Problem 1, EGMO 2026

P1

A 2026×2026 board is said to be *bordeaux* if at least one of its 2026^2 unit cells is coloured red. A rectangular region made of cells is *oddly-rectangular* if it contains an odd number of red cells. Determine the largest positive integer M such that, in every possible 2026×2026 bordeaux board, there exists an oddly-rectangular region of at least M cells.

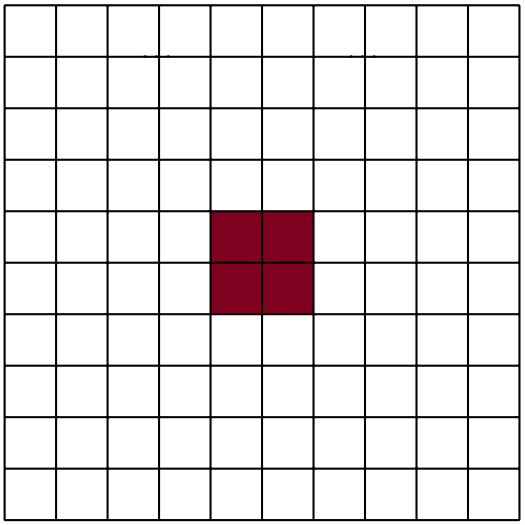
Note: A rectangular region has sides that are parallel to the sides of the board and contains all of its interior.

Solutions

Solution 1

Answer. 1013^2.

Example. Here is an example for a grid 10×10. The same can be done for the grid 2026×2026.



Colour the four central cells red. Any rectangle that contains at least two of them contains either 2 or 4 red cells. In order to contain exactly one red cell, the rectangle must have one of its vertices at the center of the square, and therefore its area is at most 1013^2.

Bound. Suppose that every rectangle of area at least 1013^2 contains an even number of red cells. Consider an arbitrary cell (x, y). WLOG assume that the cell is in the bottom right part of the board, i.e. $x, y \geq 1014$. We consider the rectangles

$$[1, x] \times [1, y], \quad [1, x-1] \times [1, y], \quad [1, x] \times [1, y-1], \quad [1, x-1] \times [1, y-1].$$

Each of these rectangles contains an even number of red cells. Every cell other than (x, y) belongs to an even number of these rectangles, and cell (x, y) belongs to exactly one of them. So by a double-counting

argument, it can't be red. Since (x, y) was arbitrary, we conclude that none of the cells are red, which is a contradiction.

Solution 2

See Solution 1 for a proof that $M \leq 1013^2$.

Lower Bound. Suppose that every rectangle of area at least 1013^2 contains an even number of red cells.

Consider a horizontal arbitrary $1 \times n$ rectangle with $n \geq 1013$. It is the extremity of a $1014 \times n$ included in the grid (above it if it is in the lower half of the grid, under it otherwise), which contains an even number of red cells as its area is at least 1013^2. The same rectangle without the $1 \times n$ rectangle also has area at least 1013^2 thus contains an even number of red cells. Thus the $1 \times n$ rectangle contains an even number of red cells.

Since every $1 \times n$ rectangle with $n \geq 1013$ contains an even number of cells, and each cell is the extremity of a 1×1014 rectangle, by the same reasoning, every cell is uncoloured, which is a contradiction.

Solution 3

See Solution 1 for a proof that $M \leq 1013^2$.

Lower Bound. Suppose by contradiction that every rectangle of area at least 1013^2 contains an even number of red cells.

Claim: Any two cells that differ by a horizontal shift of at least 1013 are either both red, or both not red.

Proof. Consider an arbitrary $n \times 1$ rectangle C, with $n \geq 1013$. Let R be a $n \times m$ rectangle, having C as its first or last column, such that $m \geq 1014$ (for m at least 1014 and at most the maximum distance from C to the vertical boundary + 1). Removing the first or last column of R gives two rectangles that have area at least 1013^2. They thus both have an even number of red cells. This shows that if we can map two columns of size at least 1013 by a horizontal shift of at least 1013, the parity of the number of red cells they contain is the same.

Consider two cells A and B that differ by a horizontal shift of at least 1013. Suppose WLOG they are in the lower half of the grid. Consider the 1014×1 columns C_1 and C_2 whose lowest cells are A and B. Then C_1 and C_2 have the same parity of the number of red cells, and so do $C_1 \setminus A$ and $C_2 \setminus B$. So A and B are either both red, or both not red.

We have thus indeed shown that any two cells that differ by a horizontal shift of at least 1013 are either both red, or both not red. This also works for vertical shifts.

□

Claim: Either all cells are red or none of them are red.

Proof. If we divide the grid in its four 1013×1013 sub grids, notice that any cell of a subgrid is linked to the opposite corner by two suitable shifts. Furthermore, all the corners are linked by suitable shifts, so either they are all red, or none of them are. □

This shows that all the cells have the same state, which has to be red because the board is bordeaux. Hence, any rectangle of area 1013^2 has an odd number of red cells, contradicting our original hypothesis.

Solution 4

See Solution 1 for a proof that $M \leq 1013^2$.

Lower Bound.

Suppose, for the sake of contradiction, that every rectangle of size at least 1013^2 contains an even number of red cells.

Claim 1. Every row and column contains an even number of red cells. **(*)**

Proof. Let $k \in \{1014, \ldots, 2026\}$. Note that $[1, 2026] \times [1, k]$ and $[1, 2026] \times [1, k-1]$ both have size greater than 1013^2, so they both contain an even number of red cells. Therefore their difference, the kth column $[1, 2026] \times [k, k]$, contains an even number of red cells.

It follows that all columns on the right side of the board contain an even number of red cells. By symmetry, we can analogously show that the same holds for columns on the left and for all rows. □

Now consider a red cell. Its row and its column both contain an even number of red cells. Therefore its cross (the union of the row and column) contains an odd number of red cells. Since the full 2026×2026 grid contains an even number of red cells, out of the four (possibly empty) remaining rectangular regions, either one or three have an odd number of red cells. Either way, at least one of these regions that contain a corner is oddly-rectangular.

Claim 2. If a rectangular region R containing a corner is oddly-rectangular, then there exists a rectangle with area at least 1013^2 that is oddly-rectangular. **(**)**

Proof. The lines of the sides of R cut the whole grid into 4 rectangular regions. By Claim 1, two rectangular regions adjacent to R have the same parity of the number of red cells, as their union is the disjoint union of some rows or columns. Hence, all 4 regions are oddly-rectangular, and at least one has area at least 1013^2, as their union is the whole grid. □

This completes the proof by contradiction.

Solution 5

See Solution 1 for a proof that $M \leq 1013^2$.

Lower Bound.

Suppose, for the sake of contradiction, that every rectangle of size at least 1013^2 contains an even number of red cells.

Claim 1. Same as Solution 4 ($*$).

Consider any red cell B, and a rectangular region containing B and a corner cell C as it's corners. If there is another red cell in that region, replace B by this new red cell. Doing so repeatedly allows us to find B and C such that the rectangular region between them doesn't contain any other red cells. In particular, this region is odd as it contains exactly one red cell.

Claim 2. Same as Solution 4 ($**$)

This completes the proof by contradiction.

Problem 2, EGMO 2026

P2

Given a positive integer n, Marie plays a game where she starts with the number 1 on a blackboard. As many times as she wants, she can choose an integer j such that $1 \leq j \leq n$ and replace the number V on the blackboard with the number $j \cdot R\left(\frac{V}{j}\right)$. Here, $R(x)$ denotes the nearest integer to x; if x is exactly halfway between two consecutive integers, it is rounded up. For example, $R(1.3) = 1$ and $R(1.5) = R(1.8) = 2$.

a) Prove that for each given n, there is a positive integer B such that Marie can never end up with a number larger than B on the blackboard.

b) For any given n, let $f(n)$ be the maximum number obtainable on the blackboard after finitely many replacements. Show that there exists a positive integer N such that for all $n \geq N$, we have that 2026 divides $f(n)$.

Solutions

Solution 1

Claim 1. Marie can never reach a number greater than $n!$.

Proof. We prove this by induction on the number of steps. Before the first step, Marie's number is 1, which is less or equal to $n!$.

If at some point, Marie's number V is at most $n!$ and she chooses $j \in \{1, \ldots, n\}$, then since $V/j \leq n!/j$ and the function R is non-decreasing, her resulting number will still be at most $j \cdot R(n!/j) = n!$ since $j \leq n$, which concludes the induction. □

Note that this finishes part (a) of the problem, since Marie can never end up with a number greater than $B = n!$.

Claim 2. For any positive integer k and $n \geqslant 2^k$, $f(n)$ is a multiple of 2^k.

Proof. Fix a positive integer k, and $n \geq 2^k$. Assume 2^k does not divide $f(n)$ and let $\ell = \nu_2(f(n))$. We have $\ell \in \{0, \ldots, k-1\}$. Since 2^ℓ divides n and $2^{\ell+1}$ does not divide n, we have $n \equiv 2^\ell \pmod{2^{\ell+1}}$. Since $\ell \leq k-1$, $2^\ell \leq n$, and $f(n)/2^{\ell+1}$ is a half-integer, if Marie chooses $j = 2^{\ell+1}$, she can increase the number on the board, which is a contradiction. □

Claim 3. For any odd positive integer A, for any integer k such that $A < 2^k$, if $n \geqslant 2^{2k}$, then $f(n)$ is a multiple of A.

Proof. Fix an odd positive integer A, an integer k such that $A \leqslant 2^k$, and n such that $n \geqslant 2^{2k}$. Note that $f(n)$ has a remainder between 0 and $2^{j-1}A$ (excluding $2^{j-1}A$) mod $2^j A$ for each $j \in \{0, \ldots, k\}$, otherwise Marie could choose $2^j A$ and increase her number, as $2^j A \leq 2^{2k} \leq n$.

Let r_j be the remainder of $f(n)$ mod $2^j A$ for all $j \in \{0, \ldots, k\}$, we thus have $0 \leq r_j \leq 2^{j-1}A$. For all $j \in \{0, \ldots, k-1\}$, we have $r_j \equiv f(n) \equiv r_{j+1} \pmod{2^j A}$, and since $-2^j A < r_{j+1} - r_j < 2^j A$, we have

$r_j = r_{j+1}$, thus $r_0 = r_k$. Since 2^k divides $f(n)$, it divides $r_k = r_0$, but $0 \leq r_0 < A \leq 2^k$, therefore $r_0 = 0$. We conclude that A divides $f(n)$. □

Choosing $A = 1013$ in Claim 3 and combining this with Claim 2, we have that $2026 \mid f(n)$ for all $n \geqslant 1024^2$.

Solution 2

Call a number *special* if the remainder it leaves modulo k is strictly less than $\frac{k}{2}$ for each $1 \leqslant k \leqslant n$. Clearly $n!$ is special.

Claim 1. Marie can never reach a number greater than the smallest positive special number.

Proof. Assume the number Marie has currently written on the board is v, and say she applies the transformation for some j; this operation will increase the number on the board if and only if v leaves a remainder at least $j/2$ modulo j.

Let M be the value of the smallest special number. If Marie could cross this value, then there exists $a \leqslant M < b$ such that Marie changes the number from a to b. Let's say this change was done using $j = t$ for some $1 \leqslant t \leqslant n$. Then a leaves a remainder at least $t/2$ when divided by t, and the next multiple of t is b. Since $a \leq M < b$, M also leaves a remainder of at least $t/2$ modulo t, which is a contradiction to it being special. □

Note that this finishes part (a) of the problem, since Marie can never write a number greater than $B = n!$.

Clearly $f(n)$ is special, otherwise Marie could increase her number.

Claim 2. For any $n \geqslant k^2$, all special numbers are multiples of k.

Proof. We prove that for $n \geqslant k^2$, all special numbers are multiples of i for each $1 \leqslant i \leqslant k$, and we do this by induction. The cases of $i = 1$ and $i = 2$ are clear. Now assume we know this for some i. Let M be a special number being considered. By the induction hypothesis, we know that $i \mid M$. Now consider simultaneously

$$M \equiv x \pmod{i+1} \quad \text{and} \quad M \equiv y \pmod{i(i+1)}.$$

Note that $i + 1 \leqslant k$, so $i(i + 1) \leqslant k^2 \leqslant n$. Since M is known to be 0 modulo i, the possible remainders for the latter are $0, i, 2i, 3i, \ldots$ which correspond to remainders $0, -1, -2, \ldots$ modulo $i + 1$. We need $y < \frac{i(i+1)}{2}$, which forces the remainder modulo $i + 1$ to be in $0, -1, \ldots, -\left\lfloor \frac{i}{2} \right\rfloor$; but the only remainder here which is also in the first half of remainders modulo $i + 1$ is 0, so we must have $i + 1 \mid M$. □

Choosing $k = 2026$ in Claim 2, we get that $2026 \mid f(n)$ for all $n \geqslant 2026^2$.

Solution 3

We provide a reformulation of the first solution of the second part of the problem.

Claim 1. A number $V > 0$ is special if and only if for all $1 \leqslant k \leqslant n$, we have $\lfloor \frac{2V}{k} \rfloor = 2 \cdot \lfloor \frac{V}{k} \rfloor$.

Proof. It suffices to notice that if $1 \leq k \leq n$, the remainder that V leaves modulo k is $V - k \cdot \left\lfloor \frac{V}{k} \right\rfloor$, and

$$\begin{aligned} & V - k \cdot \left\lfloor \frac{V}{k} \right\rfloor < \frac{k}{2} \\ \iff & 2k \cdot \left\lfloor \frac{V}{k} \right\rfloor \leqslant 2V < k + 2k \cdot \left\lfloor \frac{V}{k} \right\rfloor \\ \iff & \left\lfloor \frac{2V}{k} \right\rfloor = 2 \cdot \left\lfloor \frac{V}{k} \right\rfloor. \end{aligned}$$

□

Let us consider $v > 0$ such that $2^v \geqslant 2026$, and $n \geqslant 2^{2v}$. Let us denote $f(n)$ by V. Because V is special (or else Marie could increase her number using some j), Claim 1 shows that for all $1 \leqslant j \leqslant v$, one has $\lfloor \frac{2V}{2^j} \rfloor = 2 \cdot \lfloor \frac{V}{2^j} \rfloor$. A straightforward induction then shows that

$$V = \left\lfloor \frac{2V}{2} \right\rfloor = 2 \cdot \left\lfloor \frac{2V}{4} \right\rfloor = \cdots = 2^v \left\lfloor \frac{V}{2^v} \right\rfloor.$$

Similarly, applying Claim 1 to $2026 \cdot 2^j$ for all $1 \leqslant j \leqslant v$ (these are indeed at most n by construction) gives

$$\left\lfloor \frac{V}{2026} \right\rfloor = \left\lfloor \frac{2V}{2 \cdot 2026} \right\rfloor = \cdots = 2^v \left\lfloor \frac{V}{2^v \cdot 2026} \right\rfloor.$$

We have thus shown that $2^v \mid V$ and $2^v \mid \lfloor \frac{V}{2026} \rfloor \mid 2026 \cdot \lfloor \frac{V}{2026} \rfloor$. But

$$0 \leqslant V - 2026 \cdot \left\lfloor \frac{V}{2026} \right\rfloor < 2026 \leqslant 2^v,$$

therefore this multiple of 2^v is zero. We have thus shown that for all $n \geqslant 2^{2v}$, one has $2026 \mid f(n)$.

Solution 4

We give an alternative solution for part (b).

For a given n, let

$$f(n) = \sum_{i=1}^{\infty} a_i i!$$

with $0 \leq a_i \leq i$ and all but finitely many a_i being 0 (it is known that such a decomposition exists).We prove by induction on m that for every m, if n is large, we have $a_0 = \cdots = a_m = 0$.

- Base case: $f(n)$ is even for $n \geq 2$, so $j = 1$ works.
- Induction step: take n large enough and $m \geq 2$ such that $a_1 = \cdots = a_{m-1} = 0$. Assume that $n \geq (m+1)!$. Using the fact that $f(n)$ is special with $j = (m+1)!$, we have that $a_m m! < (m+1)!/2$, so $a_m < (m+1)/2$.

We now plug in for j numbers of the form $(m+1)!/k$ for some $k \mid (m+1)!$. Since $f(n)$ is special, we have that the remainder of $f(n)$ modulo $(m+1)!/k$ is less than $(m+1)!/2k$, which can be rewritten as

$$\left\{ \frac{a_m k}{m+1} \right\} < \frac{1}{2}.$$

In particular, for $k = 1$, we have $a_m < (m+1)/2$.

We now present 3 ways of concluding:

- We prove by induction on $\ell \in \{1, \ldots, m+1\}$ that $a_m < (m+1)/2\ell$. For $\ell = 1$, the statement is already proven.

  Induction step: if the inequality is true for some $\ell \in \{1, \ldots, m\}$, then we have that

  $$0 \leq \frac{a_m(\ell+1)}{m+1} < \frac{\ell+1}{2\ell} \leq 1.$$

  Since $\left\{ \frac{a_m(\ell+1)}{(m+1)} \right\} < \frac{1}{2}$, we have $\frac{a_m(\ell+1)}{(m+1)} < \frac{1}{2}$, as desired.

- By taking $k = m$, the fractional part of $a_m \cdot m/(m+1)$ is less than $1/2$, but it also is the fractional part of $a_m - a_m/(m+1)$, thus it is the fractional part of $1 - a_m/(m+1)$. Since $1 \geq 1 - a_m/(m+1) > 1/2$, we have $a_m = 0$.
- By taking $k = \lfloor m/a_m \rfloor$, we have $a_m k < m + 1$, and $k > \frac{m}{a_m} - 1$. Therefore

$$a_m k \geq m + 1 - a_m > \frac{m+1}{2},$$

  i.e. $\frac{1}{2} < \frac{a_m k}{m+1} < 1$ which is absurd.

Solutions
Problem 3

Problem 3. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y, the following holds:

$$f((f(x)+f(y))^2) = (x+y)f(x+y).$$

Solution

Answer: The solutions are $f(x) = x, f(x) = -x, f(x) = 0$.

Denote the equation by $(*)$. A straightforward check shows that all the above mentioned solutions satisfy the problem.

Step A.1: $f(0) = 0$

Taking $x = y = 0$ in $(*)$ we get that $f(4f(0)^2) = 0$.

Denote $T = f^{-1}(0)$ the set of all zeroes of f. In particular, $4f(0)^2 \in T$. Then putting $x = 0$, $y = 4f(0)^2$ in $(*)$ gives $f(f(0)^2) = 0$, hence $f(0)^2 \in T$. Now if $t_1, t_2 \in T$ we have that, by taking $y = t_1$ and $y = t_2$:

$$(x+t_1)f(x+t_1) = f(f(x)^2) = (x+t_2)f(x+t_2) \tag{1}$$

Plugging $x = -t_1$ in (1), we get that $t_2 - t_1 \in T$, so $4f(0)^2 - f(0)^2 = 3f(0)^2 \in T$, implying $3f(0)^2 - f(0)^2 = 2f(0)^2 \in T$.

Now put $x = y = f(0)^2$ in $(*)$ to obtain

$$f(0) = 2f(0)^2 f(2f(0)^2) = 0.$$

Step A.2: f is injective or $f(x) = 0$.

Take $y = 0$ in $(*)$ to arrive at:

$$f(f(x)^2) = xf(x) \tag{2}$$

In (2), replace $x \to f(x)^2$, so:

$$f(x^2 f(x)^2) = f(x)^2 f(f(x)^2) = xf(x)^3$$

Multiply both sides of $(*)$ by $(f(x)+f(y))^2$ to obtain:

$$(x+y)f(x+y)(f(x)+f(y))^2 = (f(x)+f(y))^2 f((f(x)+f(y))^2) = f(f((f(x)+f(y))^2)^2) =$$
$$= f((x+y)^2 f(x+y)^2) = (x+y)f(x+y)^3$$

So for all $x \neq -y$:

$$f(x+y)(f(x+y) - f(x) - f(y))(f(x+y) + f(x) + f(y)) = 0 \tag{3}$$

We now claim that $T = \{0\}$ or $T = \mathbb{R}$. Assume not and take a non-zero $t \in T$. By (1) we have that $(0-t) - t = -2t$ is also in T.

Plug $x \notin T$ and $y = -2t$ in (3):

i) If $f(x+y)=0$, then $x+y\in T$, so $x=x+y-y\in T$, absurd.

ii) If $f(x+y)=f(x)+f(y)=f(x)$, we use (1) to write $(x+y)f(x+y)=xf(x)$, which then becomes $xf(x)=(x+y)f(x)$. Hence $f(x)=0$, contradicting $x\notin T$.

iii) If $-f(x+y)=f(x)+f(y)=f(x)$, once again we use (1) to write $(x+y)f(x+y)=xf(x)$, so that $xf(x)=-(x+y)f(x)$, so either $2x=-y=2t$ or $f(x)=0$, contradicting $t\in T$ and $x\notin T$.

Hence the claim is proven. To conclude, assume f is not identically 0, so that $T=\{0\}$. Assume $f(a)=f(b)$ for some a,b. If a or b is zero, then since $T=\{0\}$, the other is also zero so $a=b=0$. Otherwise, we write that $af(a)=f(f(a)^2)=f(f(b)^2)=bf(b)$, so $(a-b)f(a)=0$, i.e. $a=b$ because $f(a)\neq 0$.

Step B: f is additive.

Using $(*)$ for (x,y) and $(x+y,0)$ gives

$$f((f(x)+f(y))^2)=(x+y)f(x+y)=f(f(x+y)^2)$$

since $f(0)=0$. So by injectivity, we have for all real x,y, that $f(x+y)^2=(f(x)+f(y))^2$, i.e.

$$f(x+y)=f(x)+f(y)\quad\text{or}\quad -f(x+y)=f(x)+f(y) \tag{4}$$

Taking $y=-x$, we obtain $-f(x)=f(-x)$ for all real numbers x.

Assume that for some real x,y:

$$f(x+y)\neq f(x)+f(y) \tag{5}$$

Then we must have $-f(x+y)=f(x)+f(y)$. Consider $x\to x+y, y\to -y$ in (4):

i) If $f(x+y-y)=f(x+y)-f(y)$, then $f(x)+f(y)=f(x+y)$, contradiction with (5).

ii) If $-f(x+y-y)=f(x+y)-f(y)$, then combining with $-f(x+y)=f(x)+f(y)$ gives $f(y)=0$, hence $y=0$, so $f(x)+f(y)=f(x+y)$, contradiction with (5).

Step C: $f(x)=x$ and $f(x)=-x$ are the remaining solutions.

We know that f is additive, injective and satisfies (2). We will show that the above mentioned solutions are the only ones that satisfy these three conditions.

By plugging in $x=1$ into (2), we get $f(f(1)^2)=f(1)$, so by injectivity $f(1)^2=1$. If f is a solution of the equation, then $-f$ is a solution as well, so without loss of generality, we can assume $f(1)=1$. Consider $x\to x+1$ in (2). Then:

$$f(f(x)^2+2f(x)+1)=xf(x)+f(x)+x+1$$

So by subtracting $f(f(x)^2)=xf(x)$ (equation (2)) we get

$$2f(f(x))=f(x)+x. \tag{6}$$

Switching $x\to f(2x)$ in (2) we obtain:

$$\begin{aligned} f\Big(f(f(2x))^2\Big)&=f(2x)f(f(2x))\\ f\Big((2f(f(x)))^2\Big)&=4f(x)f(f(x))\\ f(f(x)^2+2xf(x)+x^2)&=2f(x)^2+2xf(x)\\ f(f(x)^2)+2f(xf(x))+f(x^2)&=2f(x)^2+2xf(x)\end{aligned}$$

where we used (6) to go from the second to the third line. Using (2) and

$$2f(xf(x)) \overset{(2)}{=} 2f\Big(f(f(x)^2)\Big) \overset{(6)}{=} f(x)^2 + f(f(x)^2) \overset{(2)}{=} f(x)^2 + xf(x),$$

the previous equation becomes:

$$xf(x) + (f(x)^2 + xf(x)) + f(x^2) = 2f(x)^2 + 2xf(x)$$

and so

$$f(x^2) = f(x)^2 \geq 0$$

Hence f is non-negative for all non-negative values of x, so it is well-known it is linear. Combining with $f(1) = 1$ (and the fact that $-f$ is also a solution) we get the desired solution set.

Alternate solution

We give different solutions for each of the steps.

Alternate step A: Let us start by showing injectivity. Assume that $f(x) = f(x')$ and use $(*)$ with $(x, 0)$ and $(x', 0)$ to get

$$xf(x) = f((f(x) + f(0))^2) = x'f(x').$$

If $f(x) \neq 0$, one may divide by $f(x)$ on both sides to get $x = x'$ as desired. It therefore remains to consider the preimages of 0. Our goal will be to show that either there is at most one, or the function is identically zero. Therefore start by assuming that there are r, s distinct such that $f(r) = f(s) = 0$. Applying the condition on f to (r, y) and (s, y), one gets

$$(r + y)f(r + y) = (s + y)f(s + y)$$

so that $x \mapsto xf(x)$ is $(s - r)$-periodic. For $x = 0$ this function is clearly zero, so $(s - r)f(s - r) = 0$ and so $f(s - r) = 0$. Then repeating the argument with r, s replaced with s, r we get $f(r - s) = 0$. Using $(*)$ with $(s - r, r - s)$ gives that $f(0) = 0$.

Now assume by contradiction that there is some a such that $f(a) \neq 0$ (notably $a \neq 0$). Recall that $xf(x)$ is T-periodic for some $T > 0$. Apply $(*)$ to $(T, a - T)$ and $(0, a)$. Then both right-hand sides are equal and non zero so that by the "almost injectivity" of f proved at the start,

$$\begin{aligned}(f(T) + f(a - T))^2 &= (f(0) + f(a))^2 \\ f(a - T)^2 &= f(a)^2\end{aligned}$$

But since $(a - T)f(a - T) = af(a)$, this is a contradiction (unless $a = \frac{T}{2}$ in which case one simply runs the argument with the period T replaced by $2T$). Hence, f is identically 0.

Assume now that f is injective. Applying $(*)$ to $(0, 0)$ gives $f(4f(0)^2) = 0$. Then, applying the condition to $(0, 4f(0)^2)$ gives $f(f(0)^2) = 0$. By injectivity, one must have $f(0) = 0$ in this case as well. Note that equation (2) follows from $f(0) = 0$.

Alternate step B: As in the other proof of step B, one proves that for all x, y,

$$(f(x) + f(y))^2 = f(x + y)^2 \tag{4}$$

$$f(-x) = -f(x).$$

Take $x \neq 0$. Using the above equation with (x, x) gives $f(2x) = \pm 2f(x)$. Assume by contradiction that $f(2x) = -2f(x)$. Then applying (4) with $(2x, x)$ would give $f(3x) = \pm f(x)$, and reapplying (4) with $(3x, x)$ gives $f(4x) \in \{0, \pm 2f(x)\}$. However, applying (4) with $(2x, 2x)$ gives $f(4x) = \pm 4f(x)$ so this is impossible since $f(x) \neq 0$. Therefore, for all x, we have $f(2x) = 2f(x)$.

Now let x and y be arbitrary. If $y = 0$ or $y = -x$, the equation $f(x + y) = f(x) + f(y)$ holds because of $f(0) = 0$ or $f(-x) = -f(x)$. Otherwise, assume that one has $f(x+y) = -f(x) - f(y)$. Then using (4) with $(x + y, y)$ gives $f(x + 2y) = \pm f(x)$. But on the other hand, using (4) with $(x, 2y)$ gives $f(x+2y) = \pm(f(x)+2f(y))$ which implies either $f(y) = 0$ or $f(y) = -f(x)$. By injectivity, this can only hold if $y = 0$ or $y = -x$, which we already treated. Hence, f satisfies the Cauchy equation $f(x + y) = f(x) + f(y)$.

Second alternate step B: As in the other proofs of step B, we arrive at

$$f(x + y)^2 = (f(x) + f(y))^2 \tag{4}$$

and $f(-x) = -f(x)$. Applying this equation to the pairs $x \to x, y \to y - x$ and $x \to -x, y \to y + x$ gives

$$f(y)^2 = (f(x) + f(y - x))^2 = (f(-x) + f(y + x))^2.$$

Expanding the second equality gives

$$\begin{aligned} 2f(x)f(y - x) + f(y - x)^2 &= -2f(x)f(y + x) + f(y + x)^2 \\ 2f(x)(f(x + y) - f(x - y)) &= (f(x + y) + f(x - y))(f(x + y) - f(x - y)). \end{aligned}$$

Assuming $y \neq 0$, injectivity gives that $f(x + y) - f(x - y) \neq 0$ so

$$2f(x) = f(x + y) + f(x - y).$$

Applying this equation successively with $x \to \frac{x+y}{2}, y \to \frac{x-y}{2}$ and $x \to \frac{x+y}{2}, y \to \frac{x+y}{2}$ gives

$$f(x) + f(y) = 2f(\tfrac{x+y}{2}) = f(x + y) + f(0)$$

and so $f(x + y) = f(x) + f(y)$ for all x, y.

Alternate step C: As in the first proof of step C, we prove $f(1) = \pm 1$, and so without loss of generality, $f(1) = 1$. We also follow the previous proof to obtain

$$f(f(x)) = \tfrac{1}{2}(x + f(x)). \tag{6}$$

Using linearity, setting $r = f(x) - x$ gives $f(r) = -\frac{1}{2}r$. In order to prove that $f(x) = x$, it remains to show that $r = 0$. To do this, notice that (6) can be rewritten as $f(2f(x) - x) = x$, so that the inverse function of $f(x)$ is $2f(x) - x$. Plugging in $x = 2f(a) - a$ into (2) gives that for all a,

$$f(a^2) = a(2f(a) - a). \tag{7}$$

Plugging in $a + b$ to (7) and subtracting the two equations (7) with a and b gives the more general version

$$f(ab) = af(b) + bf(a) - ab.$$

from which plugging in $b = a^2$ and using (7) gives

$$f(a^3) = a^2(3f(a) - 2a). \tag{8}$$

It is now time to use our r such that $f(r) = -\frac{1}{2}r$. Plugging in $a = r$ to (7) gives

$$f(r^2) = r(-2 \cdot \tfrac{1}{2}r - r) = -2r^2$$

and then plugging in $a = r^{2/3}$ to the equation (8) gives

$$f(r^2) = r^{4/3}(3f(r^{2/3}) - 2r^{2/3}) = 3r^{4/3}f(r^{2/3}) - 2r^2.$$

Combining both equations gives $r^{4/3}f(r^{2/3}) = 0$ and hence $r = 0$. This concludes the proof.

Problem 4, EGMO 2026

P4

Let $1 = a_1 \geq a_2 \geq a_3 \geq \cdots$ be an infinite sequence of real numbers such that $a_n = a_{2n} + a_{2n+1}$ for all positive integers n. For $r = 2026^{2026}$, prove that

$$\frac{1}{r} \leq a_r \leq \frac{2}{r+1}.$$

Solutions

We first state and prove three auxiliary lemmas, which are used later in the proofs.

Lemma 1 (Lemma A)**.** *For every $n \in \mathbb{N}$, we have*

$$a_n + a_{n+1} + \ldots + a_{2n-1} = 1.$$

Proof. We consider the equalities $a_k = a_{2k} + a_{2k+1}$ for $k \in \{1, \ldots, n-1\}$. Taking their sum gives us

$$a_1 + a_2 + \ldots + a_{n-1} = a_2 + a_3 + \ldots + a_{2n-2} + a_{2n-1}.$$

Together with $a_1 = 1$, this implies

$$a_n + \cdots + a_{2n-1} = (a_2 + a_3 + \ldots + a_{2n-2} + a_{2n-1}) - (a_2 + \ldots + a_{n-1}) = 1.$$

□

Lemma 2 (Weaker version of Lemma A)**.** *For every $k = 2^n$, $n \in \mathbb{N}$,*

$$a_k + a_{k+1} + \ldots + a_{2k-1} = 1.$$

Remark : Showing Lemma A for a specific range $[\![n, 2n-1]\!]$ containing $r+1$ counts as a weaker version of Lemma A.

Proof. By the problem statement, the lemma holds for $k = 1$. We proceed by induction and assume that the statement holds for some $k = 2^l$ with

$$a_{2^l} + a_{2^l+1} + \ldots + a_{2^{l+1}-1} = 1.$$

We can expand each term in the left-hand side and obtain

$$(a_{2^{l+1}} + a_{2^{l+1}+1}) + (a_{2^{l+1}+2} + a_{2^{l+1}+3}) + \ldots + (a_{2^{l+2}-2} + a_{2^{l+2}-1}) = 1,$$

which is precisely the statement of the lemma for $k = 2^{l+1}$. □

Lemma 3 (Lemma B)**.** *For every $k \in \mathbb{N}$ and every $m \leq 2k$, we have*

$$a_m \geq \frac{1}{2} a_k.$$

Proof. Since $a_k = a_{2k} + a_{2k+1}$ and $a_{2k} \geq a_{2k+1}$, we get that

$$a_k \leq 2a_{2k}.$$

Therefore, for any $m \leq 2k$, we have

$$a_m \geq a_{2k} \geq \frac{a_k}{2}.$$

□

Proof of the lower bound

Using Lemma A for $n = r$, we get

$$a_r + \ldots + a_{2r-1} = 1.$$

Since the sequence is decreasing, each of the terms

$$a_{r+1}, \ldots, a_{2r-1}$$

is less than or equal to a_r, which implies

$$ra_r \geq a_r + \ldots + a_{2r-1} = 1.$$

Therefore,

$$a_r \geq \frac{1}{r}.$$

Proof of the upper bound

Using Lemma A for $n = \frac{r}{2} + 1$, we get

$$a_{r/2+1} + \cdots + a_{r+1} = 1.$$

We note that each of the terms

$$a_{r/2+1}, \ \ldots, \ a_{r-1}$$

is greater than or equal to a_r. This implies

$$\frac{r}{2}a_r + a_{r+1} = \left(\frac{r}{2} - 1\right) a_r + a_r + a_{r+1} \leq a_{r/2+1} + \cdots + a_{r+1} = 1.$$

Together with

$$a_{r+1} \geq \frac{1}{r+1},$$

which can be shown analogously to the proof of the lower bound for r above, we get

$$a_r \leq \frac{2}{r}\left(1 - \frac{1}{r+1}\right) = \frac{2}{r+1}.$$

Remark : Instead of using the lower bound for a_{r+1} in the last step, we can use $a_{r+1} \geq \frac{a_r}{2}$.

Proof of the upper bound - solution 2

Using Lemma A for $n = r$, we get

$$a_r + a_{r+1} + \cdots + a_{2r-1} = 1.$$

Using Lemma B, we get that each of the terms $a_{r+1}, \ \ldots, a_{2r-1}$ is greater than or equal to $\frac{a_r}{2}$. Summing all of these inequalities implies that

$$a_r + a_{r+1} + \cdots + a_{2r-1} \geq a_r + (r-1)\frac{a_r}{2} = \frac{r+1}{2}a_r.$$

This gives us

$$a_r \leq \frac{2}{r+1}.$$

Remark : Alternatively, we can use Lemma A starting at $r+1$

$$a_{r+1} + \cdots + a_{2r} + a_{2r+1} = 1.$$

Since the average of the terms on the left hand side of the sum is greater than or equal to the average of the two smallest terms (a_{2r} *and* a_{2r+1})*, and* $a_{2r} + a_{2r+1} = a_r$*, we get* $a_r \leq \frac{2}{r+1}$.

Proof using Lemma 2 - solution 3

Using the weaker version of Lemma A, we get

$$a_m + \cdots + a_{r-1} + a_r + a_{r+1} + \cdots + a_{2m-1} = 1 \tag{1}$$

for some integer m such that $m \leq r \leq 2m-1$ (for instance for m a power of two, but the proof works for any such m; see also the remark below). Note that each of the terms

$$a_{r+1}, \ldots, a_{2m-1}$$

is smaller than or equal to a_r. Additionally, since $m, \ldots, r-1$ are all greater than $\frac{r}{2}$, by applying Lemma B, we get that each of the terms

$$a_m, \ldots, a_{r-1}$$

is smaller than or equal to $2a_r$. This, together with (1), gives us

$$2(r-m)a_r + a_r + (2m-r-1)a_r \geq 1,$$

which implies

$$a_r \geq \frac{1}{r}.$$

The upper bound follows similarly using (1),

$$a_m, \ldots, a_{r-1} \geq a_r$$

and

$$a_{r+1}, \ldots, a_{2m-1} \geq \frac{a_r}{2}.$$

The latter follows from Lemma B since $r+1, \ldots, 2m-1 < 2r$.

Remark. *The proof above for* $m = r$ *and* $m = r/2 + 1$ *gives the previous proofs. In that case, the use of Lemma B is reduced to the minimum.*

Proof without any form of Lemma A - solution 4

For $r \in \mathbb{N}$, define a function f_r by

$$f_r(k) = \begin{cases} 1, & r \leq k \leq 2r, \\ f_r(2k) + f_r(2k+1), & k < r. \end{cases}$$

Note that for all $r \leq k \leq 2r$, $a_k \leq a_r = f_r(k)a_r$. Since $f_r(k)$ sums up in the same manner as the original sequence a_k, we have that $a_k \leq f_r(k)a_r$ for $k < r$. To show the lower bound, we aim to prove that $f_r(1) = r$, which would imply $1 = a_1 \leq r a_r$.

Let n be an integer such that $2^n \le r < 2^{n+1}$. We want to work with the binary representation of r and consider how $f_r(\cdot)$ behaves when we perform a bit-shift on r. More formally, let us define the sequences $(q_i)_{i\in[\![0,n]\!]}$ and $(s_i)_{i\in[\![0,n]\!]}$ as the quotient and the remainder of r by 2^i, $i \in [\![0, n]\!]$. Hence, for $i \in [\![0, n]\!]$, we have $r = q_i 2^i + s_i$ and (a) $0 \le s_i < 2^i$, (b) $s_i = 2^{i-1} + s_{i-1}$ if q_{i-1} is odd, (c) $s_i = s_{i-1}$ if q_{i-1} is even, and (d) $q_i = \left\lfloor \frac{q_{i-1}}{2} \right\rfloor$, that also implies the sequence q_i is strictly decreasing.

We now use descending induction on k from n to 1 to show that

$$f_r(k) = \begin{cases} 2^i + s_i, & k = q_i, \\ 2^i, & q_i < k < q_{i-1}. \end{cases}$$

Base. We have $q_0 = r$ and $s_0 = 0$, hence, $f(r) = 1 = 2^0 + s_0$.

Induction step. Assume that the hypothesis holds for all l in $[\![k + 1, r]\!]$ and let us prove it for k. We distinguish three cases:

- $\mathbf{q_i < k < q_{i-1}}$. If $i = 1$, then $f_r(k) = f_r(2k) + f_r(2k + 1) = 2^1$ as $k > q_i = \left\lfloor \frac{r}{2} \right\rfloor$. Otherwise, using the property (d), we get $q_{i-1} < 2k < 2k + 1 < q_{i-2}$ and $f(k) = f(2k) + f(2k + 1) = 2^{i-1} + 2^{i-1} = 2^i$.
- $\mathbf{k = q_i}$ and $\mathbf{q_{i-1}}$ **is odd**. Then, $f_r(k) = f_r(q_{i-1} - 1) + f_r(q_{i-1}) = 2^i + 2^{i-1} + s_{i-1} = 2^i + s_i$.
- $\mathbf{k = q_i}$ and $\mathbf{q_{i-1}}$ **is even**. Then, $f_r(x) = f_r(q_{i-1}) + f_r(q_{i-1} + 1) = 2^{i-1} + s_{i-1} + 2^{i-1} = 2^i + s_i$.

Since $q_n = 1$, then $s_n = r - 2^n$, and $f_r(1) = 2^n + r - 2^n = r$.

For the upper bound, we define a similar function g_r such that for all k, $a_k \ge g_r(k)a_r$. The initial conditions are $g_r(k) = 1$ for $\frac{r}{2} < k \le r$, and $g_r(k) = \frac{1}{2}$ for $r < k \le 2r$, which come from Lemma B. Additionally, we have the same summing pattern $g_r(k) = g_r(2k) + g_r(2k + 1)$. We then notice that the functions $2 \cdot g_r$ and f_{r+1} (introduced analogously to f_r) are defined by the same initial condition and the same recurrent formula, thus, $2 \cdot g_r(1) = f_{r+1}(1) = \frac{1}{r+1}$. Therefore, $a_r \le \frac{2}{r+1}$.

Remark. *Both estimates in the problem are sharp. To construct a sequence attaining the lower bound* $a_r = \frac{1}{r}$*, set*

$$a_r = a_{r+1} = \cdots = a_{2r-1} = \frac{1}{r}$$

and extend the sequence to the left by setting $a_n = a_{2n} + a_{2n+1}$ *and to the right by setting* $a_{2n} = a_{2n+1} = \frac{a_n}{2}$*. It is straightforward to check that the resulting sequence is non-increasing. Moreover, because* $a_n = a_{2n} + a_{2n+1}$ *again implies that* $a_n + a_{n+1} + \ldots + a_{2n-1}$ *is constant, it follows that* $a_1 = a_r + a_{r+1} + \cdots + a_{2r-1} = 1$*, so the sequence satisfies the problem conditions and has* $a_r = \frac{1}{r}$ *by construction.*

To construct a sequence attaining the lower bound $a_r = \frac{2}{r+1}$*, we similarly set*

$$a_{r+1} = a_{r+2} = \cdots = a_{2r+1} = \frac{1}{r+1}$$

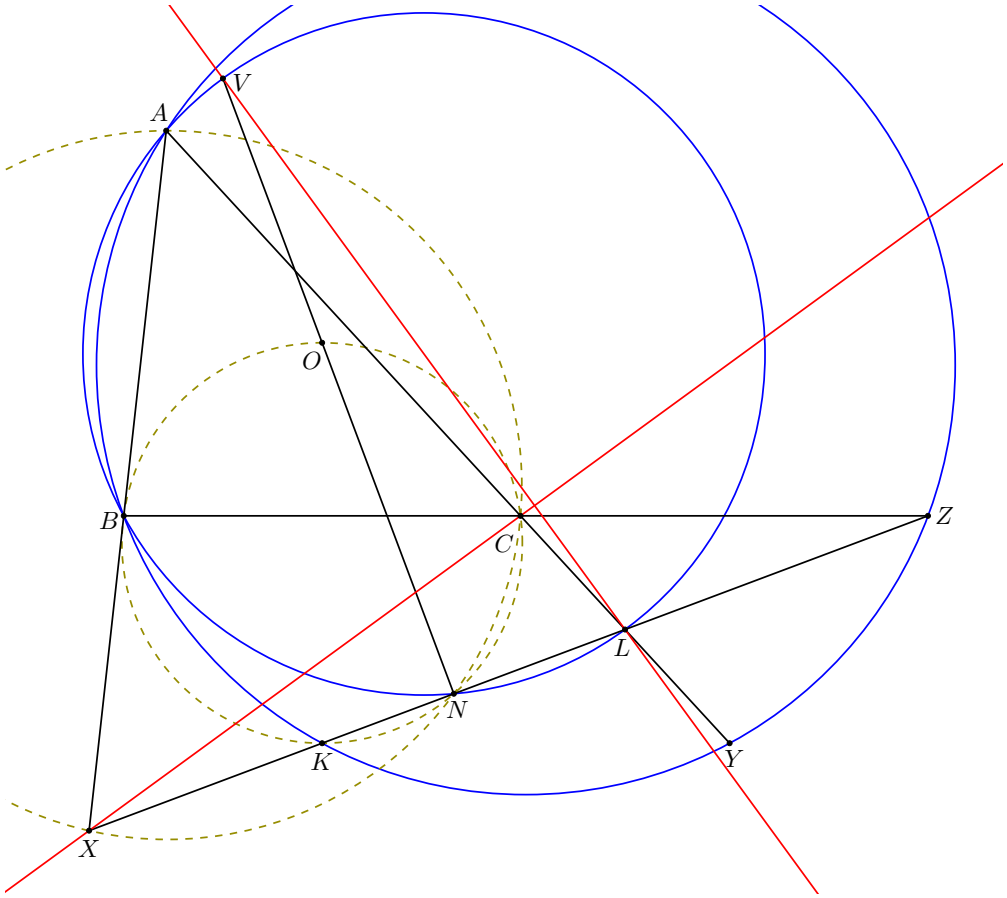
and extend it in the same way. The proof that this sequence satisfies the problem conditions is similar. Moreover, we have $a_r = a_{2r} + a_{2r+1} = 2 \cdot \frac{1}{r+1}$*, as desired.*

Problem 5, EGMO 2026

P5

Let ABC be an acute triangle with $AC > AB$. Denote by ω its circumcircle and by O its circumcentre. Let K be the intersection of the tangents to ω at B and C. Circle ABK intersects line BC again at $Z \neq B$. Let L be the midpoint of KZ. Let X be the intersection of lines KZ and AB. Let V be the point on circle ABL on the same side of BC as A such that OV is perpendicular to KZ. Prove that LV is perpendicular to CX.

Solutions



We split the proof in three claims; we show in turn.

➢ **Claim 1.** The point L is on the line AC.

➢ **Claim 2.** The foot of the altitude from O to KZ is on the circle ABL.

➢ **Claim 3.** The lines VL and CX are orthogonal.

Each claim relies on the previous ones. For each of the claim we give multiple proofs.

We denote by M the midpoint of BC, by N the foot of the perpendicular from O to KZ, by Y the intersection of the line AC with the circle ABK and by U the intersection of the line AK with the circle ABC.

Claim 1. *The point L is on the line AC.*

Proof 1 of Claim 1. Extend the line AC to meet the circle ABK again at Y. From the alternate segment theorem, we have

$$\angle ZYA = \angle ZBA = \angle KCY \text{ and } \angle AYK = 180^\circ - \angle KBA = \angle ACB.$$

Hence $ZY \parallel KC$ and $KY \parallel CZ$ so $KCZY$ is a parallelogram. In particular, as L is the midpoint of KZ, it is also the midpoint of CY. Hence L lies on AC. □

Proof 2 of Claim 1. Let AC intersect KZ at L'. Then we have

$$\angle AZL' = \angle AZK = 180^\circ - \angle KBA = \angle ACB = 180^\circ - \angle ZCA$$

and

$$\angle L'KA = \angle ZKA = \angle ZBA = 180^\circ - \angle ACK.$$

These two equalities show that the line KZ is tangent to both circles ACZ and ACK so, by power of point,

$$L'K^2 = L'C \cdot L'A = L'Z^2.$$

This implies $L'K = L'Z$ and thus L' is the midpoint of KZ. We have proved that $L' = L$. □

Proof 3 of Claim 1. Let L^* be the second intersection of the line AC with the circle MAZ. Then,

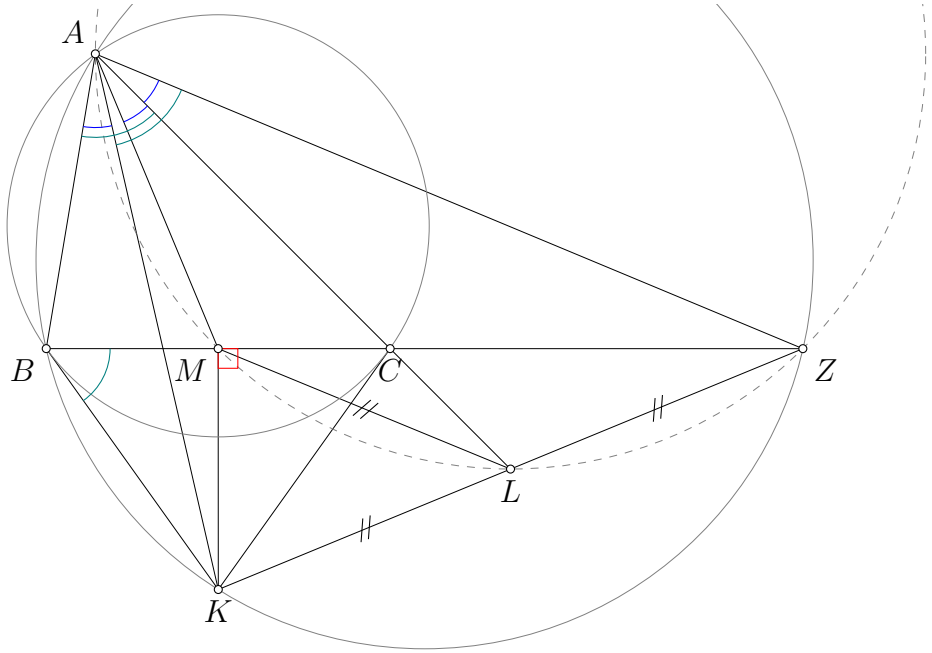
$$\angle BZL^* = \angle MAL^* = \angle BAK = \angle BZK,$$

where the second equality comes from the fact that AK is the A-symmedian of ABC. This means that K, Z and L^* are colinear.

Moreover, $\angle KAZ = \angle KBZ = \angle BAC$, so

$$\angle CAZ = \angle KAZ - \angle KAC = \angle BAC - \angle KAC = \angle BAK = \angle MAC.$$

Hence AL^* is the angle bisector of $\angle MAZ$, and $L^*M = L^*Z$. Since $KM \perp MZ$ we also have $KL^* = L^*Z$, so $L^* = L$.



□

Proof 4 of Claim 1. We prove that the triangles ABC and AKZ are similar. This can be proven in many ways but we give a simple solution by angle-chasing. We have indeed

$$\angle AKZ = \angle ABZ = \angle ABC \text{ and } \angle KAZ = \angle KBZ = \angle BAC.$$

Let M be the midpoint of BC. Since M and L are the midpoints of the sides BC and KZ this implies that $ABCM$ and $AKZL$ are similar and so $\angle MAC = \angle LAZ$.

We know that the line AK is the A-symmedian in ABC which means $\angle BAK = \angle MAC$. We have

$$\angle BAK + \angle KAZ = \angle BAZ = \angle BAC + \angle CAZ.$$

Since $\angle KAZ = \angle BAC$, this means that

$$\angle LAZ = \angle MAC = \angle BAK = \angle CAZ$$

and A, C and L are colinear. □

Proof 5 of Claim 1. As L is the midpoint of KZ, proving that A, C and L are colinear is equivalent to proving that the triangles AKC and ACZ have same area. We have now,

$$\begin{aligned} & [AKC] = [ACZ] \\ \Leftrightarrow \quad & \frac{AK \cdot AC \cdot \sin(\angle KAC)}{2} = \frac{AZ \cdot AC \cdot \sin(\angle CAZ)}{2} \\ \Leftrightarrow \quad & AK \cdot \sin(\angle KAC) = AZ \cdot \sin(\angle CAZ). \end{aligned}$$

We have $\angle KAZ = \angle KBZ = \angle KBC = \angle BAC$, so $\angle BAK = \angle CAZ$. This means that we want to prove that

$$\frac{AK}{AZ} = \frac{\sin(\angle BAK)}{\sin(\angle KAC)}.$$

We have

$$\angle AKZ = \angle ABZ = \angle ABC \text{ and } \angle KAZ = \angle KBZ = \angle BAC,$$

so the triangles ABC and AKZ are similar. This means that

$$\frac{AK}{AZ} = \frac{AB}{AC} = \frac{BU}{CU} = \frac{\sin(\angle BAU)}{\sin(\angle CAU)} = \frac{\sin(\angle BAK)}{\sin(\angle CAK)},$$

where the second equality comes from the harmonicity of the points A, B, U and C. □

Proof 6 of Claim 1. Since $\angle CUA = \angle CBA = \angle ZKA$, we have that the lines UC and KZ are parallel. Then, using the fact that the points A, U, B and C are harmonic we can project from C to the line KZ in order to obtain that the points K, Z, $CU \cap KZ$ and $AC \cap KZ$ are harmonic. From the parallelity, we have that $CU \cap KZ = \infty_{KZ}$ and thus AC intersect KZ in its midpoint. □

Proof 7 of Claim 1. One has that

$$\angle LKA = \angle ZKA = \angle ZBA = 180^\circ - \angle ACK,$$

hence, the line KZ is tangent to ACK and this means that $\angle CKA = \angle CAK$. Moreover,

$$\angle KCZ = 180^\circ - \angle BCK = 180^\circ - \angle BAK = 180^\circ - \angle KBC = 180^\circ - \angle KAZ.$$

We have thus proven that C is the A-Humpty point of triangle AZK which implies that the points A, C and L are colinear. □

Claim 2. *The foot N of the altitude from O to KZ is on the circle ABL.*

Proof 1 of Claim 2. Since $\angle OCK = \angle ONK = \angle KBO = 90°$, we have that $OCNKB$ is cyclic. From this, we get,

$$\angle BNK = \angle BCK = \angle BAC = \angle BAL,$$

where the last equality follows from the fact that L lies on the line AC (by Claim 1). □

Proof 2 of Claim 2. We start by proving the following claim.

Claim 2.1. *The circumcircles of ABC and AKL are tangent.*

Proof of Claim 2.1. Let t be the tangent to ABC at A. Then, $(t, AC) = \angle ABC = \angle AKL$. This finishes the proof of the claim. □

Now, we look at the inversion $\mathcal{I}$ of centre X and radius $r = \sqrt{XB \cdot XA}$. The inversion $\mathcal{I}$ exchanges B and A. Since, $ABKZ$ is cyclic, the inversion $\mathcal{I}$ also exchanges K and Z and in order to prove Claim 2, we need to show that the points L and N are also exchanged by $\mathcal{I}$. From Claim 2.1, and using the inversion $\mathcal{I}$, we just have to show that the circumcircles of BNZ and ABC are tangent.

This last fact follows from

$$KB^2 = KM \cdot KO = KN \cdot KZ,$$

where we used that $OMNZ$ is cyclic as $\angle OMZ = \angle ONZ = 90°$ in the last equality. □

Proof 3 of Claim 2. We prove the following.

Claim 2.2. *The points B, U and N are colinear.*

Proof 1 of Claim 2.2. We define N' as the intersection of BU with the perpendicular from O to KZ. We have first that $\angle CUA = \angle ABC = \angle AKZ$, which means that $UC \parallel KZ$ and $ON' \perp CU$. We prove now that $BKN'O$ is cyclic as

$$\angle BN'O = 90° - \angle CUN' = 90° - \angle BAC = 90° - \angle CBK = \angle OKB.$$

Moreover, $OBKN$ is also cyclic since $\angle OBK = \angle ONK = 90°$. Since O, N' and N are colinear we are done. □

Proof 2 of Claim 2.2. We have that $OBKN$ is cyclic since $\angle OBK = \angle ONK = 90°$, as is $OMNZ$ since $\angle OMZ = \angle ONZ = 90°$. We have

$$\angle NBK = \angle NOK = \angle NOM = \angle NZM = \angle BAK = \angle UBK.$$

This finishes the proof of the claim. □

Since U is on the line BN and A, U, B and C are harmonic, by projecting from B to the line KZ we have that the points X, N, K and Z are in harmonic configuration. From MacLaurin's identity we have that

$$XN \cdot XL = XK \cdot XZ = XA \cdot XB.$$

This finishes the proof of Claim 2. □

Proof 4 of Claim 2. We remark that we have that $OBKN$ is cyclic since $\angle OBK = \angle ONK = 90^\circ$, as is $OMNZ$ since $\angle OMZ = \angle ONZ = 90^\circ$.

Let t be the tangent line to the circle ABC in A. It is well-known that the lines t, AK, AB and AC are in harmonic configuration. However, we have

$$(t, AB) = \angle ACB = \angle XBK, \angle BAC = \angle KBC$$

and

$$\angle KBN = \angle NOK = \angle NOM = \angle NZM = \angle BAK,$$

which means that the lines angles between the quadruple of lines BX, BN, BK and BZ and the quadruple of lines t, AK, AB and AC are the same. Hence, the lines BX, BN, BK and BZ are also in harmonic configuration and by projecting on line KZ, that the points X, N, K and Z are in harmonic configuration. □

Remark. *The proof number 3 and 4 of Claim 2., that are very similar in spirit, do not use claim 1.*

Claim 3. *The lines VL and CX are orthogonal.*

Proof 1 of Claim 3. We start by proving the following claim.

Claim 3.1. *The quadrilateral $ACNX$ is cyclic.*

Proof 1 of Claim 3.1. Since $\angle KBO = \angle KCO = \angle KNO = 90°$, we have $OCNKB$ cyclic. So

$$\angle LNC = 180° - \angle CNK = \angle KBC = \angle BAC = \angle XAC$$

proving the claim. □

Proof 2 of Claim 3.1. We first notice that since

$$\angle LCZ = \angle ACB = 180° - \angle ABK = \angle AZL,$$

we have that the circle ACZ is tangent to the line ZK.

We have that $ABNL$ and $ABKZ$ are cyclic, hence by power of a point,

$$XN \cdot XL = XA \cdot XB = XK \cdot XZ.$$

Since L is the midpoint of segment KZ, from Newton's identity we have that X, N, K and Z are harmonic. Using now MacLaurin's identity and the above tangency, we have $LN \cdot LX = LZ^2 = LC \times LA$, which finishes the proof of the claim. □

We are now done as, from the above claims and $VN \perp KZ$,

$$\angle LXC = \angle NXC = \angle NAC = \angle NAL = \angle NVL = 90° - \angle VLN = 90° - \angle VLX$$

which gives $CX \perp LV$. □

Proof 2 of Claim 3. We prove a much needed cyclicity.

Claim 3.2. *The quadrilateral $MAZL$ is cyclic.*

Proof 1 of Claim 3.2. We have

$$\angle KZM = \angle KZB = \angle KAB = \angle CAM,$$

where the last equality comes from the fact that the line AK is the symmedian. □

Proof 2 of Claim 3.2. The third proof of Claim 1 already proves Claim 3.2. □

Now we can finish the proof of Claim 3 as follows,

$$CA \cdot CL = CM \cdot CZ$$

and

$$XB \cdot XA = XK \cdot XZ,$$

so that CX is the radical axis of the circles $ABNL$ and MKZ. Letting O' be the center of $ABNL$, we obtain that $O'L \perp CX$, and since VL is a diameter of $ABNL$, we finally obtain $VL \perp CX$. □

Proof 3 of Claim 3. We start by a small claim.

Claim 3.3. *The point L is the midpoint of CY.*

Proof 1 of Claim 3.3. We already know that L is on the line CY by Claim 1. We have thus,

$$\angle LYZ = \angle AYZ = \angle ABC = \angle KCL.$$

This implies that the lines CK and YZ are parallel and since $KL = LZ$, it implies that $CL = LY$. □

Proof 2 of Claim 3.3. Proof 1 of Claim 1 also proves Claim 3.3. □

Now we can finish the proof of Claim 3 as follows,

$$2CA \cdot CL = CA \cdot CY = CB \cdot CZ = 2CM \cdot CZ$$

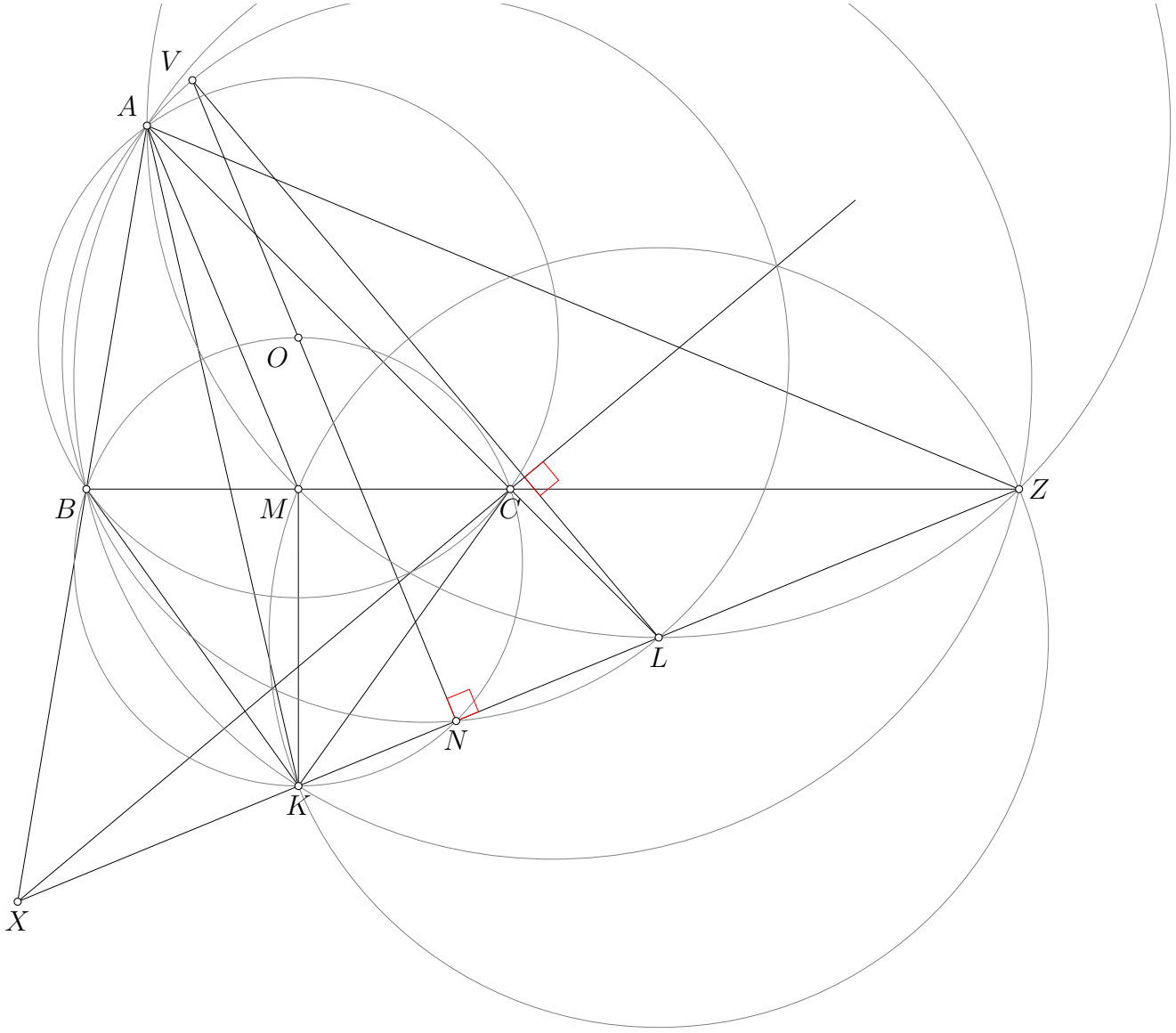
and

$$XB \cdot XA = XK \cdot XZ,$$

so that CX is the radical axis of the circles $ABNL$ and MKZ. Letting O' be the center of $ABNL$, we obtain that $O'L \perp CX$, and since VL is a diameter of $ABNL$, we finally obtain $VL \perp CX$.

□

Proof 4 of Claim 3. □


V
A
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X

Problem 6, EGMO 2026

P6

Let p be a prime number and let n be a positive integer such that p does **not** divide n. Denote by k the number of positive divisors of n, and by $1 = d_1 < d_2 < \cdots < d_k = n$ the positive divisors of n. For $i = 1, 2, \ldots, k$, let c_i be the number of positive divisors ℓ of d_i^2 such that $d_i - \ell$ is divisible by p. Prove that

$$(p-1)\,(c_1 + c_2 + \cdots + c_k) \geq k^2.$$

Solutions

Solution 1

Claim 1.1: The number of ordered pairs of divisors (d, d') such that $d \equiv d' \mod p$ (where d may be equal to d') at least $k^2/(p-1)$.

Proof of Claim 1.1: For each $t \in \{1, 2 \ldots p-1\}$, let N_t be the number of d congruent to t mod p. Consider the divisors of n modulo p. We know that $k = N_1 + N_2 + \ldots + N_{p-1}$, since $p \nmid n$, and the number of ordered pairs (d, d') such that $d \equiv d' \mod p$ is $N_1^2 + N_2^2 + \ldots + N_{p-1}^2$. By the arithmetic mean-quadratic mean inequality, we have

$$N_1^2 + N_2^2 + \ldots + N_{p-1}^2 \geq \frac{(N_1 + N_2 + \ldots + N_{p-1})^2}{p-1} = \frac{k^2}{p-1}.$$

■

Now it suffices to prove that the number of pairs (d_i, ℓ) is at least the number of ordered pairs of divisors (d, d') such that $d \equiv d' \mod p$. We do so by constructing for each such pair (d_i, d_j) a pair (a, b) with $a \mid n$, $b \mid a^2$ and $p \mid a - b$, such that $a|n$ and $b|a^2$ and all such pairs (a, b) are distinct.

For a pair (d_i, d_j) with $d_i \equiv d_j \mod p$ we define $a = \operatorname{lcm}(d_i, d_j)$ and $b = \frac{d_j^2}{\gcd(d_i, d_j)}$. Since $d_i, d_j \mid n$ we have $a = \operatorname{lcm}(d_i, d_j) \mid n$. Also, $b \mid d_j^2 \mid (\operatorname{lcm}(d_i, d_j))^2 = a^2$. Now note that $\operatorname{lcm}(d_i, d_j) = \frac{d_i d_j}{\gcd(d_i, d_j)}$ and hence $(a, b) = (d_i \cdot \frac{d_j}{\gcd(d_i, d_j)}, d_j \cdot \frac{d_j}{\gcd(d_i, d_j)})$. As $\frac{d_j}{\gcd(d_i, d_j)}$ is an integer and $p \mid d_i - d_j$, we also find $p \mid a - b$.

Now we just need to prove that all the pairs (a, b) we create in this way are distinct. Suppose (a, b) is created from both (d_i, d_j) and (d_i', d_j'). We have

$$\gcd(a, b) = \gcd\left(d_i \cdot \frac{d_j}{\gcd(d_i, d_j)}, d_j \cdot \frac{d_j}{\gcd(d_i, d_j)}\right) = \gcd(d_i, d_j) \cdot \frac{d_j}{\gcd(d_i, d_j)} = d_j,$$

while similarly $\gcd(a, b) = d_j'$, so $d_j = d_j'$. Since $\frac{d_i}{d_j} = \frac{a}{b} = \frac{d_i'}{d_j'}$ we now have $d_i = \frac{a}{b} \cdot d_j = \frac{a}{b} \cdot d_j' = d_i'$. We conclude that $(d_i, d_j) = (d_i', d_j')$. So all pairs (a, b) that we create are indeed distinct, which is the last part we had to prove. □

Remark. *The proof uses that* $(d_i, d_j) \mapsto (a, b) = \left(\text{lcm}(d_i, d_j), \frac{d_j^2}{\gcd(d_i, d_j)}\right)$ *defines an injection from the set of ordered pairs* (d_i, d_j) *with* $d_i, d_j \mid n$ *and* $p \mid d_i - d_j$ *to the set of ordered pairs* (a, b) *with* $a \mid n$, $b \mid a^2$ *and* $p \mid a - b$. *It is straightforward to show that this map is in fact a bijection, from which it follows that* $c_1 + \ldots + c_k = P$. *This also implies that equality occurs in* $(p-1)(c_1 + \ldots + c_k) \geq k^2$ *precisely when equality occurs in* $N_1^2 + \ldots + N_{p-1}^2 \geq (N_1 + \ldots + N_{p-1})^2/(p-1)$, *which happens when* $N_1 = \cdots = N_{p-1}$. *So equality holds in the desired inequality if and only if the divisors of* n *are equidistributed over the (nonzero) residue classes modulo* p.

Solution 2

Let $n = \prod_{j=1}^{s} q_j^{\alpha_j}$ be the prime factorisation of n. Then we can rephrase the conditions as follows:

- $d_i = \prod_{j=1}^{s} q_j^{\gamma_j}$ is s.t. $\gamma_j \leq \alpha_j$
- $\ell = \prod_{j=1}^{s} q_j^{\delta_j}$ is s.t. $\delta_j \leq 2\gamma_j$

Furthermore, let $r = d_i \ell^{-1} = \prod_{j=1}^{s} q_j^{\beta_j}$.

Claim 2.1: For any fixed r such that $r \equiv 1 \pmod{p}$, the number of pairs (d_i, ℓ) with d_i a divisor of n and ℓ a divisor of d_i^2 such that $r = \frac{d_i}{\ell}$ is the same as the number of pairs of divisors of n, (d, d') such that $rd = d'$.

Proof of Claim 2.1: We prove both are equal to $\prod_{j=1}^{s}(\alpha_j - |\beta_j| + 1)$. The number of pairs (d_i, ℓ), is the same as the number of d_i such that $0 \leq \gamma_i - \beta_i = \delta_i \leq 2\gamma_i$ and $\gamma_i \leq \alpha_i$ hence γ_j can take value in $\{|\beta_j|, \ldots \alpha_j\}$ hence $\prod_{j=1}^{s}(\alpha_j - |\beta_j| + 1)$ such pairs (d_i, ℓ) for each r.

The number of pairs (d, d') is the same as the number of $d = \prod_{j=1}^{s} q_j^{\gamma_j}$ such that for every j, $0 \leq \gamma_j, \gamma_j + \beta_j \leq \alpha_j$. Each γ_j can take any value in $\{0, \ldots \alpha_i - |\beta_i|\}$, hence $\prod_{j=1}^{s}(\alpha_j - |\beta_j| + 1)$ such pairs $(d_i, d_{i'})$ for each r. ■

Remark. *The claim holds without the congruence condition on* r.

Note that the sum $c_1 + \ldots + c_k$ is exactly the number of distinct pairs (d_i, ℓ) such that $r = d_i/\ell$ and $r \equiv 1 \pmod{p}$. By Claim 2.1, we know

$$c_1 + c_2 + \ldots + c_k = \text{Number of pairs of divisors } (d, d') \text{ such that } d \equiv d' \pmod{p}.$$

By Claim 1.1, we are done. □

Solution 3

This solution is similar in spirit to the previous two.

For any divisor d of n, let C_d denote the set of positive integers ℓ dividing d^2 such that $d \equiv \ell \pmod{p}$. We also define S_d as the set of pairs (u, v) of coprime divisors of d such that $u \equiv v \pmod{p}$. Finally, we let S denote the set of pairs of divisors (u', v') of n such that $u' \equiv v' \pmod{p}$.

Claim 3.1: There is a bijection between C_d and S_d.

Proof of Claim 3.1: We define $g_d : S_d \to C_d$ by $g_d((u, v)) = du/v$. This is well-defined as $v|d$ implies du/v is an integer, $u|d$ implies it divides d^2, and $du/v \equiv d \pmod{p}$ from the congruence condition on u and v. Reciprocally, $f_d : C_d \to S_d$ can be defined by sending $\ell \in C_d$ to the pair formed by the numerator and denominator of ℓ/d. ■

Remark. $g_d((u, v)) = du/v$ *is essentially the same as the map in Solution 1 the pairs of divisors* $d/u, d/v$, *is sent to* $d = lcm(d/u, d/v)$ *and* $\ell = du/v = g_d(u, v)$.

Claim 3.2: $c_1 + \ldots + c_k = \sum_{d|n} \#C_d = \sum_{d|n} \#S_d$.

Proof of Claim 3.2: This follows immediately from Claim 3.1 and the definition of the c_i. ■

Claim 3.3: $\sum_{d|n} \#S_d = \#S$.

Proof of Claim 3.3:

- We express the LHS in the following way. We fix a pair (u, v) of coprime divisors of n such that $u \equiv v$ and count the number of S_d in which it appears. Because of the coprimality condition, this is given by the number of divisors of $\frac{n}{uv}$, which we denote $\tau(\frac{n}{uv})$.
- Now look at the RHS. For each $(u', v') \in S$, note that defining $g := \gcd(u', v')$, $u = u'/g$ and $v = v'/g$, we have $(u, v) \in S_d$ for any d such that $uv|d$. The number of elements of S can again be counted by first looking at pairs (u, v) of possible coprime congruent divisors of n, which are obtained each for $\tau(\frac{n}{uv})$ possible values of d, which is exactly what was obtained previously.

■

We then conclude as in Solutions 1 and 2. □

Solution 4

Define $\alpha_j, \beta_j, \gamma_j, \delta_j$ as in Solution 2. Then, the bounds on $\alpha_j, \gamma_j, \delta_j$ are equivalent to $\alpha_j \geq \gamma_j \geq |\beta_j|$ for each j. $c_1 + \cdots + c_k$ is the number of such tuples such that $\prod_{j=1}^{s} q_j^{\beta_j} \equiv 1 \pmod{p}$.

First, we count the number of tuples without the last condition. Fixing each γ_j, there are $2\gamma_j + 1$ possible values of β_j. Hence, the number of such tuples is

$$\prod_j \sum_{\gamma_j=0}^{\alpha_j} (2\gamma_j + 1) = \prod_j (\alpha_j + 1)^2 = k^2. \tag{1}$$

Now, we show that out of such tuples, at least $\frac{1}{p-1}$ have $\prod_j q_j^{\beta_j} \equiv 1 \pmod{p}$. Let g be a primitive root modulo p. We count the number of tuples such that $\prod_j q_j^{\beta_j} \equiv g^a \pmod{p}$ for each $a = 0, \ldots, p-2$.

In the case where $n = q^\alpha$, where $q \equiv g^b \pmod{p}$, the possible values are $g^{b\beta}$, $\beta = -\alpha, -\alpha+1, \ldots, \alpha$, which have $1, 2, \ldots, \alpha+1, \ldots, 1$ possible values of γ respectively. Denote by $x^{a \pmod{p-1}}$ a term x^c, where $a \equiv c \pmod{p-1}$. Thus, the number of tuples such that $\prod_j q_j^{\beta_j} \equiv g^a \pmod{p}$ is the sum of the coefficients of the $x^{a \pmod{p-1}}$ terms in the expression

$$\left(x^{-\frac{\alpha b}{2}} + x^{-\frac{\alpha b}{2}+b} + \cdots + x^{\frac{\alpha b}{2}}\right)^2$$

(the sum in the bracket is a polynomial in $\sqrt{x}^{\pm}$, but either only has odd terms or even terms).

Hence, in the general case, the number of tuples such that $\prod_j q_j^{\beta_j} \equiv g^a \pmod{p}$ is the sum of the $x^{a \pmod{p-1}}$-coefficients of

$$\prod_j \left(\left(x^{-\frac{\alpha_j b_j}{2}} + x^{-\frac{\alpha_j b_j}{2}+b_j} + \cdots + x^{\frac{\alpha_j b_j}{2}}\right)^2\right) = \left(\prod_j \left(x^{-\frac{\alpha_j b_j}{2}} + x^{-\frac{\alpha_j b_j}{2}+b_j} + \cdots + x^{\frac{\alpha_j b_j}{2}}\right)\right)^2. \tag{2}$$

Expanding the term in the square, the coefficients are symmetric across 0. Thus, collapsing this modulo $p-1$, we are taking the sum of the $x^{a \pmod{p-1}}$-coefficients of some $\left(a_0 + a_1 x + \cdots + a_{p-2} x^{p-2}\right)^2$ or some $\left(a_{\frac{1}{2}} x^{\frac{1}{2}} + a_{\frac{3}{2}} x^{\frac{3}{2}} + \cdots + a_{p-\frac{3}{2}} x^{p-\frac{3}{2}}\right)^2$ where $a_m = a_{p-1-m}$ for each m.

When $a = 0$, this is (by QM-AM, and taking $a_{p-1} = a_0$)

$$\sum_m a_m a_{p-1-m} = \sum_m a_m^2 \geq \frac{1}{p-1} \left(\sum_m a_m \right)^2. \tag{3}$$

On the other hand, $\left(\sum_m a_m\right)^2$ is exactly the sum of all the coefficients, which is the total number of tuples, i.e. k^2. □

Remark. *This method also shows that the number of tuples such that* $r = \prod_{j=1}^{s} q_j^{\beta_j} \equiv 1 \pmod{p}$ *is at least the number of tuples such that* $r \equiv b \pmod{p}$ *for any other* b*: this boils down to showing* $\sum_m a_m^2 \geq \sum_m a_m a_{b-m}$ *(indices modulo* $p-1$*), which holds by rearrangement inequality.*