The 42nd Chinese Physics Olympiad

Final — Theoretical Examination (Problems)

25 October 2025, 9:00–12:00

Seven problems, 320 marks in total. The source document contains the problems only; no official solutions are included. Figures are reproduced from the Chinese original, with a short glossary of the Chinese labels where needed.

Note. For a function of several variables $f(x, y, z)$,

$$\partial_x f(x,y,z) \equiv \frac{\partial}{\partial x} f(x,y,z) = \frac{\mathrm{d}}{\mathrm{d}x} f(x,y,z)\Big|_{y,z\ \text{fixed}},$$

and similarly for the other variables.

Problem 1 (45 points)

In the self-assembly of many biological molecules, the electric dipole–dipole interaction between polar molecules has an important part in the arrangement of the molecules. In a protein segment the polar molecules are often limited in space: some polar molecules are fixed on a membrane or on a frame, while others can slide or turn on a track. This coupling of a geometric limit in space with the dipole interaction gives a potential-energy distribution with a strong direction dependence, and this decides the stable configuration of the biological molecule.

As in figure 1a, the situation above is modelled in the two-dimensional x-y plane. The polar molecule A, with its centre fixed at the coordinate origin O, has the permanent electric dipole moment $\boldsymbol{p}_1$, of magnitude p, in the positive y direction. The polar molecule B has the permanent electric dipole moment $\boldsymbol{p}_2$, also of magnitude p, at the angle θ ($-\pi < \theta \le \pi$) to the y axis; its centre of mass is at the position $(x, y = b)$, where $b > 0$ and x can change. A smooth thin track, parallel to the x axis, passes through $y = b$; the centre of mass of molecule B is connected to the track by a smooth insulating axle of negligible mass, and this axle is perpendicular to the x-y plane and of negligible length, so that molecule B can move along the track. The mass of molecule B is m, and its centre of mass is at the insulating axle; to make the calculation simple, take its moment of inertia about the insulating axle to be $I = \frac{1}{12}mb^2$. The unit vectors in the x and y directions are $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, and the permittivity of vacuum is ε_0. Gravity is not considered.

(1) Find the electric field strength $\boldsymbol{E}_1$ that the polar molecule A produces at the position of the polar molecule B.

(2) Find the component F_x, along the x direction, of the electric force on the polar molecule B, and the torque M on it about the insulating axle; derive the condition for molecule B to stay at rest.

(3) Analyse the stability of the polar molecule B near its equilibrium position (consider only the equilibrium position with the smaller θ). Find the general expressions for $x(t)$ and $\theta(t)$ for a small departure of molecule B from that equilibrium position (because no initial conditions are given yet, the results may contain undetermined integration constants).

(4) The polar molecule B is released from rest at the initial position $x = a$ ($0 < a \ll b$), $\theta = 0$. Find the explicit expressions for $x(t)$ and $\theta(t)$ at the time t.

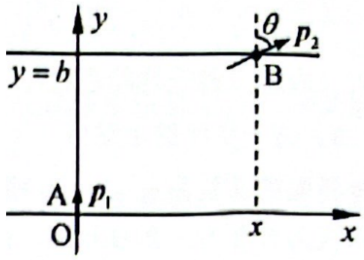

图 *1a = Fig. 1a.*

Problem 2 (45 points)

Convention: a one-dimensional simple harmonic transverse wave that travels in the positive x direction can be written as

$$y(x,t) = A\cos(kx - \omega t + \varphi_0) = \mathrm{Re}\big[\tilde{y}(x,t)\big], \qquad \tilde{y}(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t+\varphi_0)} = \tilde{A}(x)\,\mathrm{e}^{-\mathrm{i}\omega t},$$

where A, ω, k and φ_0 are the amplitude, the angular frequency, the wave vector (which may be expressed with ω and the parameters of the medium) and the initial phase; $\tilde{A}(x) = A\mathrm{e}^{\mathrm{i}(kx+\varphi_0)}$ is called the complex amplitude of the wave at x.

An elastic medium with a periodicity in space is called a “phonon crystal”. This problem discusses the propagation of a small transverse wave of angular frequency ω on a one-dimensional string. Gravity is not considered.

(1) Consider an infinitely long uniform string of linear mass density ρ and tension τ, with the x axis along the string, as in figure 2a. A simple harmonic wave of amplitude A and initial phase 0 travels in the positive x direction. Find, at the time t and the position x, the kinetic energy $\varepsilon_\mathrm{k}(x,t)$ and the potential energy $\varepsilon_\mathrm{p}(x,t)$ per unit original length of the string along the x direction (take the potential energy of the string to be zero when no wave travels on it), and also the instantaneous energy flow $I(x,t)$ and the average energy flow $\bar{I}$.

(2) When simple harmonic waves travelling in the positive and in the negative x direction are present on the string at the same time, write the complex amplitudes of these two waves at $x = x_1$ as $\tilde{A}_1$ and $\tilde{B}_1$, and at $x = x_2$ as $\tilde{A}_2$ and $\tilde{B}_2$. Then $\tilde{A}_2$, $\tilde{B}_2$ can be expressed with $\tilde{A}_1$, $\tilde{B}_1$ as

$$\begin{cases} \tilde{A}_2 = \tilde{M}_{11}\tilde{A}_1 + \tilde{M}_{12}\tilde{B}_1 \\ \tilde{B}_2 = \tilde{M}_{21}\tilde{A}_1 + \tilde{M}_{22}\tilde{B}_1 \end{cases} \qquad \text{in matrix form} \qquad \begin{pmatrix} \tilde{A}_2 \\ \tilde{B}_2 \end{pmatrix} = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix} \begin{pmatrix} \tilde{A}_1 \\ \tilde{B}_1 \end{pmatrix} \equiv \tilde{M} \begin{pmatrix} \tilde{A}_1 \\ \tilde{B}_1 \end{pmatrix}.$$

The matrix $\tilde{M}$ is called the transfer matrix from $x = x_1$ to $x = x_2$.

Consider two semi-infinite strings joined at $x = 0$. At the junction some constraint allows the magnitudes of the tension on the two sides to differ (this holds below as well). The part $x < 0$ is string 1, with linear density ρ_1 and tension τ_1; the part $x > 0$ is string 2, with linear density ρ_2 and tension τ_2, as in figure 2b.

(2.1) Find the transfer matrix $\tilde{M}_{12}$ of the system from $x = 0^-$ to $x = 0^+$.

(2.2) A simple harmonic wave travels from far away on the left in the positive x direction; at the junction $x = 0$ it is partly reflected and partly transmitted. Find the transmittance T (the ratio of the average energy flow $\bar{I}_\mathrm{t}$ of the transmitted wave to the average energy flow $\bar{I}_\mathrm{i}$ of the incident wave).

(3) Consider next a string system made of string 1 (ρ_1, τ_1) and string 2 (ρ_2, τ_2) with several junctions: the part $x < 0$ is a semi-infinite piece of string 2; the part $x > 0$ consists, from left to right, of a piece of string 1 of length a, a piece of string 2 of length b, a piece of string 1 of length a, and a semi-infinite piece of string 2, as in figure 2c.

(3.1) Find the transfer matrix $\tilde{M}_{212}$ of the system from $x = 0^-$ to $x = (2a + b)^+$ (express every complex matrix element $\tilde{z}$ in the form $\mathrm{Re}(\tilde{z}) + \mathrm{i}\,\mathrm{Im}(\tilde{z})$).

(3.2) A simple harmonic wave travels from far away on the left in the positive x direction and is reflected and transmitted at each junction. Find the transmission coefficient $\tilde{t}$ of the system (the ratio of the complex amplitude $\tilde{A}_4$ of the transmitted wave at $x = (2a + b)^+$ to the complex amplitude $\tilde{A}_1$ of the incident wave at $x = 0^-$).

(4) Consider pieces of string 1 (ρ_1, τ_1) of length a and pieces of string 2 (ρ_2, τ_2) of length b placed alternately, so that they build a periodic structure of spatial period $d = a + b$, that is, a one-dimensional phonon crystal, as in figure 2d. By the Bloch theorem, a one-dimensional wave $\tilde{y}(x, t)$ that travels in a periodic medium can be written as a plane wave whose complex amplitude is modulated periodically:

$$\tilde{y}(x, t) = \tilde{u}_K(x)\, \mathrm{e}^{\mathrm{i}(Kx - \omega t)},$$

where the modulation factor $\tilde{u}_K(x)$ is a function of period d, $\tilde{u}_K(x + d) = \tilde{u}_K(x)$, and K is the Bloch wave vector.

(4.1) For the waves that this one-dimensional phonon crystal allows to travel, find the relation between the angular frequency ω and the wave vector K. A continuous range of ω in which waves can travel is a "pass band" of the phonon-crystal energy band; a continuous range in which they cannot travel is a "stop band". For the pass band of this crystal that contains the lowest angular frequency, find the phase velocity v_p of the wave in the long-wave limit ($K \to 0$).

(4.2) If $\rho_1 = 2\rho_2$, $\tau_1 = 2\tau_2$, $a = 2b$, find the range of ω of each stop band of this one-dimensional phonon crystal, and the angular-frequency width $\Delta\omega$ of each stop band. Express the results with ρ_2, τ_2 and b.

(4.3) The quasi-particle of a propagation mode of a mechanical wave in a crystal is called a "phonon". In a one-dimensional phonon crystal the energy ε and the momentum p of a phonon are related to the angular frequency ω and the wave vector K by the de Broglie relations. If the wave vector K is near $K_0 = \dfrac{\pi}{d}$, that is $K = K_0 + \delta K$ ($\delta K \ll K_0$), the phonon energy can be written as

$$\varepsilon(K) \approx \varepsilon(K_0) + \frac{(\hbar\,\delta K)^2}{2m^*},$$

where $\hbar$ is the reduced Planck constant and m^* is called the effective mass of the phonon. For the pass band that contains the lowest angular frequency of the one-dimensional phonon crystal of part (4.2), find the effective mass m^* of a phonon of wave vector K_0. Express the result with $\hbar$, ρ_2, τ_2 and b.

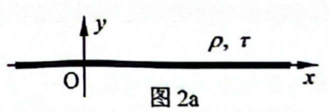

图 2a

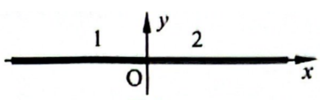

图 2b

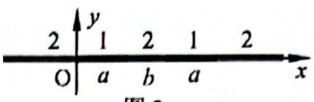

图 2c

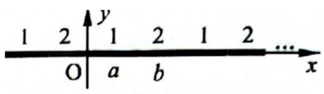

图 2d

图 *2a–2d = Fig. 2a–2d.*

Problem 3 (45 points)

There is a free electron in liquid helium. By the Pauli exclusion principle, a helium atom very close to the free electron feels a strong repulsion from the electron; a small spherical cavity without helium atoms therefore forms around the electron, and this cavity is called an "electron bubble". In addition, the helium atoms are polarized by the electric field of the electron and are therefore attracted by it, so that the density of the helium atoms outside the electron bubble rises. When the electron accelerates it carries the bubble and the attracted helium atoms with it; the "effective mass" of a free electron in liquid helium is therefore much larger than the mass m_e of a free electron.

Given: at the standard atmospheric pressure $p_0 = 1.0 \times 10^5$ Pa, the density of liquid helium is $\rho_0 = 125$ kg/m^3, the surface-tension coefficient is $\sigma = 3.7 \times 10^{-4}$ N/m, the relative permittivity is $\varepsilon_\mathrm{r} = 1.0556$, and the bulk modulus is $B \equiv -V\dfrac{\mathrm{d}p}{\mathrm{d}V} = 8.24 \times 10^8$ kg/(m $\cdot$ s^2) (where V is the volume and p the pressure); $m_\mathrm{e} = 9.11 \times 10^{-31}$ kg, the permittivity of vacuum is $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, the Planck constant is $h = 6.626 \times 10^{-34}$ m^{2}kg/s, the elementary charge is $e = 1.602 \times 10^{-19}$ C, and the mass of a helium atom is $M = 6.646 \times 10^{-27}$ kg.

Parts (1) to (6) below ask only for the expressions; do not substitute the numerical values.

(1) Take the centre of the bubble as the origin, and let the time-averaged position of the electron coincide with the origin. Find the electric field strength E at the distance R outside the bubble.

(2) Under the field E a helium atom becomes polarized; the electric dipole moment of each atom is $p = \alpha E$, where α is the polarizability of the helium atom, related to the relative permittivity approximately by $\varepsilon_\mathrm{r} - 1 \approx \dfrac{\rho_0}{M\varepsilon_0}\,\alpha$. Find the attraction that the electron exerts on each atom at the distance R.

(3) Suppose the pressure very far from the electron is one atmosphere. If the change of the density of the liquid helium is not taken into account, find the pressure $p(R)$ at the distance R from the origin in equilibrium.

(4) In fact the change of the pressure above does change the density of the liquid helium. Use the result of part (3) to find the relation between the change of the density of the liquid helium, $\Delta\rho(R) \equiv \rho(R) - \rho_0$, and R, to the lowest non-zero order in the elementary charge e.

(5) The change of the density of the liquid helium outside the bubble comes from the attraction of the liquid helium by the electron. If the radius of the bubble is r, calculate the total mass ΔM_p of the liquid helium that is attracted by the electron.

(6) By the uncertainty principle the electron moves without stopping and produces a pressure by its continual collisions with the wall of the bubble. In a bubble of radius r, the mean momentum of the electron has the magnitude $\dfrac{h}{2r}$. Find the pressure of the electron on the wall of the bubble.

(7) The range of the Pauli repulsion is of the order 10^{-9} m. Estimate the contributions of the different effects to the pressure on the wall of the bubble; keeping only the most important effects, derive the expression for the equilibrium radius r of the bubble, and find its numerical value.

(8) A sphere of volume V that accelerates in a liquid of density ρ_l carries the surrounding liquid with it; this is equivalent to an increase $\Delta M_{\mathrm{b}} = \dfrac{\rho_l V}{2}$ of the mass of the sphere. Find the ratio of the total effective mass of an electron that accelerates in liquid helium to the mass of the electron.

Problem 4 (45 points)

The Lagrange points have an important place in space science. Near such a point a probe can stay for a long time with a small consumption of fuel; the points suit the placing of telescopes and space stations for continuous astronomical observation. Three celestial bodies S_1, S_2 and S_3 have the masses m_1, m_2 and m_3, where m_3 is much smaller than m_1 and m_2, so that the effect of S_3 on the large bodies S_1 and S_2 is negligible. Suppose S_1 and S_2 move in uniform circular motion about their common centre of mass O, and the distance between them is R. When the small body S_3 is at certain special points of space, it can stay at rest relative to S_1 and S_2; these points are called the Lagrange points. Let the gravitational constant be G, and put $\alpha = \dfrac{m_2}{m_1 + m_2}$, $\beta = 1 - \alpha$. In parts (2), (3) and (4), assume $\alpha \ll 1$.

(1) Find the angular velocity ω of the rotation of S_1 and S_2 about their centre of mass O.

(2) Find the positions of all the Lagrange points.

(3) Analyse the stability of the equilibrium of S_3 at the Lagrange point nearest to S_2.

(4) Consider the motion of S_3 near the Lagrange point nearest to S_2. Suppose the initial displacement and the initial velocity of S_3 can be controlled, so that the mode in which the displacement grows exponentially with time does not appear. Find the expression for the position of S_3 as a function of time (keep only the terms of first order in the displacement in the dynamical equations), and find the conditions that the initial position and the initial velocity must satisfy.

Problem 5 (45 points)

Under conditions of extremely high temperature and high density, the reactions of atomic nuclei reach a dynamic equilibrium: the abundances of the different nuclides no longer change with time, and are decided only by the temperature, the density and the chemical composition of the system. The kinetic energy of the random thermal motion of the nuclei is much smaller than their rest energy, so they can be treated as non-relativistic particles. The number density of the particles in momentum space follows the Maxwell–Boltzmann distribution

$$f(\boldsymbol{p}) = \frac{1}{h^3} \exp\left[\frac{\mu - E(\boldsymbol{p})}{k_{\mathrm{B}} T}\right],$$

where h is the Planck constant, k_{B} is the Boltzmann constant, μ is the chemical potential of the particle, and $E(\boldsymbol{p})$ is the energy of a particle of momentum $\boldsymbol{p}$. The spin of the particles is not considered. The chemical potential of a photon is zero. When a reaction reaches equilibrium, the total chemical potential of the reactants equals the total chemical potential of the products. The speed of light in vacuum is c.

(1) The rest mass of a nucleus is m. Find the chemical potential μ of the nucleus when the temperature is T and the number density of the nuclei is n.

(2) At the temperature T, the photodisintegration of the nucleus X and its synthesis reaction reach a dynamic equilibrium:

$$^{A}_{Z}\mathrm{X} + \gamma \ \rightleftharpoons \ Z\,\mathrm{p} + N\,\mathrm{n},$$

where γ, p, n are the photon, the proton and the neutron, Z is the atomic number (the number of protons) of the nuclide, N is the number of neutrons, and $A = Z + N$ is the mass number. Find the relation between the number density of the nuclei X and the number densities of the protons and of the neutrons at this point. The mass and the total binding energy of the nucleus X are m_{X} and B_{X}.

(3) At the temperature T, the photodisintegration of the deuteron $^{2}_{1}\mathrm{H} \equiv \mathrm{D}$ and its synthesis reaction reach a dynamic equilibrium:

$$\mathrm{D} + \gamma \ \rightleftharpoons \ \mathrm{p} + \mathrm{n}.$$

Find the expression for the ratio of the number densities $\dfrac{n_{\mathrm{D}}}{n_{\mathrm{p}} n_{\mathrm{n}}}$, where n_a is the number density of the particle a ($a = \mathrm{D}, \mathrm{p}, \mathrm{n}$). The mass and the binding energy of the deuteron are m_{D} and B_{D}.

(4) When the temperature of the universe is higher than 0.8 MeV$/k_{\mathrm{B}}$ (about 9.3×10^{9} K), the protons and the neutrons of the universe can change into each other by thermal collisions; the two nuclear reactions are

$$\mathrm{n} + \nu_{\mathrm{e}} \ \rightleftharpoons \ \mathrm{p} + \mathrm{e}^{-}, \qquad \mathrm{n} + \mathrm{e}^{+} \ \rightleftharpoons \ \mathrm{p} + \bar{\nu}_{\mathrm{e}},$$

where e^{-}, e^{+} are the electron and the positron, and ν_{e}, $\bar{\nu}_{\mathrm{e}}$ are the electron neutrino and its antiparticle. At high temperature, neutrino–antineutrino pairs and electron–positron pairs are produced and annihilate continually in large numbers, and their chemical potentials are zero; the changes above between protons and neutrons reach a dynamic equilibrium very quickly. Find the expression for the ratio of the neutron number density n_{n} to the proton number density n_{p} in this state. When the temperature of the universe falls below 0.8 MeV$/k_{\mathrm{B}}$, the changes between protons and neutrons stop very quickly (they "freeze out") and the ratio of their number densities no longer changes with the temperature. Find the value of the ratio of the neutron number density to the proton number density in the frozen state.

(5) As the temperature of the universe falls further, the neutrons and the protons begin to combine and, through a series of nuclear reactions, finally form stable helium nuclei $^{4}_{2}\mathrm{He}$. Assume all the neutrons are bound into helium nuclei (the decay of the neutron is not counted); estimate the mass abundance of helium (the fraction of the mass of the helium nuclei in the mass of all nuclei).

Given: the masses of the proton and the neutron are $m_{\mathrm{p}} = 938.272\ \mathrm{MeV}/c^2$ and $m_{\mathrm{n}} = 939.565\ \mathrm{MeV}/c^2$; and the integral formula

$$\int_0^{+\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \qquad a > 0.$$

Problem 6 (50 points)

Thermal radiation is an electromagnetic wave, and an electromagnetic wave has momentum, so a body feels a force from the radiation when it absorbs, reflects or emits thermal radiation. In the discussion of the motion of an asteroid this radiation force is usually neglected. Because of the spin of the asteroid and its finite thermal conductivity, the radiation force can change the orbit of the asteroid; this is called the Yarkovsky effect. A simplified model is used below to discuss it.

Assume the Sun and the asteroid are both spherical ideal black bodies. The radiation energy flux density $\boldsymbol{J}$ follows the Stefan–Boltzmann law, $\boldsymbol{J} = \sigma T^4 \boldsymbol{n}$, where T is the surface temperature of the black body and $\boldsymbol{n}$ is the outward unit normal vector of the radiating surface, with $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/(m^2K^4)}$. The asteroid moves in a circular orbit around the Sun, with orbital radius R_a; the magnitude of its spin angular velocity is ω, its spin axis and its orbital axis are parallel to each other, and its orbital period is much longer than its spin period. Let the radius of the asteroid be r_a, its density ρ, its specific heat capacity C and its thermal conductivity κ. Given: the surface temperature of the Sun $T_\mathrm{S} = 6000$ K, its radius $r_\mathrm{S} = 7 \times 10^8$ m, its mass $M_\mathrm{S} = 2 \times 10^{30}$ kg, and $r_\mathrm{a} \ll r_\mathrm{S} \ll R_\mathrm{a}$; the gravitational constant $G = 6.67 \times 10^{-11}\ \mathrm{m^3/(kg \cdot s^2)}$; the speed of light in vacuum $c = 3.0 \times 10^8$ m/s.

(1) Assume the radiation travels perpendicular to the surface of the ideal black body. Prove that the radiation force F_r per unit area of the surface of the black body, caused by the absorption of the radiation, is related to the radiation energy flux density J_r by $F_\mathrm{r} = J_\mathrm{r}/c$.

(2) Assume the surface temperature of the asteroid stays uniform and constant. Find the relations of the magnitude J_0 of the solar radiation energy flux density received at its surface, and of the surface temperature T_0, to the orbital radius R_a; estimate the relative change of the orbital radius of a typical asteroid caused by the radiation force, for a typical asteroid of radius 10^3 m and density $5 \times 10^3\ \mathrm{kg/m^3}$.

(3) Because of the spin, the power of the solar radiation received per unit area at any given place of the asteroid surface changes periodically with time, and the temperature of that place also changes with time; but the two changes are not synchronous — the second lags the first by a phase, because the heat capacity and the thermal conductivity are finite. For a typical asteroid, heat conduction happens mainly in the direction perpendicular to the surface and is limited to a surface layer of small thickness, so a one-dimensional heat-conduction model can be used to study the size of this lag phase.

Consider first the situation on the equator of the asteroid. As in figure 6a, the solar radiation power $J_\mathrm{in}(t, \varphi)$ received per unit surface area at the longitude φ on the equator changes periodically with the time t. At $t = 0$ the longitude $\varphi = 0$ faces the Sun, and the spin direction is the direction of increasing longitude φ. Assume $J_\mathrm{in}(t, \varphi)$ can be written as

$$J_\mathrm{in}(t, \varphi) = \frac{J_0}{4} + \alpha J_0 \cos(\omega t + \varphi),$$

where α is a constant. At the time t, the temperature $T(t, x, \varphi)$ at the distance x ($x \geq 0$) from the surface inside the asteroid satisfies the heat-conduction equation

$$\rho C \frac{\partial T(t, x, \varphi)}{\partial t} = \kappa \frac{\partial^2 T(t, x, \varphi)}{\partial x^2},$$

and at the surface ($x = 0$) it satisfies the energy-flux conservation equation

$$-\kappa \left.\frac{\partial T(t, x, \varphi)}{\partial x}\right|_{x=0} + J_\mathrm{out}(t, \varphi) = J_\mathrm{in}(t, \varphi),$$

where $J_\mathrm{out}(t, \varphi)$ is the energy flux density radiated outward at that place. The solution of the heat-conduction equation is known to have the form

$$T(t, x, \varphi) = T_0 + T_1 \mathrm{e}^{-\gamma x} \cos(\omega t - \beta x + \varphi - \delta).$$

Assuming $T_1 \ll T_0$, find the expressions for T_1, γ, β and the lag phase δ of the surface temperature.

(4) Use the result of part (3) to calculate the force $F(\varphi)$ per unit area caused by the outward radiation at the different longitudes φ of the equator of the asteroid at one and the same moment, and find the average radiation force $\bar{F}$ per unit area over the whole equator.

(5) The force on the whole asteroid caused by the outward radiation is $\lambda S\bar{F}$, where S is the surface area of the asteroid and λ is a constant of the order of 1. Find the acceleration in the direction of the orbital motion caused by the outward radiation of the asteroid.

(6) Because the change of the orbit of the asteroid is very slow, its orbit can always be treated as approximately circular. Neglecting the effect of the radiation on the spin, find the rate of change with time of the orbital radius R_a of the asteroid.

(7) Consider stony and iron asteroids in the asteroid belt ($R_\mathrm{a} \approx 4 \times 10^{11}$ m). For which kind of asteroid is the Yarkovsky effect stronger? What is required of the spin direction of the asteroid, so that the orbit of the asteroid can change from the asteroid belt to near the orbit of the Earth (radius about 1.5×10^{11} m)? Given: the typical spin period of an asteroid is about 1 h; for stone, the density $\rho_\mathrm{ST} \approx 3.5 \times 10^3$ kg/m^3, the specific heat capacity $C_\mathrm{ST} \approx 750$ J/(kg · K), the thermal conductivity $\kappa_\mathrm{ST} \approx 3$ W/(m · K); for iron, the density $\rho_\mathrm{Fe} \approx 8 \times 10^3$ kg/m^3, the specific heat capacity $C_\mathrm{Fe} \approx 500$ J/(kg · K), the thermal conductivity $\kappa_\mathrm{Fe} \approx 70$ W/(m · K).

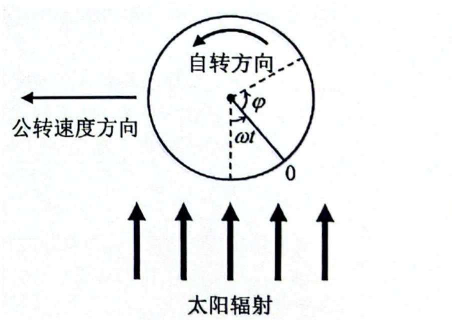

图 6a

Labels: 自转方向 = *spin direction;* 公转速度方向 = *direction of the orbital velocity;* 太阳辐射 = *solar radiation;* 图 *6a* = *Fig. 6a.*

Problem 7 (45 points)

Plasma is called the fourth state of matter; 99% of the visible matter of the universe can be treated as plasma. A plasma is an electrically neutral substance made of a large number of electrons, ions and neutral particles; it has its name because the charges of the negative and the positive ions are equal, so that the whole is quasi-neutral. The electric neutrality of a plasma also appears as a good screening of an electric field by the plasma; this is called Debye screening.

Consider now a plasma made of electrons and singly charged positive ions. In the equilibrium state, the number densities n_j of the ions and of the electrons ($j = \mathrm{i}, \mathrm{e}$ for the ion and the electron) both follow the Boltzmann distribution

$$n_j = n_{j0} \exp\left(-\frac{q_j \phi}{k_\mathrm{B} T_j}\right),$$

where n_{j0} is a constant, q_j is the charge, ϕ is the electrostatic potential, T_j is the absolute temperature and k_B is the Boltzmann constant. The weak-coupling condition is $\dfrac{q_j \phi}{k_\mathrm{B} T_j} \ll 1$, the quasi-neutrality condition is $n_{\mathrm{i}0} = n_{\mathrm{e}0} = n_0$, and the elementary charge is written e.

(1) The particles of a one-dimensional plasma satisfy the equation (the momentum equation)

$$m_j n_j\left(\partial_t v_j + v_j \partial_x v_j\right) = q_j n_j E - \partial_x p_j,$$

where m_j and v_j are the mass and the macroscopic velocity of the particle, E is the electric field strength and p_j is the pressure. Prove from this that, in the equilibrium state, the electrons follow the Boltzmann distribution (the electron temperature is then uniform in space and does not change with time).

(2) A test point charge q is now placed in the plasma described above, under the weak-coupling condition.

(2.1) Find the electrostatic potential distribution $\phi(r)$ of the three-dimensional space in the steady state, where r is the distance from the field point to the charge q.

(2.2) Derive the expression for the characteristic length of the potential distribution $\phi(r)$, that is the Debye screening length λ_{D} $\left(\lambda_{\mathrm{D}} \equiv \dfrac{-\phi r}{\mathrm{d}(\phi r)/\mathrm{d}r}\right)$, and the expression for the electron Debye screening length λ_{De} (the electron Debye screening length is the Debye screening length in the limit of an infinite ion temperature).

(2.3) Given $n_0 = 1.0 \times 10^{20}\ \mathrm{m}^{-3}$, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $k_{\mathrm{B}}T_{\mathrm{e}} = 10$ keV, $k_{\mathrm{B}}T_{\mathrm{i}} = 1.0$ keV, $e = 1.602 \times 10^{-19}$ C, give the numerical value of the Debye screening length λ_{D}.

(3) The Debye screening effect shows that the charge-separation effect of a plasma exists only on the scale λ_{D}; on scales much larger than λ_{D} the plasma can be treated as quasi-neutral. But when the macroscopic ion velocity v_{i} and the external magnetic field B are not zero, the balance of the forces on the ion flow (the pressure neglected) requires a non-zero balancing electric field, and therefore $\Delta n = n_{\mathrm{i}} - n_{\mathrm{e}} \neq 0$. Let the gradient scale length of the electric field strength E be L (if the distribution of the field along the x direction is not uniform, the gradient scale length is $\dfrac{E}{\mathrm{d}E/\mathrm{d}x}$), and let the characteristic angular frequency of the motion of the system be $\omega_0 \approx \dfrac{v_{\mathrm{i}}}{L}$. Express $\dfrac{\Delta n}{n}$ approximately with ω_{pe} $\left(= \sqrt{\dfrac{e^2 n_0}{\varepsilon_0 m_{\mathrm{e}}}}\right)$, ω_{ce} $\left(= \dfrac{eB}{m_{\mathrm{e}}}\right)$ and ω_0. Assuming the characteristic frequency $\dfrac{\omega_0}{2\pi} = 1$ GHz, the magnetic field $B = 1$ T and $n_0 = 1 \times 10^{20}\ \mathrm{m}^{-3}$, estimate the value of $\dfrac{\Delta n}{n}$.

(4) The very large difference between the masses of the electron and of the ion makes the wave phenomena of a plasma rich and complicated; waves from infrasound to ultra-high-frequency electromagnetic waves can be observed in a plasma. Consider now the wave phenomena in the frequency range of sound waves. Without a magnetic field, the components of the plasma satisfy, besides the momentum equation of part (1), also the following system of equations:

$$\begin{cases} \partial_t n_j + \partial_x\left(n_j v_j\right) = 0 \\ \left(\partial_t + v_j \partial_x\right)\left(p_j n_j^{-\gamma_j}\right) = 0 \\ \partial_x E = \dfrac{e}{\varepsilon_0}\left(n_{\mathrm{i}} - n_{\mathrm{e}}\right) \end{cases}$$

where γ_j $(j = \mathrm{i}, \mathrm{e})$ is the polytropic index of the particle. Assume that in equilibrium the plasma is distributed uniformly in free space and has no macroscopic flow, and that a small disturbance of the form $\exp(-\mathrm{i}\omega t + \mathrm{i}kx)$ appears in it. Keeping the first-order terms of the disturbance, derive from the system above the general relation between ω and k; express it with ω, k, γ_j, m_j, T_j, λ_{De} and k_{B} (use the subscript 1 for all disturbance quantities and the subscript 0 for the undisturbed quantities).

(5) In the case where ω^2 and $k^2\dfrac{k_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{i}}}$ are of the same order (T_{e} and T_{i} being of the same order), derive from the general relation of part (4) the simplified expression $\omega^2 = \omega^2(k)$, that is the so-called dispersion relation (keep only the leading term). Under what condition does this dispersion relation reduce to the dispersion relation of the classical ion sound wave, $\omega^2 = k^2\dfrac{\gamma_{\mathrm{e}}k_{\mathrm{B}}T_{\mathrm{e}} + \gamma_{\mathrm{i}}k_{\mathrm{B}}T_{\mathrm{i}}}{m_{\mathrm{i}}}$?

(6) In the case where ω^2 and $k^2\dfrac{k_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{e}}}$ are of the same order (T_{e} and T_{i} being of the same order), what simplified dispersion relation does the general relation of part (4) give (keep only the leading term)? Give also the expression for the group velocity v_{g} of the wave (express it with ω_{pe}^2 and the other known quantities).

(7) Assume the electrons follow the Boltzmann distribution under the weak-coupling condition, and replace the equation $\partial_x E = \dfrac{e}{\varepsilon_0}(n_{\mathrm{i}} - n_{\mathrm{e}})$ of part (4) by the quasi-neutrality condition $n_{\mathrm{i}} = n_{\mathrm{e}}$, the equations for the ions staying unchanged. Derive the dispersion relation $\omega^2 = \omega^2(k)$ of the ion sound wave in this case, and show how this result follows from the dispersion relation of the classical ion sound wave (see part (5)).