The 41st Chinese Physics Olympiad

Final — Theoretical Examination (Solutions)

2024 — official reference solutions and marking scheme

Translated from the official booklet "Reference solutions and marking scheme of the theoretical examination of the 41st National Physics Competition for Secondary School Students (final round)" (第 41 届全国中学生物理竞赛决赛理论试题解答与评分标准). Six problems, 320 marks in total. Equation numbers (1), (2), … *within each problem follow the circled numbers of the Chinese original; primed variants* (16′), (20′), … *belong to the alternative methods, which the original marks with* 【另解】 / 【解法一】, 【解法二】 *and which are rendered here as "Alternative solution" and "Method 1", "Method 2", … Passages that the original encloses in the brackets* 【】 *are set here as bracketed remarks. Statements marked* (∗), (∗∗), (∗ ∗ ∗) *in the original carry marks of their own and are referred to by those symbols in the marking schemes, which are translated in full at the end of each problem.*

Notation. Chinese subscripts of the original are rendered in English: 左 *(left) and* 右 *(right) as* N_{left}, N_{right}; 整体 *(whole vehicle) as* M_{tot}; 快 *(fast axis) and* 慢 *(slow axis) as* ϕ_{fast}, ϕ_{slow}; *and the superscript* 干涉 *(interference term) as* I^{int}*. The source PDF has lost many round brackets in its converted formulas; these have been restored silently where the intended grouping is unambiguous. Genuine misprints are transcribed as printed with a translator's note giving the correct form. The figures of the solutions are cropped from the original; note that the two figures belonging to Problem 4 are labelled "Fig. S1a" and "Fig. S1b" in the source (*题解图 1a, 题解图 1b*) although they belong to the fourth problem.*

Problem 1 (β^- decay of ^{210}Bi: decay energy, electron spectrum and the neutrino)

Solution

(1) By the statement of the problem, the β^- decay of the parent nucleus ${}^{A}_{Z}\mathrm{P}$ into the daughter nucleus ${}^{A}_{Z+1}\mathrm{D}$ can be written as

$${}^{A}_{Z}\mathrm{P} \to {}^{A}_{Z+1}\mathrm{D} + \mathrm{e}^- + \cdots$$

Denote the rest mass of the parent nucleus by m_{P}, that of the daughter nucleus by m_{D} and that of the electron by m_{e}, and assume the parent nucleus is at rest before the decay. By energy conservation, the decay energy of this process (which includes the energy of all particles produced besides ${}^{A}_{Z+1}\mathrm{D}$ and the electron e^-) can be expressed through the rest masses of the parent nucleus, the daughter nucleus and the electron as

$$Q_{\beta^-} = \left[m_{\mathrm{P}} - (m_{\mathrm{D}} + m_{\mathrm{e}})\right]c^2 \tag{1}$$

By the statement of the problem the rest masses of the atom corresponding to the parent nucleus and of the atom corresponding to the daughter nucleus are M_{P} and M_{D} respectively, so

$$m_{\mathrm{P}} = M_{\mathrm{P}} - Zm_{\mathrm{e}}, \tag{2}$$

$$m_{\mathrm{D}} = M_{\mathrm{D}} - (Z+1)m_{\mathrm{e}} \tag{3}$$

Hence the decay energy Q_{β^-} of the β^- decay can be written as

$$Q_{\beta^-} = (M_{\mathrm{P}} - M_{\mathrm{D}})c^2 \tag{4}$$

The condition for this β^- decay to occur is

$$Q_{\beta^-} > 0, \tag{5}$$

so, by (1)(2)(3), the condition (5) for β^- decay can be stated as

$$M_{\mathrm{P}}(Z, A) > M_{\mathrm{D}}(Z+1, A) \tag{6}$$

That is: for two isobars with atomic numbers Z and $Z+1$, β^- decay can occur only if the atomic mass of the former is larger than the atomic mass of the latter.

(2) If no new particle is produced in this decay process, then at the level of the nucleus the process is

$$\mathrm{P} \to \mathrm{D} + \mathrm{e}^-$$

The total energy of P is $E_{\mathrm{P}} = m_{\mathrm{P}}c^2$; let the total energy of D be E_{D} and its momentum $\boldsymbol{p}_{\mathrm{D}}$, and let the total energy of the electron be E_{e} and its momentum $\boldsymbol{p}_{\mathrm{e}}$. Then energy conservation and momentum conservation read

$$m_{\mathrm{P}}c^2 = E_{\mathrm{D}} + E_{\mathrm{e}} \tag{7}$$

$$0 = \boldsymbol{p}_{\mathrm{D}} + \boldsymbol{p}_{\mathrm{e}} \tag{8}$$

From (8),

$$|\boldsymbol{p}_{\mathrm{D}}| = |\boldsymbol{p}_{\mathrm{e}}| \equiv p \tag{9}$$

Using (9), equation (7) becomes

$$\frac{p^2}{2m_{\mathrm{D}}} + K_{\mathrm{e}} = (m_{\mathrm{P}} - m_{\mathrm{D}} - m_{\mathrm{e}})c^2 \tag{10}$$

where

$$\frac{p^2}{2m_{\mathrm{D}}} = E_{\mathrm{D}} - m_{\mathrm{D}}c^2 \tag{11}$$

$$K_{\mathrm{e}} = E_{\mathrm{e}} - m_{\mathrm{e}}c^2 \tag{12}$$

From the relativistic energy–momentum relation,

$$E_{\mathrm{e}} = \sqrt{(pc)^2 + m_{\mathrm{e}}^2c^4} \tag{13}$$

Substituting (13) into (12) gives

$$p^2 = \frac{1}{c^2}K_{\mathrm{e}}(K_{\mathrm{e}} + 2m_{\mathrm{e}}c^2) \tag{14}$$

Eliminating p between (10) and (14), K_{e} satisfies the equation

$$K_{\mathrm{e}}^2 + 2(m_{\mathrm{D}} + m_{\mathrm{e}})c^2K_{\mathrm{e}} - 2m_{\mathrm{D}}(m_{\mathrm{P}} - m_{\mathrm{D}} - m_{\mathrm{e}})c^4 = 0 \tag{15}$$

The solution of (15) is

$$\begin{aligned}
K_{\mathrm{e}} &= \left[-(m_{\mathrm{D}} + m_{\mathrm{e}}) + \sqrt{(m_{\mathrm{D}} + m_{\mathrm{e}})^2 + 2m_{\mathrm{D}}(m_{\mathrm{P}} - m_{\mathrm{D}} - m_{\mathrm{e}})}\,\right]c^2 \\
&= \left[\sqrt{m_{\mathrm{D}}(2m_{\mathrm{P}} - m_{\mathrm{D}}) + m_{\mathrm{e}}^2} - (m_{\mathrm{D}} + m_{\mathrm{e}})\right]c^2 \\
&= \Big[\sqrt{(M_{\mathrm{D}} - Z_{\mathrm{D}}m_{\mathrm{e}})\big[2(M_{\mathrm{P}} - Z_{\mathrm{P}}m_{\mathrm{e}}) - (M_{\mathrm{D}} - Z_{\mathrm{D}}m_{\mathrm{e}})\big] + m_{\mathrm{e}}^2} \\
&\qquad - \big[(M_{\mathrm{D}} - Z_{\mathrm{D}}m_{\mathrm{e}}) + m_{\mathrm{e}}\big]\Big]c^2 \\
&= \left[\sqrt{M_{\mathrm{D}}(2M_{\mathrm{P}} - M_{\mathrm{D}}) + 2\big[M_{\mathrm{D}} - Z_{\mathrm{D}}M_{\mathrm{P}}\big]m_{\mathrm{e}} + Z_{\mathrm{P}}^2m_{\mathrm{e}}^2} - M_{\mathrm{D}} + Z_{\mathrm{P}}m_{\mathrm{e}}\right]c^2
\end{aligned} \tag{16}$$

The negative root has been discarded, and use has been made of the approximate relations valid when the binding energy of the electrons in the atom is neglected,

$$M_{\mathrm{P}} = m_{\mathrm{P}} + Z_{\mathrm{P}} m_{\mathrm{e}} \tag{17}$$

$$M_{\mathrm{D}} = m_{\mathrm{D}} + Z_{\mathrm{D}} m_{\mathrm{e}} \tag{18}$$

together with

$$Z_{\mathrm{P}} = Z_{\mathrm{D}} - 1 \tag{19}$$

Substituting the data given in the problem into (16),

$$\begin{aligned} K_{\mathrm{e}} = \Big\{ \big[& 209.982883 \times (2 \times 209.984130 - 209.982883) \\ & + 2 \times (209.982883 - 84 \times 209.984130) \\ & \times \frac{0.511}{931.494} + \left(83 \times \frac{0.511}{931.494} \right)^2 \big]^{1/2} \\ & - 209.982883 + 83 \times \frac{0.511}{931.494} \Big\} \times 931.494 \text{ MeV} \approx 1.161 \text{ MeV} \end{aligned} \tag{20}$$

[Alternative solution:
Neglecting the binding energy of the electrons in the atom, the mass of the ${}^{210}_{83}\mathrm{Bi}$ nucleus is

$$m_{\mathrm{P}} = M_{\mathrm{P}} - 83 m_e = 209.984130 - 83 \times 0.000549 = 209.938563 \text{ amu} \tag{16$'$}$$

The mass of the nucleus ${}^{210}_{84}\mathrm{Po}$ produced by its β^- decay is

$$m_{\mathrm{D}} = M_{\mathrm{D}} - 84 m_e = 209.982883 - 84 \times 0.000549 = 209.936767 \text{ amu} \tag{17$'$}$$

The positive root of (15) is

$$\begin{aligned} K_{\mathrm{e}} &= \left(\sqrt{(m_{\mathrm{D}} + m_{\mathrm{e}})^2 + 2 m_{\mathrm{D}} (m_{\mathrm{P}} - m_{\mathrm{D}} - m_{\mathrm{e}})} - m_{\mathrm{D}} - m_{\mathrm{e}} \right) c^2 \\ &= \left(\sqrt{m_{\mathrm{D}} (2 m_{\mathrm{P}} - m_{\mathrm{D}}) + m_e^2} - m_{\mathrm{D}} - m_{\mathrm{e}} \right) c^2 \\ &= \left(\sqrt{m_{\mathrm{P}}^2 - (m_{\mathrm{P}} - m_{\mathrm{D}})^2 + m_e^2} - m_{\mathrm{D}} - m_{\mathrm{e}} \right) c^2 \end{aligned} \tag{18$'$}$$

Here

$$\frac{(m_{\mathrm{P}} - m_{\mathrm{D}})^2}{m_{\mathrm{P}}^2} \sim 10^{-10}, \qquad \frac{m_e^2}{m_{\mathrm{P}}^2} \sim 10^{-11}$$

so (18$'$) becomes

$$K_{\mathrm{e}} = (m_{\mathrm{P}} - m_{\mathrm{D}} - m_{\mathrm{e}}) c^2 \tag{19$'$}$$

The physical meaning of this expression is very clear: the ${}^{210}_{84}\mathrm{Po}$ nucleus is so much heavier than the electron that its recoil kinetic energy $\dfrac{p^2}{2m_d}$ is negligible compared with the kinetic energy K_{e} of the electron. Inserting the numbers,

$$\begin{aligned} K_{\mathrm{e}} &= (209.938563 - 209.936767 - 0.000549) \text{ amu} \cdot c^2 \\ &= 0.001247 \text{ amu} \cdot c^2 = 1.161 \text{ MeV} \end{aligned} \tag{20$'$}$$

In fact, going back to the atomic masses (rather than the nuclear masses), this is exactly

$$\begin{aligned} K_{\mathrm{e}} &= \left[M({}^{210}_{83}\mathrm{Bi}) - M({}^{210}_{84}\mathrm{Po}) \right] c^2 = (209.984130 - 209.982883) \text{ amu} \cdot c^2 \\ &= 0.001247 \text{ amu} \cdot c^2 = 1.161 \text{ MeV} \end{aligned}$$

]

(3) The kinetic energy of the electron after the decay obtained under the assumption $P \to D + e^-$ is comparable with the maximum value of the electron kinetic energy seen in experiment. Therefore the rest mass of the third particle (the neutrino) should be very close to 0 (at most about $0.03\,\mathrm{MeV}/c^2$; any value close to this is accepted),*(∗)* otherwise the maximum kinetic energy in the measured kinetic-energy spectrum of the electrons would have to be clearly smaller than the maximum value shown in the figure. *(∗∗)*

(4) By the statement of the problem, the elementary process of the β^- decay is

$$\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \mathrm{X} \tag{21}$$

The charges of the neutron, the proton and the electron are

$$Q_\mathrm{n} = 0, \qquad Q_\mathrm{p} = +1, \qquad Q_{\mathrm{e}^-} = -1 \tag{22}$$

Charge conservation before and after the decay gives the charge of X as

$$Q_\mathrm{X} = Q_\mathrm{n} - (Q_\mathrm{p} + Q_{\mathrm{e}^-}) = 0 \tag{23}$$

By the statement of the problem, the angular momentum of the system before the decay is the spin of the neutron:

$$l_\mathrm{i} = S_\mathrm{n} = \frac{1}{2}\hbar \tag{24}$$

Suppose that the only particles after the decay were the proton p and the electron e^-. The spins of the proton and of the electron are both

$$S_\mathrm{p} = \frac{1}{2}\hbar, \qquad S_{\mathrm{e}^-} = \frac{1}{2}\hbar$$

By the composition rule for two angular momenta given in the problem, the total spin of the proton and the electron, i.e. the total angular momentum of the decay products,

$$\boldsymbol{L}_\mathrm{f} = \boldsymbol{S} = \boldsymbol{S}_\mathrm{p} + \boldsymbol{S}_{\mathrm{e}^-} \tag{25}$$

can take all the possible values

$$l_\mathrm{f} = s_\mathrm{p} - s_\mathrm{e}, \quad s_\mathrm{p} + s_\mathrm{e} \tag{26}$$

The first possible value in (26) corresponds to the spins of the proton and the electron adding in opposite directions, and the total spin is

$$l_\mathrm{f} = S = 0 \tag{27}$$

The second possible value in (26) corresponds to the spins of the proton and the electron adding in the same direction, and the total spin is

$$l_\mathrm{f} = S = 1\hbar \tag{28}$$

Neither of the two possible values (27) and (28) equals the initial value (24), which contradicts angular-momentum conservation. In order that the total angular momentum of the system be conserved, a particle X must exist, and its spin must be $\frac{1}{2}\hbar$:

$$S_\mathrm{X} = \frac{1}{2}\hbar \tag{29}$$

$$\frac{3}{2}\hbar \quad \text{(not yet excluded)} \tag{30}$$

In summary: if, besides the proton and the electron, one further particle is produced in the β^- decay, that particle must be electrically neutral and have spin $\frac{1}{2}\hbar$ (spin $\frac{3}{2}\hbar$ is not yet excluded); that is, an electrically neutral fermion must exist [namely the particle usually called the neutrino (if $S_{\mathrm{X}} = \frac{1}{2}\hbar$); more precisely, it is the electron antineutrino $\bar{\nu}_{\mathrm{e}}$].

Marking scheme.
This problem carries 50 points.
Part (1) 13 points: (1) 3 points, (2)(3) 1 point each, (4) 2 points, (5)(6) 3 points each;
Part (2) 19 points: (7)(8) 3 points each, (9) 1 point, (10) 2 points, (11)(12)(13)(14) 1 point each; (15) 2 points, (16) 2 points, (20) 2 points;
Part (3) 5 points: (∗) 3 points, (∗∗) 2 points;
Part (4) 13 points: (22)(23)(25) 2 points each, (26) 2 points, (27)(28) 1 point each, (29) 2 points, (30) 1 point.

Problem 2 (one-dimensional diatomic chain on a ring: eigenfrequencies, energy and boundary conditions)

Solution

(1) For convenience put $x_0 = x_{2N}$ and $x_{2N+1} = x_1$. The equations of motion of the system are then easily written down:

$$0 = M\ddot{x}_{2i-1} + k(2x_{2i-1} - x_{2i} - x_{2i-2}), \qquad (i = 1, \ldots, N) \tag{1}$$

$$0 = m\ddot{x}_{2i} + k(2x_{2i} - x_{2i+1} - x_{2i-1}), \qquad (i = 1, \ldots, N) \tag{2}$$

(2) By the statement of the problem, when the spring ring performs small vibrations in an eigenmode all the beads vibrate with the same angular frequency; call it ω. The small vibrations of all the beads on the ring are steady, and the only differences between them should be in amplitude and in phase. By symmetry the amplitudes of the vibrations of the large beads must all be equal; call it A. For the same reason the amplitudes of the small beads must all be equal; call it B. Likewise, by symmetry the phase difference between the vibrations of adjacent beads must be the same; call it α. The solution can therefore be taken in the form

$$x_{2i-1} = A\cos[\omega t + \varphi_0 + 2(i-1)\alpha], \qquad i = 1, 2, \cdots, N \tag{3}$$

$$x_{2i} = B\cos[\omega t + \varphi_0 + (2i-1)\alpha], \qquad i = 1, 2, \cdots, N \tag{4}$$

where φ_0 is an arbitrary constant. From (3),

$$x_1 = A\cos\omega t. \tag{5}$$

Since the solution must satisfy the condition

$$x_0 = x_{2N} \tag{6}$$

we must have

$$2N\alpha = 2n\pi, \qquad n \in Z \tag{7}$$

where Z denotes the set of integers. Since the values of α are periodic with period 2π, an independent range of values can be taken as

$$-\pi < \alpha \leq \pi \tag{8}$$

Correspondingly there are $2N$ choices of n:

$$n = -N+1,\ -N+2, \cdots, N-1,\ N \tag{9}$$

Substituting the solution (3)(4) into the equation of motion (1) gives

$$\cos\alpha = \frac{2k - M\omega^2}{2k} \cdot \frac{A}{B} \tag{10}$$

Substituting the solution (3)(4) into the equation of motion (2) gives

$$\cos\alpha = \frac{2k - m\omega^2}{2k} \cdot \frac{B}{A} \tag{11}$$

From (10) and (11),

$$mM\omega^4 - 2k(m+M)\omega^2 + 4k^2\sin^2\alpha = 0 \tag{12}$$

Solving (12), the eigenfrequency ω is

$$\omega = \sqrt{\frac{k}{mM}\left[(m+M) \pm \sqrt{(m-M)^2 + 4mM\cos^2\alpha}\right]} \tag{13}$$

Substituting (7) into (13),

$$\omega = \sqrt{\frac{k}{mM}\left[(m+M) \pm \sqrt{(m-M)^2 + 4mM\cos^2\frac{n\pi}{N}}\right]}, \qquad n = 0, 1, 2, \cdots, N-1 \tag{14}$$

Note that a pair of values of n opposite in sign gives the same frequency. Equation (14) gives all the possible eigenfrequencies. (In a student's answer it is accepted either to exclude or to keep the solution $n = 0$.)

(3)

(3.1) The trial solution above already gives a set of solutions satisfying the requirements. Dividing equation (10) by equation (11) side by side gives a quadratic algebraic equation for $\frac{A}{B}$,

$$\left(\frac{A}{B}\right)^2 = \frac{2k - m\omega^2}{2k - M\omega^2} \tag{15}$$

It has two roots. Since A and B are both amplitudes, the negative root is discarded, and

$$\frac{A}{B} = \sqrt{\frac{2k - m\omega^2}{2k - M\omega^2}} \tag{16}$$

(3.2) Clearly the spring ring under consideration is a conservative system. Its total energy E does not change with time, and it equals the average energy over one period of oscillation, $E = \langle E \rangle$, which is easy to find. First, the average kinetic energy is

$$\langle E_\mathrm{k} \rangle = \sum_{i=1}^{N} \frac{1}{2} M \frac{1}{T} \int_0^T \mathrm{d}t\, \dot{x}_{2i-1}^2(t) + \sum_{i=1}^{N} \frac{1}{2} m \frac{1}{T} \int_0^T \mathrm{d}t\, \dot{x}_{2i}^2(t) \tag{17}$$

Substituting (3)(4) into (17),

$$\langle E_\mathrm{k}\rangle = \sum_{i=1}^{N}\frac{1}{2}M\omega^2A^2\,\frac{1}{T}\int_0^T \mathrm{d}t\ \sin^2[\omega t+(2i-1)\alpha]+\sum_{i=1}^{N}\frac{1}{2}m\omega^2B^2\,\frac{1}{T}\int_0^T \mathrm{d}t\ \sin^2[\omega t+2(i-1)\alpha] \tag{18}$$

Using

$$\frac{1}{T}\int_0^T \mathrm{d}t\ \sin^2(\omega t+\varphi)=\frac{1}{2} \tag{19}$$

where φ may be any phase, and substituting (19) into (18),

$$\langle E_\mathrm{k}\rangle = \frac{1}{4}N\omega^2(MA^2+mB^2) \tag{20}$$

The average potential energy $\langle E_\mathrm{p}\rangle$ of this system equals its average kinetic energy $\langle E_\mathrm{k}\rangle$,

$$\langle E_\mathrm{p}\rangle = \langle E_\mathrm{k}\rangle \tag{21}$$

Therefore the total energy E_tot is

$$E_\mathrm{tot} = \langle E_\mathrm{p}\rangle + \langle E_\mathrm{k}\rangle = 2\langle E_\mathrm{k}\rangle \tag{22}$$

From (15)(20)(22),

$$E_\mathrm{tot} = \frac{1}{2}N\omega^2(MA^2+mB^2) = \frac{(m+M)k-mM\omega^2}{2k-m\omega^2}\,N\omega^2A^2 \tag{23}$$

From (22)(23),

$$\langle E_\mathrm{k}\rangle = \frac{(m+M)k-mM\omega^2}{2\big(2k-m\omega^2\big)}\,N\omega^2A^2 \tag{24}$$

(4) This situation amounts to introducing the boundary condition

$$x_1 = 0 \tag{25}$$

To find a solution satisfying the boundary condition (25), note that the solutions (3)(4) of the equations (1)(2) are in fact the piecewise representation of a travelling wave running around the ring,

$$x_{2i-1} = A\cos[\omega t+\varphi_0+k_nl_{2i-1}] \qquad i=1,2,\cdots,N \tag{26}$$

$$x_{2i} = B\cos[\omega t+\varphi_0+k_nl_{2i}] \qquad i=1,2,\cdots,N \tag{27}$$

where k_n is the wave number and α is fixed by (7),

$$\alpha = \frac{n\pi}{N} \tag{28}$$

and

$$k_nl_{2i-1} = [(2i-1)-1]\alpha \tag{29}$$

$$k_nl_{2i} = (2i-1)\alpha \tag{30}$$

Take a pair of values of n opposite in sign (hence $n\neq 0$) and superpose the two travelling waves running in opposite directions:

$$x_{2i-1} = A\big\{\cos[\omega t+\varphi_0-k_nl_{2i-1}]-\cos[\omega t+\varphi_0+k_nl_{2i-1}]\big\} \tag{31}$$

$$x_{2i} = B\big\{\cos[\omega t + \varphi_0 - k_n l_{2i}] - \cos[\omega t + \varphi_0 + k_n l_{2i}]\big\} \tag{32}$$

This obviously satisfies the boundary condition (25). From (28)(29)(30)(31)(32),

$$x_{2i-1} = 2A\sin[2(i-1)\alpha]\sin(\omega t + \varphi_0) \tag{33}$$

$$x_{2i} = 2B\sin[(2i-1)\alpha]\sin(\omega t + \varphi_0) \tag{34}$$

From (16)(34),

$$x_{2i} = 2A\sqrt{\frac{2k - M\omega^2}{2k - m\omega^2}}\,\sin[(2i-1)\alpha]\sin(\omega t + \varphi_0) \tag{35}$$

Substituting the solutions (33)(34) into the equations of motion (1)(2), the result obtained for the eigenfrequencies is as given by (14), except that now $n \neq 0$ (that is, $\alpha \neq 0$). *(∗)*

(5) In an eigenmode that does not depend on μ, bead number 1 must not vibrate, and this is equivalent to the situation of part (4). *(∗)* Therefore the answer is the same as that of part (4). *(∗∗)*

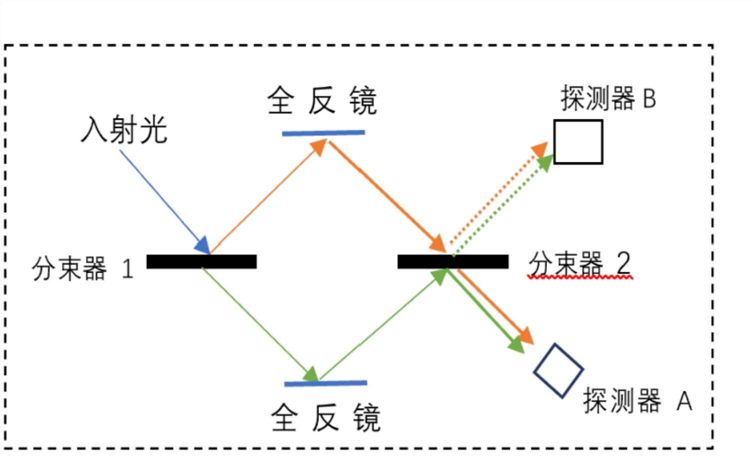
Marking scheme.
This problem carries 50 points.
Part (1) 4 points: (1)(2) 2 points each;
Part (2) 15 points: (3)(4) 2 points each, (6)(7) 1 point each; (10)(11)(12)(13) 2 points each, (14) 1 point;
Part (3) 12 points: (15) 1 point, (16) 2 points, (17) 1 point, (18)(19)(20)(21)(22) 1 point each, (23) 2 points, (24) 1 point;
Part (4) 17 points: (25)(26)(27) 1 point each, (29) 2 points, (30) 1 point, (31)(32) 2 points each, (∗) 2 points, (33)(34) 2 points each, (35) 1 point;
Part (5) 2 points: (∗) 1 point, (∗∗) 1 point.

Problem 3 (matrix optics and the quadrant interferometer)

Solution

(1)

(1.1) A sketch of the complete optical path of the interferometer is shown in Fig. S3a. When the light travelling downwards after beam splitter 2 interferes destructively, the intensity at detector A is 0; there is also the light travelling upwards after beam splitter 2, which interferes constructively, and by energy conservation the intensity of this beam must be a maximum, i.e. detector B receives the maximum intensity. *(∗)*



题解图 3a

Labels: 入射光 = *incident light;* 全反镜 = *total reflector;* 分束器 1, 2 = *beam splitters 1, 2;* 探测器 A, B = *detectors A, B;* 题解图 3a = *Fig. S3a.*

(1.2) When the energy at detector A is at its minimum, the energy is concentrated in the optical path leading to detector B.

Method 1. Follow the phase changes of the various fields along the optical paths (the amplitudes are all the same, so only the phases need be considered). Let $\tilde{E}_1$, $\tilde{E}_2$ denote the fields that travel along the upper and the lower path and enter detector A downwards after the second beam splitter. Then

$$\tilde{E}_1 = \exp[i(\delta_r + \delta_\mathrm{m} + \delta_t + kl_\mathrm{top})] \tag{1}$$

$$\tilde{E}_2 = \exp[i(\delta_t + \delta_\mathrm{m} + \delta_r + kl_\mathrm{bottom})] \tag{2}$$

(only the phases are written; the amplitudes are equal) Here δ_r and δ_t are the phase increments produced by reflection and by transmission at a beam splitter, and δ_m is the phase increment produced by reflection at a total reflector. Let $\tilde{E}_1'$, $\tilde{E}_2'$ denote the fields that travel along the upper and the lower path and enter detector B upwards after the second beam splitter. Likewise,

$$\tilde{E}_1' = \exp i(\delta_r + \delta_\mathrm{m} + \delta_r + kl_\mathrm{top}) \tag{3}$$

$$\tilde{E}_2' = \exp i(\delta_t + \delta_\mathrm{m} + \delta_t + kl_\mathrm{bottom}) \tag{4}$$

When the fields entering detector A cancel each other, the phase difference of $\tilde{E}_1$ and $\tilde{E}_2$ is

$$kl_\mathrm{top} - kl_\mathrm{bottom} = k\Delta l = \pi \tag{5}$$

$$(\Delta l = l_\mathrm{top} - l_\mathrm{bottom})$$

At the same time $\tilde{E}_1'$ and $\tilde{E}_2'$ reinforce each other, so the phase difference of $\tilde{E}_1'$ and $\tilde{E}_2'$ must be 0: (6)

$$2(\delta_r - \delta_t) + k\Delta l = 0 \tag{7}$$

From this,

$$\delta_r - \delta_t = -\phi = -\frac{\pi}{2} \tag{8}$$

(1 point) (using the formula given in the problem, $\phi = \delta_t - \delta_r$)

$$\frac{t}{r} = e^{i\frac{\pi}{2}} = i \tag{9}$$

[**Method 2.** Or, more briefly: the superposition of the two fields entering the lower detector A is

$$rt + rte^{ik\Delta l} \qquad \text{(omitting } E_0\text{)}$$

The superposition of the two fields entering the upper detector B is

$$r^2 + t^2 e^{ik\Delta l} = R + Te^{i2\phi} e^{ik\Delta l} \qquad (R = r^2;\ T = t^2)$$

If the light cancels at detector A,

$$k\Delta l = \pi,$$

then at the upper detector B

$$\left| R - Te^{i2\phi} \right| = 1,$$

and since

$$R + T = 1,$$

it follows that

$$\phi = \frac{\pi}{2}$$

]

The above calculation applies to each of the P and S components.

(2) For incident S polarization the reflected and the transmitted light are also S only; that is, the P and S components do not mix. Hence the diagonal matrix

$$\tilde{R} = \begin{pmatrix} \tilde{r}_\mathrm{P} & 0 \\ 0 & \tilde{r}_\mathrm{S} \end{pmatrix} = \begin{pmatrix} r_\mathrm{P} \mathrm{e}^{i\delta_\mathrm{P}} & 0 \\ 0 & r_\mathrm{S} \mathrm{e}^{i\delta_\mathrm{S}} \end{pmatrix} \tag{10}$$

is the reflection matrix of the beam splitter. Using (9),

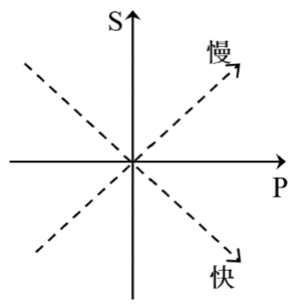
$$\tilde{T} = \begin{pmatrix} i\tilde{r}_\mathrm{P} & 0 \\ 0 & i\tilde{r}_\mathrm{S} \end{pmatrix} = \begin{pmatrix} ir_\mathrm{P} \mathrm{e}^{i\delta_\mathrm{P}} & 0 \\ 0 & ir_\mathrm{S} \mathrm{e}^{i\delta_\mathrm{S}} \end{pmatrix} \tag{11}$$

is the transmission matrix of the beam splitter.

(3) The matrix $\tilde{M}$ by which a total reflector changes the polarization state. The incident field must satisfy the electromagnetic boundary conditions at the surface of the total reflector: the field (i.e. the electric field) is continuous along the tangential direction of the mirror surface, i.e. the sum of the tangential components of the incident field and of the reflected field at the mirror surface must be 0. Hence, if the P and S components of the incident light at the mirror lie along the positive P and S directions of the incident light of Fig. 3b(b), then after reflection the components of the light along the prescribed positive P and S directions of the reflected light are $+1$ and -1, that is

$$\tilde{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

(4) The matrix $\tilde{Q}$ of the quarter-wave plate. Its fast axis is along the $-45°$ direction; set up a fast–slow axis coordinate system as in Fig. S3b.



题解图 3b

Labels: 快 *= fast (axis);* 慢 *= slow (axis);* 题解图 3b *= Fig. S3b.*

By definition, the vector representation of the polarization state obtained after the P component passes through the quarter-wave plate is the first column of $\tilde{Q}$; taking the S component through the quarter-wave plate gives the polarization-state vector that is its second column.

Assume the unknown 2×2 matrix

$$\tilde{Q} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \tag{13}$$

acts on the basis vector

$$\boldsymbol{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

so that

$$\tilde{Q}\boldsymbol{P} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{a} \\ \tilde{c} \end{pmatrix} \tag{15}$$

Below, $\hat{1}, \hat{2}$ denote the basis vectors along the P and S directions, and $\hat{1}', \hat{2}'$ the basis vectors along the fast and slow axes; in the $\hat{1}, \hat{2}$ basis,

$$\hat{1}' = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \hat{2}' = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{16}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\left[\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \frac{\sqrt{2}}{2}\hat{1}' + \frac{\sqrt{2}}{2}\hat{2}' \tag{17}$$

$$\boldsymbol{S} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\left[-\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = -\frac{\sqrt{2}}{2}\hat{1}' + \frac{\sqrt{2}}{2}\hat{2}' \tag{18}$$

The polarization form of P polarization (basis vector $\hat{1}$, horizontal polarization) after this quarter-wave plate (expressed in the original P–S basis) is

$$\begin{aligned} &\frac{\sqrt{2}}{2}e^{i\phi_{\text{fast}}}\hat{1}' + \frac{\sqrt{2}}{2}e^{i\phi_{\text{slow}}}\hat{2}' = \frac{1}{2}e^{i\phi_{\text{fast}}}(\hat{1}-\hat{2}) + \frac{1}{2}e^{i\phi_{\text{slow}}}(\hat{1}+\hat{2}) \\ &= \frac{1}{2}e^{i\phi_{\text{fast}}}(\hat{1}-\hat{2}) + \frac{1}{2}e^{i\phi_{\text{fast}}}e^{i(\phi_{\text{slow}}-\phi_{\text{fast}})}(\hat{1}+\hat{2}) \\ &= \frac{1}{2}e^{i\phi_{\text{fast}}}(1+i)\hat{1} + \frac{1}{2}e^{i\phi_{\text{fast}}}(i-1)\hat{2} = \frac{1}{2}e^{i\phi_{\text{fast}}}\begin{pmatrix} 1+i \\ i-1 \end{pmatrix} = \frac{1}{2}e^{i\phi_{\text{fast}}}\sqrt{2}\begin{pmatrix} e^{i\pi/4} \\ e^{i3\pi/4} \end{pmatrix} \\ &= \frac{\sqrt{2}}{2}e^{i\phi_{\text{fast}}}e^{i\frac{\pi}{4}}\begin{pmatrix} 1 \\ i \end{pmatrix} \end{aligned} \tag{19}$$

Similarly, for S polarization (basis vector $\hat{2}$, vertical polarization) after the quarter-wave plate,

$$\begin{aligned}
&-\frac{\sqrt{2}}{2}e^{i\phi_{\text{fast}}}\hat{1}'+\frac{\sqrt{2}}{2}e^{i\phi_{\text{slow}}}\hat{2}'=-\frac{1}{2}e^{i\phi_{\text{fast}}}(\hat{1}-\hat{2})+\frac{1}{2}e^{i\phi_{\text{slow}}}(\hat{1}+\hat{2})\\
&=-\frac{1}{2}e^{i\phi_{\text{fast}}}(\hat{1}-\hat{2})+\frac{1}{2}e^{i\phi_{\text{fast}}}e^{i(\phi_{\text{slow}}-\phi_{\text{fast}})}(\hat{1}+\hat{2})\\
&=\frac{1}{2}e^{i\phi_{\text{fast}}}(i-1)\hat{1}+\frac{1}{2}e^{i\phi_{\text{fast}}}(i+1)\hat{2}=\frac{1}{2}e^{i\phi_{\text{fast}}}\begin{pmatrix}i-1\\ i+1\end{pmatrix}=\frac{1}{2}e^{i\phi_{\text{fast}}}\sqrt{2}\begin{pmatrix}e^{i3\pi/4}\\ e^{i\pi/4}\end{pmatrix}\\
&=\frac{\sqrt{2}}{2}e^{i\phi_{\text{fast}}}e^{i\frac{\pi}{4}}\begin{pmatrix}i\\ 1\end{pmatrix}
\end{aligned}\tag{20}$$

Hence the matrix of the wave plate is

$$\tilde{Q}=\frac{\sqrt{2}}{2}e^{i\phi_{\text{fast}}}e^{i\frac{\pi}{4}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\tag{21}$$

In the standard form required by the problem,

$$a=\frac{\sqrt{2}}{2}e^{i(\phi_{\text{fast}}+\frac{\pi}{4})},\qquad b=i,\quad c=i,\quad d=1$$

The overall common phase factor in front is unimportant, so the following forms are all accepted as correct (that is, all three ways of writing it score):

$$\tilde{Q}=\frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\qquad\text{or}\qquad\tilde{Q}=\frac{\sqrt{2}}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\tag{22}$$

(5) For the linear polarizer whose transmission direction is horizontal, use the method of part (4), acting on the basis vectors.
Acting with $\tilde{L}_{\text{H}}$ on P polarization (basis vector 1, horizontal polarization) — i.e. P polarization passing through a horizontal polarizer — one still gets P polarization $\begin{pmatrix}1\\0\end{pmatrix}$; acting on S polarization (basis vector 2, vertical polarization) the result is 0, i.e. $\begin{pmatrix}0\\0\end{pmatrix}$. Therefore

$$\tilde{L}_{\text{H}}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\tag{23}$$

Similarly, for the linear polarizer transmitting the vertical direction,

$$\tilde{L}_{\text{V}}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\tag{24}$$

For the linear polarizer transmitting the $+45°$ direction, P light (horizontal polarization, amplitude 1) passing through it becomes $+45°$ linear polarization of amplitude $\frac{\sqrt{2}}{2}$; the matrix expression of the normalized $+45°$ linear-polarization basis vector (in the P–S basis) is

$$\boldsymbol{U}_{+45°}=\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 1\end{pmatrix}\tag{25}$$

Hence

$$\tilde{L}_{+45°}\boldsymbol{P}=\frac{\sqrt{2}}{2}\boldsymbol{U}_{+45°}=\frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}$$

and this is the first column of the matrix. Similarly, for S polarization passing through the 45° linear polarizer,

$$\tilde{L}_{+45^\circ}\boldsymbol{S} = \frac{\sqrt{2}}{2}\boldsymbol{U}_{+45^\circ} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

and this is the second column of the matrix. Therefore

$$\tilde{L}_{+45^\circ} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \tag{26}$$

(6) First write down the fields that reach detector A along path 1 and along path 2. Matrix form is used here. The light after the initial polarizer is called the incident field; in vector form it is (that is, vertical polarization of amplitude E_0)

$$\begin{pmatrix}0\\1\end{pmatrix}E_0$$

The field arriving at detector A through path 1:

$$\begin{aligned}\tilde{E}_{1\mathrm{A}} = &\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}r_\mathrm{P}e^{i\phi_\mathrm{P}} & 0\\0 & r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\frac{\sqrt{2}}{2}e^{i(\phi_\mathrm{fast}+\frac{\pi}{4})}\begin{pmatrix}1 & i\\i & 1\end{pmatrix}\\&\times\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}r_\mathrm{P}e^{i\phi_\mathrm{P}} & 0\\0 & r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}E_0\end{aligned} \tag{27}$$

Reading the above expression from right to left, the factors are the incident light, the reflection, the first beam splitter, the mirror, the quarter-wave plate, the second beam splitter and finally the vertical polarizer. Here and below r_P, r_S, t_P, t_S denote the absolute values of the reflection and transmission coefficients. Carrying out the calculation (the lines below show in turn the field after each element; in practice the calculation is quicker if one uses the fact that some of the matrices commute):

$$\begin{aligned}\tilde{E}_{1\mathrm{A}} &= \left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}r_\mathrm{P}e^{i\phi_\mathrm{P}} & 0\\0 & r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\right]\frac{\sqrt{2}}{2}e^{i(\phi_\mathrm{fast}+\frac{\pi}{4})}\begin{pmatrix}1 & i\\i & 1\end{pmatrix}\\&\quad\times\left[\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}r_\mathrm{P}e^{i\phi_\mathrm{P}} & 0\\0 & r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\right]\begin{pmatrix}0\\1\end{pmatrix}E_0\\&= \frac{\sqrt{2}}{2}e^{i(\phi_\mathrm{fast}+\frac{\pi}{4})}\begin{pmatrix}0 & 0\\0 & r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\begin{pmatrix}1 & i\\i & 1\end{pmatrix}\begin{pmatrix}r_\mathrm{P}e^{i\phi_\mathrm{P}}\ (\text{or}*) & 0\\0 & -r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}E_0\\&= -\frac{\sqrt{2}}{2}e^{i(\phi_\mathrm{fast}+\frac{\pi}{4})}r_\mathrm{S}e^{i\phi_\mathrm{S}}\begin{pmatrix}0 & 0\\0 & r_\mathrm{S}e^{i\phi_\mathrm{S}}\end{pmatrix}\begin{pmatrix}1 & i\\i & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}E_0\\&= -\frac{\sqrt{2}}{2}r_\mathrm{S}^2e^{i(2\phi_\mathrm{S}+\phi_\mathrm{fast}+\frac{\pi}{4})}\begin{pmatrix}0\\1\end{pmatrix}E_0\end{aligned} \tag{28}$$

For the field reaching detector A along path 2:

$$\tilde{E}_{2\mathrm{A}} = \left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}ir_\mathrm{P}e^{i\delta_\mathrm{P}} & 0\\0 & ir_\mathrm{S}e^{i\delta_\mathrm{S}}\end{pmatrix}\right]\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^3\begin{pmatrix}ir_\mathrm{P}e^{i\delta_\mathrm{P}} & 0\\0 & ir_\mathrm{S}e^{i\delta_\mathrm{S}}\end{pmatrix}\right]\begin{pmatrix}0\\1\end{pmatrix}E_0 \tag{29}$$

Reading from right to left, the above expression represents the incident light passing in turn through the transmission at the beam splitter, the 3 mirrors, the 45° linear polarizer, the transmission at the second beam splitter and the vertical polarizer in front of detector A. Calculating as before,

$$\tilde{E}_{2\mathrm{A}} = \frac{1}{2}r_\mathrm{S}^2e^{i2\delta_\mathrm{S}}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}E_0 = \frac{1}{2}r_\mathrm{S}^2e^{i(2\delta_\mathrm{S})}\begin{pmatrix}0\\1\end{pmatrix}E_0 \tag{30}$$

Taking into account the different optical paths travelled, the complete expressions of the fields arriving at detector A are

$$\tilde{E}_{1\mathrm{A}}(t) = -\frac{\sqrt{2}}{2} r_{\mathrm{S}}^2 e^{i(2\phi_{\mathrm{S}}+\phi_{\mathrm{fast}}+\frac{\pi}{4})} \begin{pmatrix} 0 \\ 1 \end{pmatrix} E_0 \mathrm{e}^{i[k(l_1+l_{\mathrm{A}})-\omega t]} \tag{31}$$

$$\tilde{E}_{2\mathrm{A}}(t) = \frac{1}{2} r_{\mathrm{S}}^2 e^{i(2\phi_{\mathrm{S}})} \begin{pmatrix} 0 \\ 1 \end{pmatrix} E_0 \mathrm{e}^{i[k(l_2+l_{\mathrm{A}})-\omega t]} \tag{32}$$

[a] where l_{A} is the optical path from beam splitter 2 to detector A. The intensity of the interference term is

$$I_{\mathrm{A}}^{\mathrm{int}} = 2 \cdot \frac{1}{2} \mathrm{Re}\left[\tilde{E}_{1\mathrm{A}}^*(t) \cdot \tilde{E}_{2\mathrm{A}}(t)\right] = \frac{\sqrt{2}}{4} r_{\mathrm{S}}^4 E_0^2 \cos\left[k(l_2 - l_1) + \frac{3}{4}\pi - \phi_{\mathrm{fast}}\right] \tag{33}$$

(A prefactor $\frac{\sqrt{2}}{2}$ in front of (33) is also accepted; and so on for the rest.)
From the numerical value given in the problem

$$r_{\mathrm{S}} = \frac{\sqrt{2}}{2}$$

one obtains

$$I_{\mathrm{A}}^{\mathrm{int}} = \frac{\sqrt{2}}{16} E_0^2 \cos\left[k(l_2 - l_1) + \frac{3}{4}\pi - \phi_{\mathrm{fast}}\right] \tag{34}$$

[Alternative solution:
If in this part of the calculation the quarter-wave plate is used without the common phase, i.e. if one takes

$$\tilde{Q} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

then in the calculation above there is no term $\left(\phi_{\mathrm{fast}} + \frac{\pi}{4}\right)$ in (28), and the final equation (34) becomes

$$I_{\mathrm{A}}^{\mathrm{int}} = \frac{\sqrt{2}}{16} E_0^2 \cos\left[k(l_2 - l_1) + \pi\right] \tag{33$'$}$$

This expression is equivalent to (34); on the surface there is a phase difference, but it is only a shift of the optical-path difference, which amounts to choosing a different origin for the optical path. **Hence (34) and (33′) are both accepted as correct.**]

The field arriving at detector B along path 1:

$$\begin{aligned} \tilde{E}_{1\mathrm{B}} = &\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} ir_{\mathrm{P}}e^{i\delta_{\mathrm{P}}} & 0 \\ 0 & ir_{\mathrm{S}}e^{i\delta_{\mathrm{S}}} \end{pmatrix} \frac{\sqrt{2}}{2} e^{i(\phi_{\mathrm{fast}}+\frac{\pi}{4})}\right] \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ &\times \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r_{\mathrm{P}}e^{i\delta_{\mathrm{P}}} & 0 \\ 0 & r_{\mathrm{S}}e^{i\delta_{\mathrm{S}}} \end{pmatrix}\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} E_0 \end{aligned} \tag{35}$$

Calculating (with the optical path included),

$$\begin{aligned} \tilde{E}_{1\mathrm{B}}(t) &= -ir_{\mathrm{P}}e^{i\delta_{\mathrm{P}}} \frac{\sqrt{2}}{2} e^{i(\phi_{\mathrm{fast}}+\frac{\pi}{4})} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} r_{\mathrm{S}}e^{i\delta_{\mathrm{S}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} E_0 \mathrm{e}^{i[k(l_1+l_B)-\omega t]} \\ &= -ir_{\mathrm{S}}r_{\mathrm{P}} \frac{\sqrt{2}}{2} e^{i(\delta_{\mathrm{S}}+\delta_{\mathrm{P}}+\phi_{\mathrm{fast}}+\frac{\pi}{4})} \begin{pmatrix} i \\ 0 \end{pmatrix} E_0 \mathrm{e}^{i[k(l_1+l_B)-\omega t]} \\ &= \frac{\sqrt{2}}{2} r_{\mathrm{S}}r_{\mathrm{P}} e^{i(\phi_{\mathrm{S}}+\phi_{\mathrm{P}}+\phi_{\mathrm{fast}}+\frac{\pi}{4})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} E_0 \mathrm{e}^{i[k(l_1+l_B)-\omega t]} \end{aligned} \tag{36}$$

where l_{B} is the optical path from beam splitter 2 to detector B. The field arriving at B along path 2:

$$\begin{aligned}\tilde{E}_{2\mathrm{B}}(t) = &\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}r_{\mathrm{P}}e^{i\delta_{\mathrm{P}}} & 0\\ 0 & r_{\mathrm{S}}e^{i\delta_{\mathrm{S}}}\end{pmatrix}\right]\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &\times\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^3\begin{pmatrix}ir_{\mathrm{P}}e^{i\delta_{\mathrm{P}}} & 0\\ 0 & ir_{\mathrm{S}}e^{i\delta_{\mathrm{S}}}\end{pmatrix}\right]\begin{pmatrix}0\\ 1\end{pmatrix}E_0\mathrm{e}^{i[k(l_2+l_B)-\omega t]}\end{aligned} \tag{37}$$

$$\begin{aligned}&= -\frac{1}{2}ir_{\mathrm{P}}r_{\mathrm{S}}e^{i(\phi_{\mathrm{S}}+\phi_{\mathrm{P}})}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}E_0\mathrm{e}^{i[k(l_2+l_B)-\omega t]}\\ &= \frac{1}{2}r_{\mathrm{P}}r_{\mathrm{S}}e^{i(\phi_{\mathrm{S}}+\phi_{\mathrm{P}}+\frac{3\pi}{2})}\begin{pmatrix}1\\ 0\end{pmatrix}E_0\mathrm{e}^{i[k(l_2+l_B)-\omega t]}\end{aligned} \tag{38}$$

The interference term of the intensity detected at B:

$$I_{\mathrm{B}}^{\mathrm{int}} = 2\cdot\frac{1}{2}\mathrm{Re}\big[\tilde{E}_{1\mathrm{B}}^*(t)\cdot\tilde{E}_{2\mathrm{B}}(t)\big] = \frac{\sqrt{2}}{4}r_{\mathrm{S}}^2r_{\mathrm{P}}^2E_0^2\cos\Big[k(l_2-l_1)-\frac{3}{4}\pi-\phi_{\mathrm{fast}}\Big] \tag{39}$$

$$= \frac{\sqrt{2}}{16}E_0^2\cos\Big[k(l_2-l_1)-\frac{3}{4}\pi-\phi_{\mathrm{fast}}\Big] \tag{40}$$

[Alternative solution:
As in the discussion of (34′), if the quarter-wave plate matrix used does not include $\big(\phi_{\mathrm{fast}}+\frac{\pi}{4}\big)$, then

$$\begin{aligned}I_{\mathrm{B}}^{\mathrm{int}} &= 2\,\mathrm{Re}\big(\tilde{E}_{1\mathrm{B}}^*\cdot\tilde{E}_{2\mathrm{B}}\big) = \frac{\sqrt{2}}{2}r_{\mathrm{S}}r_{\mathrm{P}}t_{\mathrm{S}}t_{\mathrm{P}}E_0^2\cos\Big[k(l_2-l_1)-\frac{1}{2}\pi\Big]\\ &= \frac{\sqrt{2}}{16}E_0^2\cos\Big[k(l_2-l_1)-\frac{1}{2}\pi\Big]\end{aligned} \tag{40′}$$

(40) and (40′) are equivalent.]

In summary: (34) and (40) are one correct set of results for the interference terms at detectors A and B; (33′) and (40′) are an equivalent set.
The intensity at detector A, equation (34),

$$I_{\mathrm{A}}^{\mathrm{int}} = \frac{\sqrt{2}}{8}E_0^2\cos\Big[k(l_2-l_1)+\frac{3}{4}\pi-\phi_{\mathrm{fast}}\Big]$$

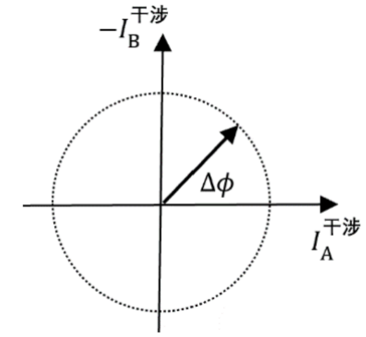
[b] Denoting the adjustable phase difference in it by $\Delta\phi$, we have

$$I_{\mathrm{A}}^{\mathrm{int}} = \frac{\sqrt{2}}{8}E_0^2\cos(\Delta\phi) \tag{41}$$

and correspondingly the intensity at B, equation (40),

$$I_{\mathrm{B}}^{\mathrm{int}} = \frac{\sqrt{2}}{8}E_0^2\cos\Big(\Delta\phi+\frac{\pi}{2}\Big) = -\frac{\sqrt{2}}{8}E_0^2\sin(\Delta\phi) \tag{42}$$

Writing the intensities using (34)(40), or using (33′)(40′), gives the same results.



题解图 3c

Labels: 干涉 *= interference (term), written* I^int *here;* 题解图 3c *= Fig. S3c.*

(7) Since detector A gives $\cos(\Delta\phi)$ and B gives $\sin(\Delta\phi)$, the phase difference can be determined uniquely. *(∗)*

In an actual experiment one may:

a) record $I_\mathrm{A}^\mathrm{int}$, $-I_\mathrm{B}^\mathrm{int}$ and plot Fig. S3c (or feed them into the X and Y inputs of an oscilloscope to obtain a Lissajous figure); *(∗∗)*

or

b) obtain $\tan\Delta\phi$ from the ratio of $I_\mathrm{A}^\mathrm{int}$ and $I_\mathrm{B}^\mathrm{int}$, and then obtain the sign of the angle from the sign of I_B. *(∗ ∗ ∗)*

Marking scheme.

This problem carries 50 points.

Part (1) 6 points: (∗) and Fig. S3a 1 point each, (1) or (2) 1 point, (3) or (4) 1 point, (5)(8) 1 point each;

Part (2) 2 points: (10)(11) 1 point each;

Part (3) 2 points: (12) 2 points;

Part (4) 8 points: (17)(18)(19)(20) 1 point each, (21) 4 points;

Part (5) 5 points: (23)(24) 1 point each, (26) 3 points;

Part (6) 24 points: (27)(29)(35)(37) 3 points each, (33)(40) 2 points each, (41)(42) 4 points each;

Part (7) 3 points: (∗) or (∗∗) or (∗ ∗ ∗) 3 points.

[a] *Translator's note:* The source prints the exponent of (32) as $e^{ik[(l_2+l_\mathrm{A})-\omega t]}$; the form given here matches (31).

[b] *Translator's note:* As printed, the prefactor here and in (41)(42) is $\frac{\sqrt{2}}{8}$, whereas (34) and (40) give $\frac{\sqrt{2}}{16}$. The factor $\frac{\sqrt{2}}{16}$ of (34)(40) is the consistent one.

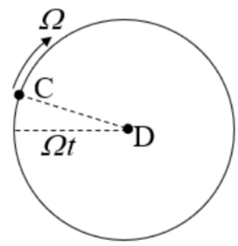
Problem 4 (two-wheeled self-balancing vehicle: cornering, accelerating and small oscillations)

Solution

(1)

(1.1) Let D be the centre of the circular trajectory of C, and take the line CD as a rotating reference frame s [this frame may be regarded as a rigid body rigidly attached to the line CD and rotating about the vertical axis through D with angular velocity Ω (but not corotating with CD itself)]; a top view is shown in Fig. S1a. In this frame the magnitude of the torque about the end point C of the rod produced by the centrifugal inertial force acting on the rod is

$$\int_0^L \mathrm{d}l\,\frac{M}{L}\cdot\Omega^2(R-l\sin\phi)\cdot l\cos\phi = M\Omega^2 L\cos\phi\cdot\left(\frac{R}{2}-\frac{L}{3}\sin\phi\right) \tag{1}$$



题解图 1a

Labels: 题解图 1a *= Fig. S1a (top view of part (1.1)).*

From the condition of torque balance of the rod about its end point C in s,

$$M\Omega^2 L\cos\phi\cdot\left(\frac{R}{2}-\frac{L}{3}\sin\phi\right) = Mg\cdot\frac{L}{2}\sin\phi \tag{2}$$

whence

$$R = \frac{2L}{3}\sin\phi + \frac{g}{\Omega^2}\tan\phi \tag{3}$$

(1.2) Since the vertical component of the total momentum of the balancing vehicle is always zero,

$$N_{\text{left}} + N_{\text{right}} = Mg \tag{4}$$

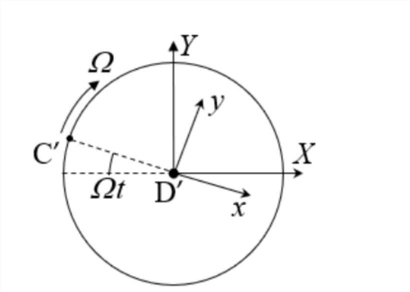
We now obtain the expression for the difference of these two normal forces from their torques. The angular momentum of the rod is found below by two methods.

Method 1: calculation in the ground frame. Since the mass of the wheels may be neglected in this part, only the angular momentum of the rod is needed. Let the vertical projections of D and C on the ground be D′ and C′, and set up a coordinate system $S(\mathrm{D}'\text{-}XYZ)$ fixed to the ground with origin D′, as shown in Fig. S1b. The three coordinate axes of the frame s ($\mathrm{D}'\text{-}xyz$) still point along the directions stated in the problem, but the origin O is taken to coincide with D′; the frame s rotates about the origin D′(O) relative to the ground with angular velocity $\boldsymbol{\Omega} = -\Omega\boldsymbol{k}$, see Fig. S1b. The frame s is at rest relative to the rotating frame s of part (1.1), which is why the same symbol s is used. In the ground frame the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the axes of the frame s can be written in terms of their components in the frame S as

$$\boldsymbol{i} = [\cos(\Omega t), -\sin(\Omega t), 0]$$

$$\boldsymbol{j} = [\sin(\Omega t), \cos(\Omega t), 0]$$

$$\boldsymbol{k} = (0, 0, 1)$$



题解图 1b

Labels: 题解图 1b = *Fig. S1b (the ground frame S and the rotating frame s).*

The first of these expressions gives the components, in the frame S, of the unit vector $\boldsymbol{i}$ of the x axis of the co-moving frame s; the meaning of the other two is likewise obvious. Note that

$$\frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t} = -\Omega\boldsymbol{j}, \qquad \frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}t} = \Omega\boldsymbol{i}, \qquad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = 0 \tag{5}$$

We now calculate the angular momentum of the rod about D'**.**

Method 1a: decompose the total angular momentum into the angular momentum of the centre of mass about the origin D' **plus the angular momentum of the rigid body about its centre of mass.**

The principal axes of the rod are $\boldsymbol{b}_1 = \boldsymbol{i}\cos\phi - \boldsymbol{k}\sin\phi$, $\boldsymbol{b}_2 = \boldsymbol{j}$, $\boldsymbol{b}_3 = \boldsymbol{i}\sin\phi + \boldsymbol{k}\cos\phi$, and the moments of inertia along the principal axes are $I_{\text{rod},1} = I_{\text{rod},2} = \dfrac{1}{12}ML^2$, $I_{\text{rod},3} = 0$. The angular momentum of the rod about its centre of mass is

$$\begin{aligned}\boldsymbol{L}_{\text{rod}} &= I_{\text{rod},1}(\boldsymbol{\Omega}\cdot\boldsymbol{b}_1)\boldsymbol{b}_1 + I_{\text{rod},2}(\boldsymbol{\Omega}\cdot\boldsymbol{b}_2)\boldsymbol{b}_2 + I_{\text{rod},3}(\boldsymbol{\Omega}\cdot\boldsymbol{b}_3)\boldsymbol{b}_3 = \frac{1}{12}ML^2\big[(-\Omega\boldsymbol{k})\cdot\boldsymbol{b}_1\big]\boldsymbol{b}_1 \\ &= \frac{1}{12}ML^2(\Omega\sin\phi)(\boldsymbol{i}\cos\phi - \boldsymbol{k}\sin\phi)\end{aligned} \tag{6}$$

The total angular momentum of the rod is

$$\begin{aligned}\boldsymbol{L} &= \boldsymbol{L}_{\text{c.m.}} + \boldsymbol{L}_{\text{rod}} \\ &= -M\Big(R - \frac{L}{2}\sin\phi\Big)\Omega\boldsymbol{j} \times \Big[-\Big(R - \frac{L}{2}\sin\phi\Big)\boldsymbol{i} + \Big(r + \frac{L}{2}\cos\phi\Big)\boldsymbol{k}\Big] + \boldsymbol{L}_{\text{rod}} \\ &= -M\Omega\Big(R - \frac{L}{2}\sin\phi\Big)^2\boldsymbol{k} - M\Omega\Big(R - \frac{L}{2}\sin\phi\Big)\Big(r + \frac{L}{2}\cos\phi\Big)\boldsymbol{i} \\ &\quad + \frac{1}{12}ML^2(\Omega\sin\phi)(\boldsymbol{i}\cos\phi - \boldsymbol{k}\sin\phi) \\ &= -M\Omega\Big(R^2 - RL\sin\phi + \frac{L^2}{3}(\sin\phi)^2\Big)\boldsymbol{k} \\ &\quad - M\Omega\Big(Rr + \frac{RL\cos\phi}{2} - \frac{rL\sin\phi}{2} - \frac{L^2}{3}\sin\phi\cos\phi\Big)\boldsymbol{i}\end{aligned} \tag{7}$$

[**Method 1b: obtain the total angular momentum by integration.** Consider the mass element of the rod at a distance l from the point C; its coordinates in the frame s are

$$-(R - l\sin\phi)\boldsymbol{i} + (r + l\cos\phi)\boldsymbol{k}$$

and its velocity is

$$(R - l\sin\phi)\Omega\boldsymbol{j}$$

Hence the angular momentum of the whole rod about D′ is

$$\begin{aligned}\boldsymbol{L} &= -\int_0^L \frac{M}{L}\mathrm{d}l\cdot(R-l\sin\phi)\Omega\boldsymbol{j}\times\left[-(R-l\sin\phi)\boldsymbol{i}+(r+l\cos\phi)\boldsymbol{k}\right]\\ &= -M\Omega\Big(R^2-RL\sin\phi+\frac{L^2}{3}(\sin\phi)^2\Big)\boldsymbol{k}\\ &\quad -M\Omega\Big(Rr+\frac{RL\cos\phi}{2}-\frac{rL\sin\phi}{2}-\frac{L^2}{3}\sin\phi\cos\phi\Big)\boldsymbol{i}\end{aligned} \tag{6′(7′)}$$

The result is the same as (7) of Method 1a.]
The rate of change of the angular momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = M\Big(Rr+\frac{RL\cos\phi}{2}-\frac{rL\sin\phi}{2}-\frac{L^2}{3}\sin\phi\cos\phi\Big)\Omega^2\boldsymbol{j}$$

Note that the friction force can produce no torque about the origin D′; hence the condition $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}=$ total external torque reads

$$\begin{aligned}&M\Omega^2\Big(Rr+\frac{RL\cos\phi}{2}-\frac{rL\sin\phi}{2}-\frac{L^2}{3}\sin\phi\cos\phi\Big)\\ &\quad = N_{\text{left}}\Big(R+\frac{d}{2}\Big)+N_{\text{right}}\Big(R-\frac{d}{2}\Big)-Mg\Big(R-\frac{L\sin\phi}{2}\Big)\end{aligned} \tag{8}$$

[**Method 2: calculation in the rotating frame.** In the frame s that rotates together with the centre of mass of the balancing vehicle, the centre line of the axle and the rod are both at rest, while the ground and the wheels move relative to the vehicle body. The problem then becomes a problem of statics for the rod.
We still use the coordinate system and the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of Method 1. Note that in the rotating frame $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are constant vectors. The angular-velocity vector of the rotating frame relative to the stationary frame is $(-\Omega\boldsymbol{k})$.
Consider the torque balance of the whole balancing vehicle about the point D′; in this way the friction force need not be found.
Similarly to the calculation in part (1.1), the magnitude of the torque of the inertial force on the rod about the point D′ is

$$\begin{aligned}\tau_{\text{inertial force, rod}} &= \int_0^L \mathrm{d}l\,\frac{M}{L}\cdot\Omega^2(R-l\sin\phi)\cdot(l\cos\phi+r)\\ &= M\Omega^2\cdot\Big(\frac{RL}{2}\cos\phi-\frac{L^2}{3}\sin\phi\cos\phi+Rr-\frac{rL}{2}\sin\phi\Big)\end{aligned} \tag{9}$$

The condition of torque balance of the rod is

$$\begin{aligned}&N_{\text{left}}\Big(R+\frac{d}{2}\Big)+N_{\text{right}}\Big(R-\frac{d}{2}\Big)-Mg\Big(R-\frac{L\sin\phi}{2}\Big)\\ &\quad = M\Omega^2\cdot\Big(\frac{RL}{2}\cos\phi-\frac{L^2}{3}\sin\phi\cos\phi+Rr-\frac{rL}{2}\sin\phi\Big)\end{aligned} \tag{10}$$

The result is the same as that of Method 1.]
Using (3)(4)(8),

$$N_{\text{left}}-N_{\text{right}} = M\cdot\frac{2r}{d}\left[g\tan\phi+\frac{L\Omega^2\sin\phi}{6}\right] \tag{11}$$

From (4)(11),

$$N_{\text{left}|\text{right}} = \frac{M}{2}g\pm M\frac{r}{d}\left[g\tan\phi+\frac{L\Omega^2\sin\phi}{6}\right] \tag{12}$$

(2) From the theorem of motion of the centre of mass, the total friction force f exerted by the ground on the two wheels of the balancing vehicle satisfies

$$f = (M + 2m)a(\theta) \tag{13}$$

The moment of inertia of each wheel about the axle is

$$I = mr^2 \tag{14}$$

By the statement of the problem the wheels roll without slipping on the ground, so the angular acceleration of a wheel about the axle is

$$\beta = \frac{a(\theta)}{r} \tag{15}$$

By the angular-momentum theorem, the torque $\tau(\theta)$ that the motor inside the vehicle delivers to the two wheels about the axle satisfies

$$\tau(\theta) - fr = 2I\beta \tag{16}$$

From (13)(15)(16),

$$\tau(\theta) = \Big(M + 2m + \frac{2I}{r^2}\Big)ra(\theta) \tag{17}$$

By the statement of the problem the inclination angle θ of the rod stays constant, so f and $\tau(\theta)$ also stay constant, and the point C moves in a straight line with constant acceleration. In the translating accelerated frame moving with C the acceleration of the rod is zero, so the torques on the rod about its end point C satisfy the equilibrium condition

$$Mg\frac{L}{2}\sin\theta - Ma(\theta)\frac{L}{2}\cos\theta + \tau'(\theta) = 0 \tag{18}$$

where $\tau'(\theta)$ and $\tau(\theta)$ are an action–reaction pair of torques,

$$\tau'(\theta) = -\tau(\theta) \tag{19}$$

From (17)(18)(19),

$$a(\theta) = \frac{MLg\sin\theta}{ML\cos\theta + 2(M+4m)r} \tag{20}$$

$$\tau(\theta) = \frac{(M+4m)rMLg\sin\theta}{ML\cos\theta + 2(M+4m)r} \tag{21}$$

(3) Take the origin O of the coordinate system $s'(\mathrm{O}xyz)$ to be the vertical projection on the ground of the initial position of the point C. At time t the Cartesian coordinates of C are $(0, y(t), r)$, and the position of the centre of mass of the rod at time t is

$$\Big(0,\ y(t) + \frac{L}{2}\sin\theta(t),\ r + \frac{L}{2}\cos\theta(t)\Big) \tag{22}$$

[One may either derive the final equation of motion first and then expand in small quantities, or keep only first-order small quantities throughout; here θ, y and their derivatives of all orders, as well as τ, are all first-order small quantities.]

Method 1. In the translating accelerated frame moving with C, the rotational equation of the vehicle-body rod about the point C is

$$\frac{1}{3}M'L^2\ddot{\theta} = M'g\frac{L}{2}\theta(t) - M'\ddot{y}\frac{L}{2} - \tau(\theta) \tag{23}$$

whence

$$\ddot{y} = g\theta - \frac{2L\ddot{\theta}}{3} - \frac{2\tau(\theta)}{M'L} \tag{24}$$

From (22), the y component of the acceleration of the centre of mass of the rod is

$$a_y(\theta) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[y(t) + \frac{L}{2}\sin\theta(t)\right] = \ddot{y} + \frac{L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = \ddot{y} + \frac{L}{2}\ddot{\theta} \tag{25}$$

The total friction force f exerted by the ground on the two wheels of the balancing vehicle is

$$f = (M' + 2m)\ddot{y} + \frac{M'L}{2}\ddot{\theta} \tag{26}$$

From the condition of pure rolling of the wheels on the ground, the angular velocity of a wheel about the axle is

$$\omega = \frac{\dot{y}}{r} \tag{27}$$

and the angular acceleration is

$$\beta = \frac{\ddot{y}}{r} \tag{28}$$

By the angular-momentum theorem, the torque $\tau(\theta)$ that the motor inside the vehicle delivers to the two wheels about the axle satisfies

$$2I\beta = \tau(\theta) - fr \tag{29}$$

Equation (29) has the same form as (16), but it is obtained under different conditions. From (26)(28)(29),

$$M'_{\text{tot}}\, r \cdot \ddot{y} = \tau(\theta) - \frac{M'Lr}{2}\ddot{\theta} \tag{30}$$

where

$$M'_{\text{tot}} = M' + 2m + \frac{2I}{r^2} = M' + 4m \tag{31}$$

Substituting (24) into (30) to eliminate $\ddot{y}$,

$$-\Big(\frac{2}{3}M'_{\text{tot}} - \frac{M'}{2}\Big)Lr\ddot{\theta} = -M'_{\text{tot}}rg\theta + \Big(M'_{\text{tot}} \cdot \frac{2r}{M'L} + 1\Big)\tau(\theta) \tag{32}$$

Substituting (21) into (32),

$$-\Big(\frac{2}{3}M'_{\text{tot}} - \frac{M'}{2}\Big)L \cdot \ddot{\theta} = g \cdot \left(\frac{M_{\text{tot}} - M}{M_{\text{tot}} + \dfrac{ML}{2r}} \cdot \frac{M_{\text{tot}}}{M'} + 1\right)(M - M') \cdot \theta \tag{33}$$

[**Method 2.** In the translating accelerated frame moving with C, the rotational equation of the vehicle-body rod about the point C is

$$\frac{1}{3}M'L^2\ddot{\theta} = M'g\frac{L}{2}\sin\theta(t) - M'\ddot{y}\frac{L}{2}\cos\theta(t) - \tau(\theta) \tag{23$'$}$$

From (23′),

$$\ddot{y} = g\tan\theta - \frac{2}{3}\frac{L\ddot{\theta}}{\cos\theta} - \frac{2\tau(\theta)}{M'L\cos\theta} \tag{24$'$}$$

From (10), the y component of the acceleration of the centre of mass of the rod is

$$a_y(\theta) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[y(t) + \frac{L}{2}\sin\theta(t)\right] = \ddot{y} + \frac{L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \tag{25$'$}$$

The total friction force f exerted by the ground on the two wheels is

$$\begin{aligned} f &= 2m\ddot{y} + M' a_y(\theta) = 2m\ddot{y} + M'\Big[\ddot{y} + \frac{L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\Big] \\ &= (M'+2m)\ddot{y} + \frac{M'L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \end{aligned} \tag{26'}$$

Likewise, by the statement of the problem the wheels roll without slipping on the ground, so the angular velocity of a wheel about the axle is

$$\omega = \frac{\dot{y}}{r} \tag{27'}$$

and the angular acceleration is

$$\beta = \frac{\ddot{y}}{r} \tag{28'}$$

By the angular-momentum theorem, the torque $\tau(\theta)$ delivered by the motor to the two wheels about the axle satisfies

$$2I\beta = \tau(\theta) - fr \tag{29'}$$

From (26′)(28′)(29′),

$$M'_{\text{tot}} r\ddot{y} = \tau(\theta) - \frac{M'Lr}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \tag{30'}$$

where

$$M'_{\text{tot}} = M' + 2m + \frac{2I}{r^2} = M' + 4m \tag{31'}$$

Substituting (24′) into (30′) to eliminate $\ddot{y}$,

$$M'_{\text{tot}} r\left(g\tan\theta - \frac{2L\ddot{\theta}}{3\cos\theta} - \frac{2\tau(\theta)}{M'L\cos\theta}\right) = \tau(\theta) - \frac{M'Lr}{2}\big(\cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2\big)$$

Simplifying gives the differential equation for θ:

$$\begin{aligned} &-\Big(M'_{\text{tot}}\cdot\frac{2}{3\cos\theta} - \frac{M'\cos\theta}{2}\Big)Lr\ddot{\theta} \\ &= -M'_{\text{tot}} rg\tan\theta + \Big(M'_{\text{tot}}\cdot\frac{2r}{M'L\cos\theta} + 1\Big)\tau(\theta) + \frac{M'Lr}{2}\sin\theta\cdot\dot{\theta}^2 \end{aligned} \tag{32'}$$

Substituting (21) into (32′),

$$\begin{aligned} &-\Big(M'_{\text{tot}}\cdot\frac{2}{3\cos\theta} - \frac{M'\cos\theta}{2}\Big)L\ddot{\theta} \\ &= -M'_{\text{tot}} g\tan\theta - \frac{\big(M'L\cos\theta + 2M'_{\text{tot}} r\big)M_{\text{tot}} Mg\tan\theta}{\big(ML\cos\theta + 2M_{\text{tot}} r\big)M'} + \frac{M'L}{2}\sin\theta\cdot\dot{\theta}^2 \end{aligned}$$

After simplification,

$$\begin{aligned} &-\Big(M'_{\text{tot}}\cdot\frac{2}{3\cos\theta} - \frac{M'\cos\theta}{2}\Big)L\ddot{\theta} \\ &= M_{\text{tot}} g\tan\theta\cdot\left(\frac{M_{\text{tot}} - M}{M_{\text{tot}} + \dfrac{ML\cos\theta}{2r}}\cdot\frac{1}{M'} + \frac{1}{M_{\text{tot}}}\right)(M - M') + \frac{M'L}{2}\sin\theta\cdot\dot{\theta}^2 \end{aligned}$$

In the small-quantity approximation, keeping the above equation to first order,

$$-\Big(\frac{2}{3}M'_{\text{tot}}-\frac{M'}{2}\Big)L\cdot\ddot{\theta}=g\cdot\left(\frac{M_{\text{tot}}-M}{M_{\text{tot}}+\dfrac{ML}{2r}}\cdot\frac{M_{\text{tot}}}{M'}+1\right)(M-M')\cdot\theta \tag{33$'$}$$

which is the same as (33).]
Since

$$\frac{2}{3}M'_{\text{tot}}-\frac{M'}{2}>0$$

and

$$\frac{M_{\text{tot}}-M}{M_{\text{tot}}+\dfrac{ML}{2r}}>0$$

the condition for small oscillations is

$$M>M' \tag{34}$$

In that case the equation is the equation of simple harmonic motion, and the frequency of the small oscillations is

$$\begin{aligned}\omega&=\sqrt{\frac{\left(\dfrac{M_{\text{tot}}-M}{M_{\text{tot}}+\dfrac{ML}{2r}}\cdot\dfrac{M_{\text{tot}}}{M'}+1\right)(M-M')}{\dfrac{2}{3}M'_{\text{tot}}-\dfrac{M'}{2}}}\cdot\sqrt{\frac{g}{L}}\\&=\sqrt{\frac{\left(\dfrac{4m}{M+4m+\dfrac{ML}{2r}}\cdot\dfrac{M+4m}{M'}+1\right)(M-M')}{\dfrac{1}{6}M'+\dfrac{8m}{3}}}\cdot\sqrt{\frac{g}{L}}\end{aligned} \tag{35}$$

Here

$$M_{\text{tot}}=M+2m+\frac{2I}{r^2}=M+4m$$

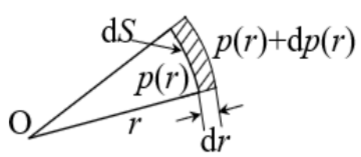
Marking scheme.
This problem carries 70 points.
Part (1)
(1.1) 7 points: (1) 2 points, (2) 3 points, (3) 2 points;
(1.2) 16 points: (4) 2 points, (5) 3 points, (6)(7) 2 points each, (8) 3 points, (11)(12) 2 points each;
Part (2) 16 points: (13) 2 points, (14)(15) 1 point each, (16) 3 points, (17)(18) 2 points each, (19) 1 point, (20)(21) 2 points each;
Part (3) 31 points: (22) 3 points, (23) 4 points, (24) 1 point, (25) 3 points, (26) 3 points, (27)(28)(29) 1 point each, (30) 3 points, (32)(33) 2 points each, (34) 3 points, (35) 4 points.
(Primed equations are marked in the same way as the corresponding unprimed ones.)

Problem 5 (structure of a star: hydrostatic equilibrium, radiation pressure and the Eddington limit)

Solution

(1) As in Fig. S5a, take an area element $\mathrm{d}S$ at radius r perpendicular to the radial direction (only the side view of the area element is drawn in the figure), and construct a cone with apex O and cross-section $\mathrm{d}S$. For a star with a spherically symmetric mass distribution, when the height of the lateral surface of the cone increases by $\mathrm{d}r$ the cross-section of the cone increases from $\mathrm{d}S$ to $\mathrm{d}S'$ (not marked in the figure). Inside this cone, the volume element of the region between the two cross-sections $\mathrm{d}S$ and $\mathrm{d}S'$ is $\mathrm{d}r\,\mathrm{d}S$ (its side surface is shown hatched). Taking the outward radial direction as positive, the force-balance equation for the matter inside this volume element is

$$-\mathrm{d}p(r)\mathrm{d}S - \frac{Gm(r)\rho(r)\mathrm{d}r\,\mathrm{d}S}{r^2} = 0 \tag{1}$$



题解图 5a

Labels: 题解图 5a = *Fig. S5a.*

Here $\rho(r)$ is the density of the stellar matter at r, and $m(r)$ is

$$m(r) = \int_0^r \mathrm{d}r'\, 4\pi r'^2 \rho(r') \tag{2}$$

From the definition of $\rho(r)$, or by differentiating both sides of (2) with respect to r,

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r) \tag{3}$$

From (1)(3), in the equilibrium state $m(r)$ and $p(r)$ satisfy

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} + \frac{Gm(r)}{4\pi r^4}\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 0 \tag{4}$$

Substituting (3) into (4) gives

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2} \tag{5}$$

Equations (3) and (5) are the differential equations satisfied by $m(r)$ and $p(r)$ in the equilibrium state.

(2) By direct computation, using (4),

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[p(r) + \frac{Gm^2(r)}{8\pi r^4}\right] = \frac{\mathrm{d}p(r)}{\mathrm{d}r} + \frac{Gm(r)}{4\pi r^4}\frac{\mathrm{d}m(r)}{\mathrm{d}r} - \frac{Gm^2(r)}{2\pi r^5} = -\frac{Gm^2(r)}{2\pi r^5} < 0 \tag{6}$$

From (2),

$$\lim_{r\to 0} m(r) \propto \rho(0) r^3 \tag{7}$$

and hence, from (7),

$$\lim_{r\to 0} \frac{m^2(r)}{r^4} \propto r^2 \to 0 \tag{8}$$

At the surface of the star, $r = R$,

$$p(R) = 0 \tag{9}$$

$$m(R) = M \tag{10}$$

From (6),

$$\left[p(r) + \frac{Gm^2(r)}{8\pi r^4}\right]_{r<R} > p(R) + \frac{Gm^2(R)}{8\pi R^4} \tag{11}$$

Taking $r \to 0$ on the left of (11) and using (8)(9)(10),

$$p(0) \geq \frac{GM^2}{8\pi R^4} \tag{12}$$

The minimum value (lower bound) of the pressure at the centre of the star is the right-hand side of (12).

(3) The gravitational potential energy of the star is

$$E_{\text{grav}} = -\int_0^R \frac{Gm(r)}{r} 4\pi r^2 \rho(r)\, \mathrm{d}r \tag{13}$$

From (5)(13),

$$E_{\text{grav}} = \int_0^R 4\pi r^3 \mathrm{d}p(r) \tag{14}$$

Integrating the right-hand side by parts,

$$E_{\text{grav}} = 4\pi r^3 p(r)\Big|_0^R - 12\pi \int_0^R p(r) r^2 \mathrm{d}r = -12\pi \int_0^R p(r) r^2 \mathrm{d}r \tag{15}$$

Because $r = 0$ and $p(R) = 0$, the first term after the first equals sign above vanishes.

(4) The condition for the star not to fall apart is that its total energy satisfies

$$E_{\text{tot}} = E_{\text{int}} + E_{\text{grav}} < 0, \tag{16}$$

where the total potential energy E_{grav} is given by (15) and the total internal energy E_{int} is

$$E_{\text{int}} = \int_0^R u(r) 4\pi r^2\, \mathrm{d}r \tag{17}$$

From the condition given in the problem,

$$(\gamma - 1)u(r) = p(r)$$

and from (17),

$$E_{\text{int}} = \frac{1}{\gamma - 1}\int_0^R p(r) 4\pi r^2 \mathrm{d}r \tag{18}$$

From (15)(16)(18), the total energy of the star is

$$E_{\text{total}} = E_{\text{int}} + E_{\text{grav}} = -\frac{1}{3(\gamma-1)} E_{\text{grav}} + E_{\text{grav}} = \frac{\gamma - \frac{4}{3}}{\gamma - 1} E_{\text{grav}} \tag{19}$$

If the star is stable and will not fall apart, then (16) must be satisfied. Noting that $E_{\text{grav}} < 0$, equation (19) shows that γ must satisfy

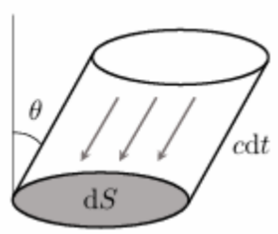
$$\gamma > \frac{4}{3} \tag{20}$$

(5) The internal-energy density u_{rad} of the photon gas can be obtained by integrating the Planck black-body radiation energy spectrum over frequency. From the formula given in the problem,

$$u_{\text{rad}} = \int_0^\infty \mathrm{d}\nu\, \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1} = \int_0^\infty \mathrm{d}(h\nu)\,\frac{8\pi}{c^3h^3}\frac{(h\nu)^3}{e^{h\nu/(kT)}-1} = \frac{8\pi^5k^4T^4}{15h^3c^3} \tag{21}$$

To compute the light pressure of the photon gas, take in the radiation field an oblique cylindrical volume element as shown in Fig. S5b, where $\mathrm{d}S$ is the area of the base of the volume element and $c\,\mathrm{d}t$ is the length of the generator of its lateral surface. If the light pressure of the photon gas is p_{rad}, the impulse it delivers to the area element $\mathrm{d}S$ in the time $\mathrm{d}t$ is

$$\mathrm{d}I = p_{\text{rad}}\mathrm{d}S\mathrm{d}t \tag{22}$$



题解图 5b

Labels: 题解图 5b = *Fig. S5b.*

Take as polar axis the normal at the centre of the base of the volume element (positive downwards, see Fig. S5b). In the time $\mathrm{d}t$ the projection, along the direction of $\mathrm{d}S$, of the momentum transferred to the area element $\mathrm{d}S$ by one photon of frequency ν is

$$\mathrm{d}p_\nu = \left[\frac{h\nu}{c} - \left(-\frac{h\nu}{c}\right)\right]\cos\theta = \frac{2\cos\theta}{c}h\nu \tag{23}$$

which is proportional to the energy $h\nu$ of a photon of frequency ν. The sum over all frequencies of the projections, along the direction of $\mathrm{d}S$, of the momenta transferred to $\mathrm{d}S$ by the photons in the time $\mathrm{d}t$ is

$$\mathrm{d}p = \frac{2\cos\theta}{c}\frac{1}{4\pi}\int_0^\infty \mathrm{d}\nu\,\frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1} = \frac{2\cos\theta}{c}\frac{u_{\text{rad}}}{4\pi} \tag{24}$$

where $\dfrac{u_{\text{rad}}}{4\pi}$ is the internal-energy density of the photon gas per unit solid angle. The impulse I is in fact the sum of the projections, along the direction of $\mathrm{d}S$, of the total momentum transferred by the photons to the area element $\mathrm{d}S$ in the time $\mathrm{d}t$:

$$\begin{aligned} I &= -\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}\cos\theta\,\frac{2\cos\theta}{c}\frac{u_{\text{rad}}}{4\pi}\mathrm{d}S\,c\,\mathrm{d}t\cos\theta \\ &= -2\frac{u_{\text{rad}}}{4\pi}\mathrm{d}S\mathrm{d}t\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi/2}\cos^2\theta\,\mathrm{d}\cos\theta = \frac{1}{3}u_{\text{rad}}\mathrm{d}S\mathrm{d}t \end{aligned} \tag{25}$$

From (25),

$$p_{\text{rad}} = \frac{1}{3}u_{\text{rad}} = \frac{8\pi^5k^4T^4}{45h^3c^3} \tag{26}$$

From (26) and the definition of the adiabatic index,

$$\gamma = 1 + \frac{p_{\text{rad}}}{u_{\text{rad}}} = \frac{4}{3} \tag{27}$$

(6) Consider again the volume element of Fig. S5a. The momentum of the radiation it absorbs gives the pressure difference of the light pressure between its inner and outer sides:

$$-\mathrm{d}p_{\mathrm{rad}}(r)\mathrm{d}S = \frac{\kappa(r)\rho(r)\mathrm{d}r\,\mathrm{d}S}{\mathrm{d}S} \times \frac{1}{c}\frac{L(r)}{4\pi r^2}\mathrm{d}S \tag{28}$$

On the right-hand side, the factors to the left and to the right of the multiplication sign are respectively the number of photons absorbed per unit area over the distance $\mathrm{d}r$, and the total momentum of the photons crossing the area element $\mathrm{d}S$ per unit time. From (28),

$$\frac{\mathrm{d}p_{\mathrm{rad}}(r)}{\mathrm{d}r} = -\frac{\kappa(r)\rho(r)L(r)}{4\pi r^2 c} \tag{29}$$

Substituting (26) into (29) and using (4),

$$T^3(r)\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{45h^3c^2\kappa(r)}{512\pi^7k^4r^4}L(r)\frac{\mathrm{d}m(r)}{\mathrm{d}r} \tag{30}$$

(7) The total pressure at each point inside the star equals the sum of the partial pressure p_{atom} of the atomic matter and the light pressure p_{rad}:

$$p(r) = p_{\mathrm{atom}}(r) + p_{\mathrm{rad}}(r) \tag{31}$$

Once radiation is included, the pressure p in equation (5) of part (1) must be understood as the total pressure. Hence, from (5)(29) and the assumption given in the problem,

$$\frac{\mathrm{d}p_{\mathrm{atom}}(r)}{\mathrm{d}r} = \frac{\mathrm{d}p(r)}{\mathrm{d}r} - \frac{\mathrm{d}p_{\mathrm{rad}}(r)}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2} + \frac{\kappa(r)\rho(r)L(r)}{4\pi r^2 c} < 0 \tag{32}$$

From (32), at the surface of the star $r = R$,

$$\left.\frac{\mathrm{d}p_{\mathrm{atom}}(r)}{\mathrm{d}r}\right|_{r=R} = -\frac{G\rho(R)m(R)}{R^2} + \frac{\kappa(R)\rho(R)L(R)}{4\pi R^2 c} < 0 \tag{33}$$

[a] From (33),

$$L(R) < \frac{4\pi cGM}{\kappa(R)} \tag{34}$$

Marking scheme.
This problem carries 50 points.
Part (1) 7 points: (1) 2 points, (2)(3)(4) 1 point each, (5) 2 points;
Part (2) 10 points: (6) 2 points, (7)(8)(9)(10) 1 point each, (11)(12) 2 points each;
Part (3) 5 points: (13) 2 points, (14) 1 point, (15) 2 points;
Part (4) 8 points: (16)(17) 2 points each, (18) 1 point, (19) 2 points, (20) 1 point;
Part (5) 10 points: (21) 2 points, (22) 1 point, (23) 2 points, (24)(25)(26) 1 point each, (27) 2 points;
Part (6) 5 points: (28) 2 points, (29) 1 point, (30) 2 points;
Part (7) 5 points: (31)(32) 1 point each, (33) 2 points, (34) 1 point.

[a] *Translator's note:* The source prints $\kappa(r)$ in (33); it should be $\kappa(R)$, as in (34).

Problem 6 (one-dimensional Casimir force between conducting plates)

Solution

(1) Between the conducting plates A and B, let λ be the wavelength of the vacuum electromagnetic wave. Then

$$n\frac{\lambda}{2} = a, \qquad n = 1, 2, 3, \cdots \tag{1}$$

whence

$$\lambda = \frac{2\pi c}{\omega} = \frac{2a}{n}$$

Therefore, between the conducting plates A and B, the angular frequencies of the vacuum electromagnetic waves are

$$\omega = \frac{n\pi c}{a}, \qquad n = 1, 2, 3, \cdots \tag{2}$$

Likewise, between the conducting plates B and C, the angular frequencies of the vacuum electromagnetic waves are

$$\omega = \frac{n\pi c}{L-a}, \qquad n = 1, 2, 3, \cdots \tag{3}$$

(2) The vacuum electromagnetic field energy between two adjacent conducting plates separated by a distance r is

$$E(r) = \sum_{\omega} \frac{1}{2}\hbar\omega = \sum_{n=1}^{\infty} \frac{1}{2}\hbar\frac{\pi n c}{r} \tag{4}$$

The total vacuum electromagnetic field energy of the system formed by the three separated conducting plates is

$$E_{\text{tot}}(a) = E(a) + E(L-a) = \left(\frac{1}{a} + \frac{1}{L-a}\right)\sum_{n=1}^{\infty} \frac{1}{2}\hbar\pi n c \tag{5}$$

(3) By the work–energy principle, the interaction force on the conducting plate B is

$$F(a) = -\frac{\mathrm{d}E_{\text{tot}}(a)}{\mathrm{d}a} = \left(\frac{1}{a^2} - \frac{1}{(L-a)^2}\right)\sum_{n=1}^{\infty} \frac{1}{2}\hbar\pi n c \tag{6}$$

The result is divergent.

(4) By the method given in the problem, the vacuum electromagnetic field energy between two adjacent conducting plates separated by a distance r is

$$E(r) = \sum_{\omega} \frac{1}{2}\hbar\omega e^{-\frac{\omega}{\pi\Lambda}} = \frac{\pi\hbar c}{2r}\sum_{n=1}^{\infty} n\, e^{-\frac{nc}{\Lambda r}} = \frac{\pi\hbar c}{2r}\sum_{n=1}^{\infty} n\, e^{-\varepsilon n} \tag{7}$$

where $\varepsilon = \dfrac{c}{\Lambda r} \ll 1$. Take the formula for the sum of a geometric series,

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\varepsilon n} = \frac{\mathrm{e}^{-\varepsilon}}{1-\mathrm{e}^{-\varepsilon}} \tag{8}$$

and differentiate both sides with respect to ε:

$$-\sum_{n=1}^{\infty} n\, \mathrm{e}^{-\varepsilon n} = \frac{-\mathrm{e}^{-\varepsilon}}{(1-\mathrm{e}^{-\varepsilon})^2} \tag{9}$$

Expanding in the small quantity ε,

$$\sum_{n=1}^{\infty} n\,\mathrm{e}^{-\varepsilon n} = \frac{\mathrm{e}^{-\varepsilon}}{(1-\mathrm{e}^{-\varepsilon})^2} = \frac{1}{\mathrm{e}^{\varepsilon}+\mathrm{e}^{-\varepsilon}-2} = \frac{1}{\varepsilon^2} - \frac{1}{12} + \frac{\varepsilon^2}{240} + \cdots \tag{10}$$

so the energy is

$$E(r) = \frac{\pi\hbar c}{2r}\frac{\mathrm{e}^{-\frac{c}{\Lambda r}}}{\left(1-\mathrm{e}^{-\frac{c}{\Lambda r}}\right)^2} = \frac{\pi\hbar}{2c}\Lambda^2 r - \frac{\pi\hbar c}{24r} + \mathrm{O}\Big(\frac{1}{\Lambda^2}\Big) \tag{11}$$

The total vacuum electromagnetic field energy of the system formed by the three separated conducting plates is

$$E_{\text{tot}}(a) = E(a) + E(L-a) = \frac{\pi\hbar}{2c}\Lambda^2 L - \frac{\pi\hbar c}{24}\Big(\frac{1}{a} + \frac{1}{L-a}\Big) + \mathrm{O}\Big(\frac{1}{\Lambda^2}\Big) \tag{12}$$

By the work–energy principle, the interaction force on the conducting plate B is

$$F(a) = -\frac{\mathrm{d}E_{\text{tot}}(a)}{\mathrm{d}a} = -\frac{\mathrm{d}}{\mathrm{d}a}[E(a)+E(L-a)] = \frac{\pi\hbar c}{24}\left[\frac{1}{(L-a)^2} - \frac{1}{a^2}\right] + \mathrm{O}\Big(\frac{1}{\Lambda^2}\Big) \tag{13}$$

From (13),

$$\lim_{\Lambda\to\infty}\lim_{L\to\infty} F(a) = -\frac{\pi\hbar c}{24a^2} \tag{14}$$

The negative sign shows that A and B attract each other.

(5) By the method given in the problem, the vacuum electromagnetic field energy between two adjacent conducting plates separated by a distance r is

$$E(a) = \sum_{\omega}\frac{1}{2}\hbar\omega f\Big(\frac{\omega}{\pi\Lambda}\Big) = \frac{\pi\hbar c}{2a}\sum_{n=1}^{\infty} n\, f\Big(\frac{nc}{\Lambda a}\Big) \tag{15}$$

Likewise

$$E(L-a) = \frac{\pi\hbar c}{2(L-a)}\sum_{n=1}^{\infty} n\, f\Big(\frac{nc}{\Lambda(L-a)}\Big) \tag{16}$$

When $L\to\infty$, the sum over n turns into an integral over x (with $\dfrac{nc}{\Lambda(L-a)} \to x$):

$$\begin{aligned} E(L-a) &= \frac{\pi\hbar c}{2(L-a)}\sum_{n=1}^{\infty} n\, f\Big(\frac{nc}{\Lambda(L-a)}\Big) = \frac{\pi\hbar}{2c}(L-a)\Lambda^2\int_0^{\infty}\mathrm{d}x\, x\, f(x) \\ &= \rho - \frac{\pi\hbar}{2c}a\Lambda^2\int_0^{\infty}\mathrm{d}x\, x\, f(x) \end{aligned} \tag{17}$$

where

$$\rho = \frac{\pi\hbar}{2c}L\Lambda^2\int_0^{\infty}\mathrm{d}x\, x\, f(x)$$

The total vacuum electromagnetic field energy of the system is

$$\begin{aligned} E_{\text{tot}}(a) &= E(a) + E(L-a) \\ &= \rho L + \frac{\pi\hbar c}{2a}\left[\sum_{n=1}^{\infty} n\, f\Big(\frac{nc}{\Lambda a}\Big) - \frac{a^2\Lambda^2}{c^2}\int_0^{\infty}\mathrm{d}x\, x\, f(x)\right] \\ &= \rho L + \frac{\pi\hbar c}{2a}\left[\sum_{n=1}^{\infty} n\, f\Big(\frac{nc}{\Lambda a}\Big) - \int_0^{\infty}\mathrm{d}n\, n\, f\Big(\frac{nc}{\Lambda a}\Big)\right] \end{aligned} \tag{18}$$

[a] Define $G(n) \equiv n\, f\left(\frac{nc}{\Lambda a}\right)$ ($G(n)$ is not an electromagnetic force). By the formula given in the problem,

$$\sum_{n=1}^{N} G(n) - \int_0^N \mathrm{d}n\, G(n) = \frac{G(N)-G(0)}{2} + \frac{G'(N)-G'(0)}{12} + \cdots$$

Because $G^{(k\geq 2)}(n)$ contains a factor $\frac{1}{\Lambda^{k-1}}$, it follows that

$$\lim_{x\to\infty} G^{(j)}(x) = 0 \tag{19}$$

Hence when $x \to \infty$ (i.e. $\Lambda \to \infty$),

$$\sum_{n=1}^{N} G(n) - \int_0^N \mathrm{d}n\, G(n) = -\frac{G(0)}{2} - \frac{G'(0)}{12} + \cdots \tag{20}$$

From $G(n) \equiv n\, f\left(\frac{nc}{\Lambda a}\right)$,

$$G(0) = 0 \tag{21}$$

$$G'(n) = f\left(\frac{nc}{\Lambda a}\right) + \frac{nc}{\Lambda a} f'\left(\frac{nc}{\Lambda a}\right) \tag{22}$$

and from (22),

$$G'(0) = f(0) = 1 \tag{23}$$

From (17)(20)(21),

$$E_{\text{tot}}(a) = \rho L + \frac{\pi\hbar c}{2a}\left(-\frac{1}{12}\right) + O\left(\frac{1}{\Lambda}\right) \tag{24}$$

So when $L \to \infty$ and $\Lambda \to \infty$, equation (24) gives

$$F(a) = -\frac{\mathrm{d}E_{\text{tot}}(a)}{\mathrm{d}a} = -\frac{\pi\hbar c}{24a^2} \tag{25}$$

This result is independent of the functional form of $f(x)$. *(∗)*

Marking scheme.
This problem carries 50 points.
Part (1) 5 points: (1) 2 points, (2)(3) 3 points together;
Part (2) 4 points: (4)(5) 2 points each;
Part (3) 4 points: (6) 4 points;
Part (4) 15 points: (7) 2 points, (8) 1 point, (9)(10)(11)(12)(13)(14) 2 points each;
Part (5) 22 points: (15)(16) 3 points together, (17)(18)(19)(20)(21)(22)(23)(24)(25) 2 points each, (∗) 1 point.

[a] *Translator's note:* The source writes ρL here and in (24), although ρ as defined above already contains the factor L; the term should read ρ. It is independent of a and therefore does not affect the force.