pi D}{k_0 d} + \frac{eD}{\hbar k_0 d}\Phi \tag{25}$$

The distance through which the (central) bright fringe moves is

$$\Delta x = x' - x = \frac{eD}{\hbar k_0 d}\Phi = \frac{eD}{pd}\mu_0 nIS \tag{26}$$

Marking scheme.
This problem carries 50 points.
Part (1) 24 points:
(i) 8 points: (1) 1 point, (2) 3 points, (4) 4 points;
(ii) 16 points: (5) 1 point, (6)(7)(8)(9)(10) 2 points each ((8′)(9′)(10′) 2 points each), (11) 2 points, (13) 3 points;
Part (2) 14 points:
(i) 2 points: (14) 2 points;
(ii) 12 points: (15) 4 points, (16)(17)(18)(19) 2 points each;
Part (3) 12 points: (20)(21) 4 points in total, (22)–(25) 4 points in total, (26) 4 points.