International Mathematical Olympiad
Hong Kong Preliminary Selection Contest 2023

Outline of solutions

Answers:

1.	449	**2.**	1799	**3.**	362	**4.**	24
5.	$\dfrac{3\sqrt{2}}{2}$	**6.**	15	**7.**	$111\sqrt{3}$	**8.**	81
9.	$\dfrac{1+3\sqrt{5}}{4}$	**10.**	120	**11.**	128	**12.**	3 : 1
13.	$48°$	**14.**	14	**15.**	32	**16.**	$\dfrac{29}{128}$
17.	$1+2\sqrt{2}$	**18.**	1326	**19.**	18	**20.**	67473

Solutions:

1. The number is the recurring decimal $0.\dot{0}7692\dot{3}$. Note that $0+7+6+9+2+3=27$ and $2023=27\times 75-2$. Hence there should be 75 periods of '307692' with the last '2' removed. The answer is thus $75\times 6-1=449$.

2. Note that $\sqrt{2023}=17\sqrt{7}$. Hence we must have $\sqrt{m}=p\sqrt{7}$ and $\sqrt{n}=q\sqrt{7}$ (see remark below), where p and q are positive integers with sum 17. We have

$$m+n=7p^2+7q^2=7p^2+7(17-p)^2=14\left[\left(p-\frac{17}{2}\right)^2+\frac{289}{4}\right],$$

which is maximised when p is as far away from $\frac{17}{2}$ as possible. We should take either $(p,q)=(1,16)$ or $(16,1)$, and the answer is thus $7(1)^2+7(16)^2=1799$.

Remark. It is intuitive that both $\sqrt{m}$ and $\sqrt{n}$ should be integer multiples of $\sqrt{7}$. For a rigorous proof, note that $2023=(\sqrt{m}+\sqrt{n})^2=m+n+2\sqrt{mn}$, so mn must be a perfect square. It follows that $m=ab^2$ and $n=ac^2$ for some positive integers a, b, c. Then $\sqrt{m}+\sqrt{n}=\sqrt{2023}$ simplifies to $(b+c)\sqrt{a}=\sqrt{2023}$, or $a(b+c)^2=7\cdot 17^2$. It follows that a can only be 7 as $b+c>1$.

3. For convenience we shall use the term 'red number' to the number on a card with a red sticker, and similarly for 'blue number'. Clearly, a red number, being a row maximum, is at least 20. Also, since A is the smallest red number, there are 19 other red numbers greater than A, so A is at most 381. The same is true for B. That is, both A and B are between 20 and 381 (inclusive), and so $|A-B|$ is at most 361.

We shall prove that all these 362 possibilities (from 0 to 361) for $|A - B|$ are attainable. Indeed, we can make $A = 381$ and B can be equal to any value between 20 and 381. To make $A = B = 381$ we can use the following configuration (the arrangements of the starred entries are unimportant):

381	*	*	$\cdots$	*
*	382	*	$\cdots$	*
*	*	383	$\cdots$	*
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
*	*	*	$\cdots$	400

On the other hand for any $B \in \{20, 21, ..., 380\}$, we can use the following configuration to obtain such value of B while having $A = 381$:

1	*	*	$\cdots$	*	381
2	382	*	$\cdots$	*	*
3	*	383	$\cdots$	*	*
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
19	*	*	$\cdots$	399	*
B	*	*	$\cdots$	*	400

It follows that the answer is 362.

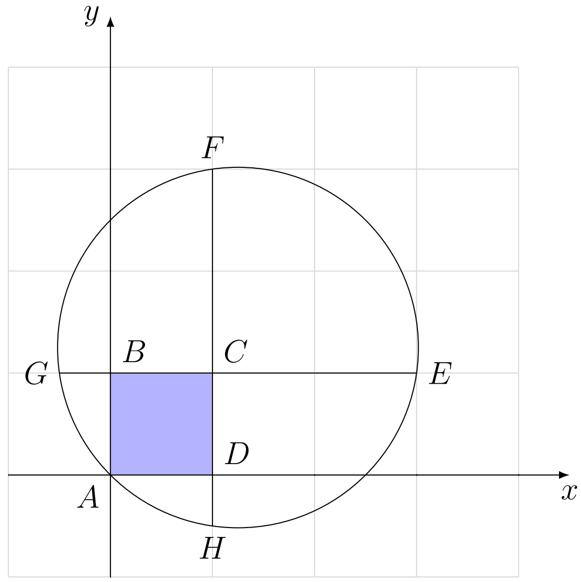
4. Let the three consecutive positive integers be n, $n + 1$ and $n + 2$. Then $1 \leq n \leq 2021$. Since multiples of 6 and 8 must be even, we must have $n+1$ being divisible by 7 whereas n and $n + 2$ are divisible by 6 and 8 in either order.

   - If n is divisible by 6, then n leaves a remainder of 6 when divided by each of 7 and 8 (so that $n + 1$ is divisible by 7 and $n + 2$ is divisible by 8). Hence $n = 56k + 6$ for some integer k, and for n to be divisible by 6, k must be divisible by 3. Since $1 \leq n \leq 2021$, there are 12 possible values of k, namely, 0, 3, 6, ..., 33.
   - If n is divisible by 8, then n leaves a remainder of 6 when divided by 7 and a remainder of 4 when divided by 6 (so that $n+1$ is divisible by 7 and $n+2$ is divisible by 6). Hence $n = 42k + 34 = 8(5k + 4) + 2(k + 1)$ for some integer k, and for n to be divisible by 8, we must have $k \equiv 3 \pmod 4$. Since $1 \leq n \leq 2021$, there are 12 possible values of k, namely, 3, 7, 11, ..., 47.

   It follows that the answer is $12 + 12 = 24$.

   Remark. If one knows the *Chinese remainder theorem*, one can deduce more quickly that there are exactly two choices of n among any 168 consecutive positive integers subject to the upper bound of 2023 (here 168 is the L.C.M. of 6, 7 and 8), and hence the answer can be obtained more quickly.

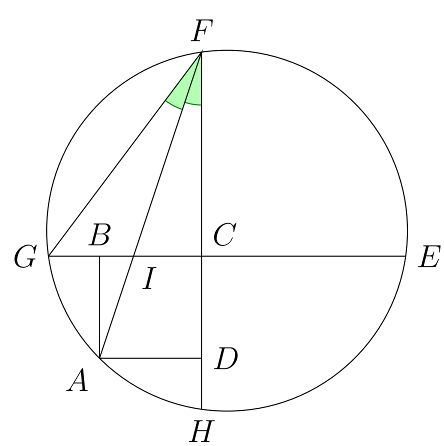
5. Set the origin at A and the directions of AD and AB be the positive x-axis and positive y-axis respectively. Then the coordinates of E and F are $(3,1)$ and $(1,3)$ respectively.



The circumcircle of $\triangle AEF$ has equation $x^2+y^2+kx+ky=0$ for some k (note that the constant term vanishes as the circle passes through the origin, and that the coefficients of x and y are the same by symmetry). Setting $x=3$ and $y=1$ gives $k=-\frac{5}{2}$.

Let the coordinates of G be $(g,1)$. Putting this into the equation of the circle, we get $g^2+1^2-\frac{5}{2}g-\frac{5}{2}(1)=0$, or $\frac{1}{2}(g-3)(2g+1)=0$. Hence $g=-\frac{1}{2}$, i.e. G has coordinates $(-\frac{1}{2},1)$. By symmetry, the coordinates of H are $(1,-\frac{1}{2})$. It follows that $GH=\frac{3}{2}\sqrt{2}$.

Alternative solution. We may also solve the problem using pure geometry. Note that $AG=AH$ by symmetry. Hence they subtend equal angles on the circumference of the circumcircle, i.e. we have $\angle AFG=\angle AFH$. Suppose FA meets GE at I.

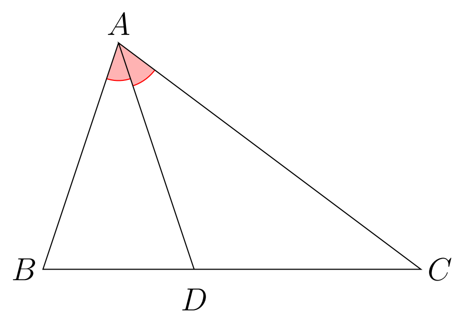


By similar triangles, we have $BI:IC=AB:CF=1:2$. Hence $IC=\frac{2}{3}$. Let $GC=x$. Using the angle bisector theorem (see remark below), we have $GI:IC=FG:FC$, i.e.

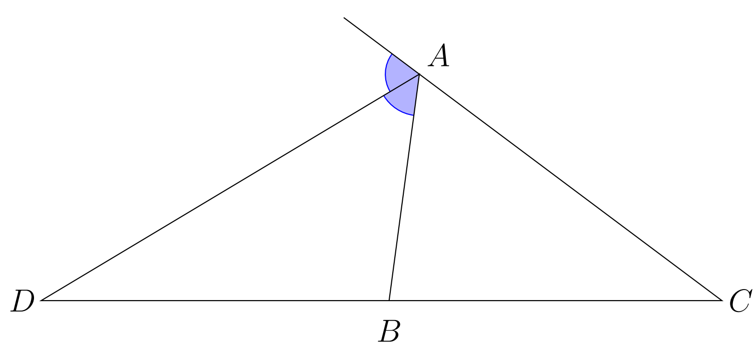
$$\frac{x-\frac{2}{3}}{\frac{2}{3}}=\frac{\sqrt{x^2+2^2}}{2}.$$

Solving gives $x = \frac{3}{2}$. By symmetry, HC also has length $\frac{3}{2}$ and so $GH = \frac{3}{2}\sqrt{2}$.

Remark. The *angle bisector theorem* asserts that the internal bisector of an angle of a triangle divides the opposite side in a ratio proportional to the lengths of the other two sides, i.e. in the figure below we have $BD : DC = AB : AC$.



The theorem can be proved by using the sine formula in $\triangle ADB$ and $\triangle ADC$, or by considering a point E on the extension of AD such that AB and CE are parallel (and then making use of similar triangles). There is also an external version, which says essentially the same thing except that we are considering the bisector of an external angle meeting the extension of the opposite side, dividing it in the same (external) ratio. More precisely, it says that in the figure below we again have $BD : DC = AB : AC$.



The proof can be obtained by slightly modifying a proof of the internal version of the theorem.

6. Note that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Hence, for $n+3$ to divide $1^3 + 2^3 + \cdots + n^3$, it is necessary for $n^2(n+1)^2$ to be a multiple of $n+3$, or equivalently, by setting $m = n+3$,

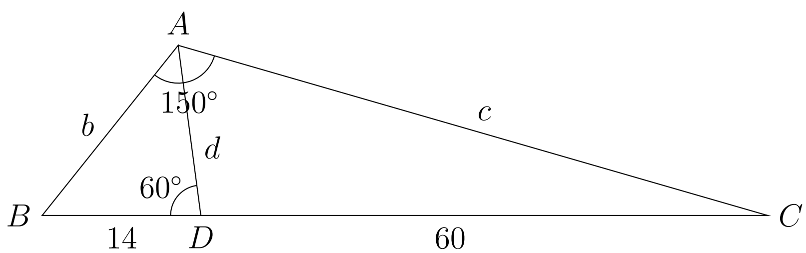
$$n^2(n+1)^2 = (m-3)^2(m-2)^2 = m^4 - 10m^3 + 37m^2 - 60m + 36$$

must be a multiple of m. Hence $m = n+3$ must be a factor of 36. To find the greatest possible value of n, we proceed as follows:

- If $m = 36$ (i.e. $n = 33$), we have $1^3 + 2^3 + \cdots + 33^3 = \dfrac{33^2 \cdot 34^2}{4} = 33^2 \cdot 17^2$ which is odd (and hence not divisible by $n + 3 = 36$).
- The next largest possible value of m is 18 (corresponding to $n = 15$), in which case we have
$$1^3 + 2^3 + \cdots + 15^3 = \frac{15^2 \cdot 16^2}{4} = 15^2 \cdot 8^2 = 2^6 \cdot 3^2 \cdot 5^2,$$
which is divisible by $n + 3 = 2 \cdot 3^2$.

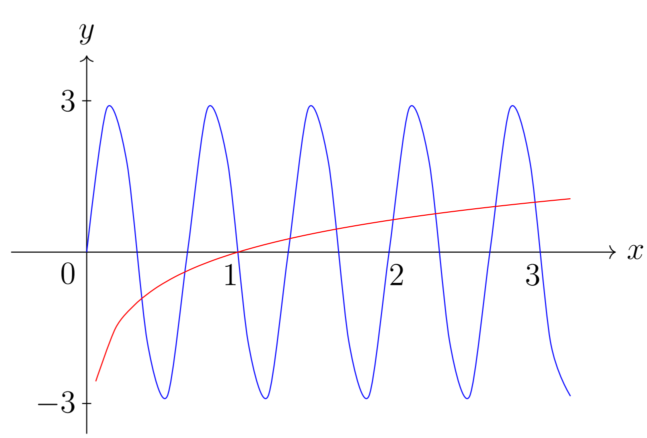
It follows that the answer is 15.

7. Let the lengths of AB, AC and AD be b, c, d respectively.



The area of $\triangle ABC$ is $\frac{1}{2}bc\sin 150°$. Alternatively, using BC as base, the height of the triangle is $d\sin 60°$ and so the area is also equal to $\frac{1}{2}(74)d\sin 60°$. This gives
$$\frac{1}{2}bc\sin 150° = \frac{1}{2}(74)d\sin 60°,$$
which simplifies to $bc = 74\sqrt{3}d$. Using this and the cosine formula, we have
$$\begin{aligned}74^2 &= b^2 + c^2 - 2bc\cos 150° \\ &= [d^2 + 14^2 - 2d(14)\cos 60°] + [d^2 + 60^2 - 2d(60)\cos 120°] - 2(74\sqrt{3}d)\cos 150° \\ &= 2d^2 + 268d + 3796,\end{aligned}$$
which simplifies to $2(d-6)(d+140) = 0$. Hence $d = 6$ and so the area of $\triangle ABC$ is $\frac{1}{2}(74)d\sin 60° = 111\sqrt{3}$.

8. A sketch of the graphs of $y = \log_3 x$ and $y = 3\sin(3\pi x)$ is as follows:



Note that the right hand side of the equation (i.e. $3\sin(3\pi x)$) must be between -3 and 3. Hence any solution satisfies $-3 \le \log_3 x \le 3$, or $\frac{1}{27} \le x \le 27$. There are three solutions for $x \le 1$ as shown in the graph above. For $x \in (1, 27]$, we have $\log_3 x > 0$ and the function $3\sin(3\pi x)$ is positive in 39 intervals, namely, $(\frac{4}{3}, \frac{5}{3})$, $(\frac{6}{3}, \frac{7}{3})$, ..., $(\frac{80}{3}, \frac{81}{3})$. In each of these intervals the graph of $\log_3 x > 0$ intersects the graph of $3\sin(3\pi x)$ at exactly two points. The answer is thus $3 + 39(2) = 81$.

9. Note that when $x = 1$, the expressions inside both cube roots are equal to 0. In view of the factor theorem, we can factorise the expressions to get

$$\sqrt[3]{(x-1)(x+2)^2} - x = \sqrt[3]{(x-1)^2(x+2)} - 1.$$

If we let $u = \sqrt[3]{x-1}$ and $v = \sqrt[3]{x+2}$, the equation becomes $uv^2 = u^2v + u^3$, or

$$u(u^2 + uv - v^2) = 0.$$

If $u = 0$, then $x = 1$. If there is another root $x > 1$, then u, v must both be positive, so $u^2 + uv - v^2 = 0$ gives $u = \lambda v$ where $\lambda = \frac{1}{2}(-1 + \sqrt{5})$. Hence $u^3 = (\lambda v)^3$, i.e. $x - 1 = \lambda^3(x + 2)$, so we have

$$x = \frac{1 + 2\lambda^3}{1 - \lambda^3} = \frac{1 + 2(\sqrt{5} - 2)}{1 - (\sqrt{5} - 2)} = \frac{1 + 3\sqrt{5}}{4}.$$

It is easy to check that such x indeed satisfies the original equation, and hence is the largest real root to the equation.

10. Let $f(n) = n(n+1)(n+2)(n+3)(n+4)$. Note that

$$\begin{aligned} f(n+1) - f(n) &= (n+1)(n+2)(n+3)(n+4)(n+5) - n(n+1)(n+2)(n+3)(n+4) \\ &= 5(n+1)(n+2)(n+3)(n+4). \end{aligned}$$

Hence we have

$$1 \times 2 \times 3 \times 4 = \frac{f(1) - f(0)}{5}, \quad 2 \times 3 \times 4 \times 5 = \frac{f(2) - f(1)}{5}, \quad \text{and so on.}$$

It follows that the sum in the question is equal to

$$\frac{f(1) - f(0)}{5} + \frac{f(2) - f(1)}{5} + \cdots + \frac{f(2023) - f(2022)}{5} = \frac{f(2023) - f(0)}{5} = \frac{f(2023)}{5},$$

which in turn is equal to

$$\frac{2023 \times 2024 \times 2025 \times 2026 \times 2027}{5} = 405 \times 2023 \times 2024 \times 2026 \times 2027.$$

Taking modulo 1000, it suffices to consider

$$\begin{aligned} 405 \times 23 \times 24 \times 26 \times 27 &= 405 \times (25^2 - 2^2) \times (25^2 - 1^2) \\ &= 405 \times 624 \times 621 \\ &= 10 \times (81 \times 312 \times 621). \end{aligned}$$

It thus suffices to consider the last two digits of $81 \times 312 \times 621$, or simply the last two digits of $81 \times 12 \times 21$. Direct computation shows that they are 12, and so the answer is 120.

11. With some initial trials, we can list the numbers of the students who get candies:

$$1,\ 2,\ 4,\ 5,\ 10,\ 11,\ 13,\ 14,\ 28,\ 29,\ \ldots$$

The numbers seems to be related to powers of 3 (this becomes more evident when more numbers are listed). Indeed, the base 3 representation of the numbers are

$$1,\ 2,\ 11,\ 12,\ 101,\ 102,\ 111,\ 112,\ 1001,\ 1002,\ \ldots,$$

and if we subtract 1 from each number, they become the following

$$0,\ 1,\ 10,\ 11,\ 100,\ 101,\ 110,\ 111,\ 1000,\ 1001,\ \ldots,$$

which are precisely those whose base 3 representations consist of the digits 0 and 1 only. Indeed, we can easily prove by induction that student n gets a candy if and only if the base 3 representation of $n-1$ consists of the digits 0 and 1 only (call such a positive integer n 'good'). This statement holds for the numbers we have listed. Now suppose n is larger than 29 and the statement holds for smaller n.

- Suppose n is good. If student n does not get a candy, there exist students ℓ and m who both got candies (where $\ell < m < n$) and such that ℓ, m, n form an arithmetic sequence. Consider the numbers $\ell - 1$, $m - 1$, $n - 1$ (in base 3, same for below). They all consist of the digits 0 and 1 only, and $2(m-1) = (\ell - 1) + (n - 1)$. This is impossible, since each digit on the left is 0 or 2, while at least one digit on the right is 1 (there is no carry in the addition, and there must be at least a place which gives $0 + 1$ since $\ell - 1 \neq n - 1$). This contradiction establishes the fact that student n gets a candy.
- Suppose n is not 'good', so at least one digit (say, the one whose place value is 3^k of the base 3 representation of $n-1$ is 2. Then student n cannot get a candy since students $n - 2 \cdot 3^k$ and $n - 3^k$ both get candies. (For example, if $n = 47$ then $n - 1$ has base 3 representation 1201. Both students 29 and 38 got candies since the base 3 representations of 28 and 37 are respectively 1001 and 1101. As 29, 38, 47 form an arithmetic sequence, student 47 cannot get a candy.)

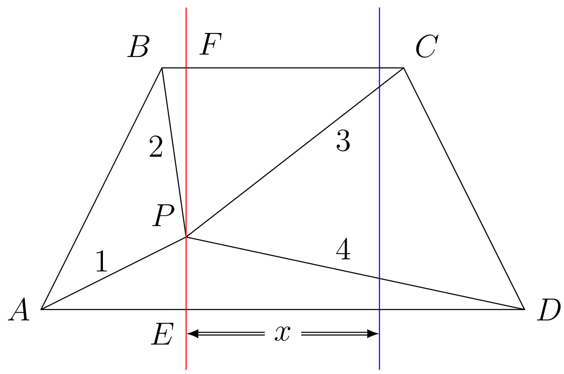
This completes the induction. Note that $2023_{10} = 2202221_3$. So the largest n which is 'good' has the property that the base 3 representation of $n-1$ is 1111111. The answer is thus $2^7 = 128$.

Remark. As long as only the answer is needed for the purpose of the contest, one can reasonably expect that the pattern observed will likely hold in general without going through the inductive proof. Of course the latter is needed if a rigorous treatment is required.

12. Suppose the line through P perpendicular to BC and AD (the red line in the figure) meets AD at E and BC at F. Then we have

$$4^2 - 1^2 = (DE^2 + PE^2) - (AE^2 + PE^2) = DE^2 - AE^2 = (DE + AE)(DE - AE).$$

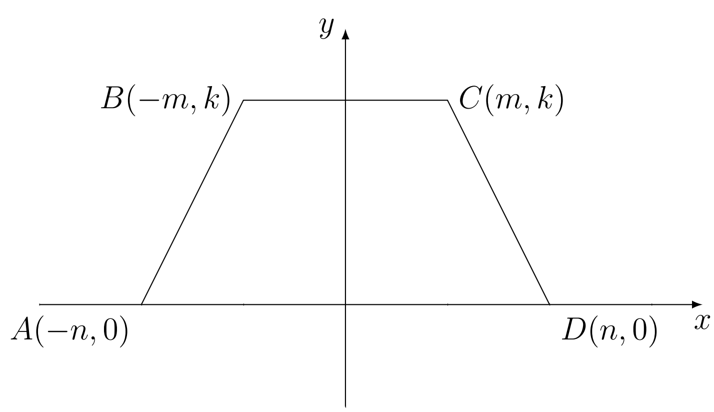
If we draw another line perpendicular to BC and AD (the blue line in the figure) such that its distance from D is equal to AE, then $DE - AE$ is simply the distance between the red and blue lines, which we denote by x.



Hence we have $AD \cdot x = 4^2 - 1^2$. By symmetry we have $BC \cdot x = 3^2 - 2^2$. It follows that

$$AD : BC = (4^2 - 1^2) : (3^2 - 2^2) = 3 : 1.$$

Remark. The solution works regardless of whether P is inside or outside the trapezium. Alternatively, we could avoid worrying about the relative position of P by using coordinate geometry. Set the x-axis along AD and the origin at the mid-point of AD. Then we may let the coordinates of A, B, C, D be $(-n, 0)$, $(-m, k)$, (m, k) and $(n, 0)$ respectively.

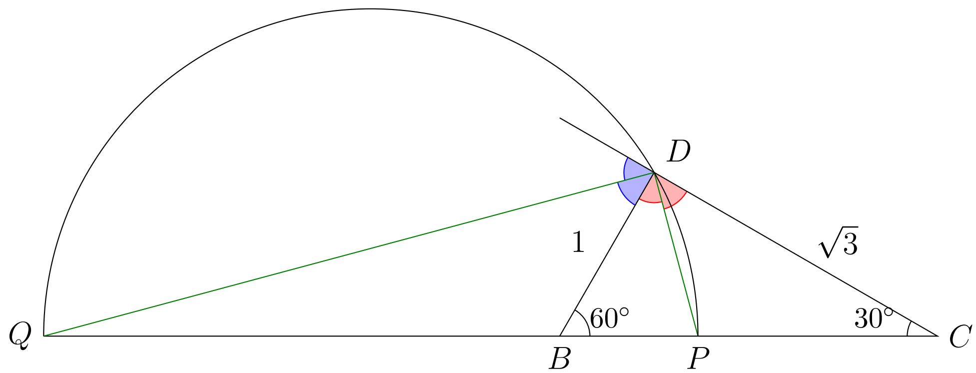


Let the coordinates of P be (p, q). Using $PA = 1$ and $PD = 4$, we get

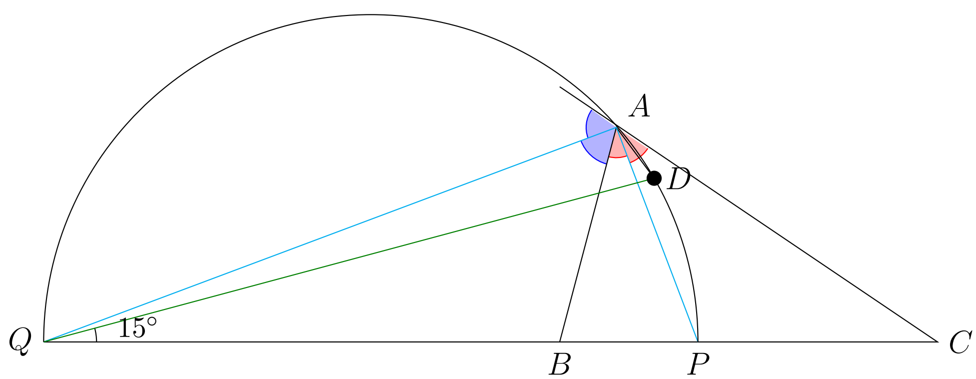
$$(p + n)^2 + q^2 = 1^2 \quad \text{and} \quad (p - n)^2 + q^2 = 4^2.$$

Upon subtraction we get $4pn = -15$. Similarly, using $PB = 2$ and $PC = 3$ we get $4pm = -5$. Hence $AD : BC = n : m = 3 : 1$.

13. Note that $DC = \sqrt{3}$, $\angle DBC = 60^\circ$ and $\angle DCB = 30^\circ$. In particular, we notice that $AB : AC = DB : DC$, so the angle bisector theorem (see remark after Question 5) seems relevant. Let the internal and external bisector of $\angle BDC$ meet CB and its extension at P and Q respectively.



Note that the internal and external bisector are always perpendicular. Hence the circle with PQ as diameter passes through D. As we know that $\angle QDB = 45^\circ$ and $\angle BDC = 90^\circ$, we have $\angle DQP = 180^\circ - \angle QDC - \angle DCQ = 15^\circ$. Now the key observation comes in. Since $AB : AC = DB : DC$, the internal and external bisector of $\angle BAC$ meet CB and its extension at the same P and Q as a consequence of the angle bisector theorem, and A also lies on the same circle with PQ as diameter:



Note that $\angle DAP = \angle DQP = 15^\circ$. Hence $\angle PAB = \angle PAC = 15^\circ + 18^\circ = 33^\circ$. It follows that $\angle DAB = \angle DAP + \angle PAB = 15^\circ + 33^\circ = 48^\circ$.

Remark. More generally, whenever D is a point inside $\triangle ABC$ such that $AB : AC = DB : DC$, then we have $\angle BAD - \angle CAD = \angle DBC - \angle DCB$. The proof can be worked out in essentially the same way as in the above solution. The circle with PQ as diameter is known as the *Apollonius circle* with foci B, C and ratio $BP : PC$.

14. Note that 74 and 111 have a common factor of 37. Also, 87 and 124 are 13 greater than 74 and 111 respectively. We call the coins with denominations \$87 and \$124 ‘bad coins’.

Note that $2023 \equiv 25 \pmod{37}$. Hence we must use k bad coins such that $13k \equiv 25 \pmod{37}$. The smallest such k is 19 (see remark), and hence we muse have $k = 19$ (because the next k would be $19 + 37$, and the total amount will surely exceed \$2023).

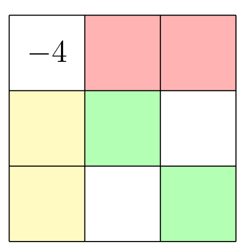
We note that $2023 = 19 \times 87 + 370$. To pay exactly \$2023, we may start with 19 coins with denomination \$87, and add 10 units of \$37. This can be done by using a \$74 coin (which contributes 2 units), a \$111 coin (which contributes 3 units), or by replacing a \$87 coin by a \$124 coin (which contributes 1 unit). Hence the problem is reduced to solving the equation $2a + 3b + c = 10$ in non-negative integers. Since each choice of (a, b) for which $2a + 3b \leq 10$ gives a unique choice of c, we only have to solve the inequality $2a + 3b \leq 10$ in non-negative integers. We find that there are 14 solutions as shown below, and so the answer is 14.

Value of b	Possible corresponding values of a
0	0, 1, 2, 3, 4, 5
1	0, 1, 2, 3
2	0, 1, 2
3	0

Remark. To solve the congruence $13k \equiv 25 \pmod{37}$, the formal way is to multiply both sides by the inverse of 13 modulo 37, which is 20 (as $13 \cdot 20 \equiv 1 \pmod{37}$), but in this case it may be easier to just use trial and error, i.e. we consider the arithmetic sequence starting with 25 and with common difference 37, namely, 25, 62, 99, 136, ..., and look for the first term that is divisible by 13. The first such term turns out to be 247, so k can be found by solving $13k = 247$.

15. We first make a few observations.

- The sum of the numbers on all cards is 0. Hence each row sum being non-negative is equivalent to each row sum being 0.
- For the same reason as above, each column sum must be 0.
- The card -4 cannot be placed in a corner cell. Indeed, if -4 is placed in a corner cell as shown below, then the sum of the numbers in the two red cells is 4, the sum in the yellow cells is 4 and the sum of the green cells is at least 4. This is impossible since the sum of the positive numbers on the cards is only $1 + 2 + 3 + 4 = 10$.



- In essentially the same way we can prove that the card -4 cannot be placed in the central cell.

From the above observations, there are 4 choices for the position of -4. After the position of -4 is fixed as shown below, there are 4 choices for the red cells (we either put in $\{0, 4\}$ or $\{1, 3\}$ in either order), and then 2 choices for the yellow cells (we put in the remaining pair in either order).

	-4	

This gives a total of $4 \times 4 \times 2 = 32$ ways to fill in five cells. There is then only one way to fill in the remaining cells using the numbers -3, -2, -1 and 2. For instance, suppose we have filled the cells in the following way, with a, b, c, d denoting the numbers yet to be filled in:

1	-4	3
a	0	b
c	4	d

Considering the rows we need $a + b = 0$ and $c + d = -4$, so $\{a, b\} = \{2, -2\}$ and $\{c, d\} = \{-1, -3\}$. Considering the columns we need $a + c = -1$ and $b + d = -3$ too. Hence the only possibility is $(a, b, c, d) = (2, -2, -3, -1)$. It remains to check whether these 32 ways also satisfy the requirement that the two diagonal sums are non-negative. We start with the following 4 cases with the position of -4 fixed:

1	-4	3
2	0	-2
-3	4	-1

1	-4	3
-3	4	-1
2	0	-2

0	-4	4
2	1	-3
-2	3	-1

0	-4	4
-2	3	-1
2	1	-3

All cases have both diagonal sums being non-negative. The other 4 cases with -4 in the same position must therefore also have the same property, since they are obtained by swapping the first and third columns from the 4 cases above:

3	-4	1
-2	0	2
-1	4	-3

3	-4	1
-1	4	-3
-2	0	2

4	-4	0
-3	1	2
-1	3	-2

4	-4	0
-1	3	-2
-3	1	2

The other cases (where the position of -4 is different) can be obtained by rotating one of the above configurations. Hence all 32 ways of filling in the table in which all row and column sums are zero will also have both diagonal sums being non-negative. It follows that the answer is 32.

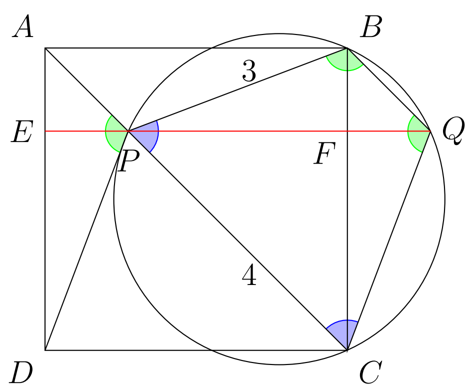
16. There are $2^{12} = 4096$ possible outcomes and we want to count how many of these enable the team to win. To do this, we let the 12 players form 6 pairs so that the members of each pair are diametrically opposite. Note that the positions of four players form a rectangle if and only if they form two pairs.

Call a pair 'bad' if the two players put up flags of the same colour. Note that if there are three or more bad pairs, then at least two pairs will both have chosen the same colour, so the team loses. Hence for the team to win, there are three possibilities:

- There is no bad pair — that means for each pair the two players will put up flags of different colours (2 possibilities for each pair), so there are altogether $2^6 = 64$ possibilities for this case.
- There is one bad pair — there are 6 choices for the bad pair, and regardless of whether a pair is good or bad, there are 2 possible combinations of flag colours for each pair. Hence there are $6 \times 2^6 = 384$ possibilities for this case.
- There are two bad pairs — there are $\binom{6}{2} = 15$ choices for the bad pairs, and one bad pair must put up red flags and the other bad pair must put up blue flags (otherwise the team loses), giving 2 possibilities. There are 2 possible combinations of flag colours for each of the other 4 pairs, and so there $15 \times 2 \times 2^4 = 480$ possibilities for this case.

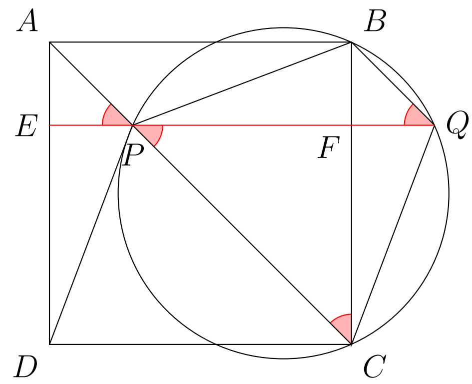
It follows that the answer is $\dfrac{64 + 384 + 480}{4096} = \dfrac{29}{128}$.

17. To make use of the condition $\angle APD + \angle BPC = 180°$, we translate $\triangle APD$ to $\triangle BQC$ so that the condition becomes $\angle BQC + \angle BPC = 180°$. This means $BQCP$ is a cyclic quadrilateral. Draw a line through P parallel to AB. This line passes through Q, and let E and F be the points where this line meets AD and BC respectively.

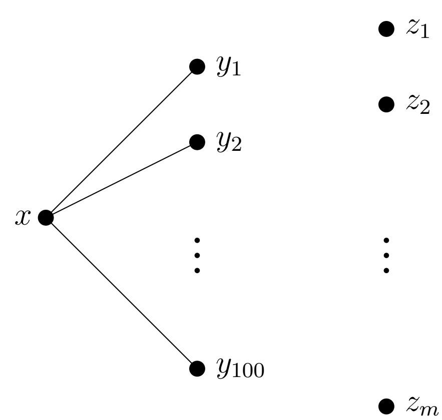


Note that $AB = PQ$ due to the translation. Hence $BC = PQ$, so they subtend equal angles on the circumference of the circle. Equating the angles subtended on the major arcs gives $\angle BPC = \angle QCP$ (note that $\angle BPC$ is given to be acute, and that $\angle QCP$ is acute since $PB < PC$). Equating the angles subtended on the minor arcs gives $\angle BQC = \angle PBQ$. Since we also have $\angle BQC + \angle BPC = 180°$, it follows that the cyclic quadrilateral $BQCP$ is actually an isosceles trapezium with BQ and PC parallel. Hence we have $FP = FC = 2\sqrt{2}$, and so $BF = \sqrt{3^2 - (2\sqrt{2})^2} = 1$. It follows that $BC = BF + FC = 1 + 2\sqrt{2}$.

Remark. P actually lies on AC since $\angle FPC = \angle FCP = \angle FQB = \angle APE$.



18. Pick any contestant x. Suppose he has played against y_1, y_2, ..., y_{100} (call these Group Y contestants). Denote the contestants who have not played against x by z_1, z_2, ..., z_m (call these Group Z contestants).



A Group Y contestant (who has played against x) has exactly 50 common opponents with x (who must be Group Y contestants), and so has played against exactly $100-1-50 = 49$ Group Z contestants. On the other hand, since a Group Z contestant has not played against x, he has exactly 4 common opponents with x (who must be Group Y contestants). Thus the number of games between a Group Y contestant and a Group Z contestant is equal to 100×49 as well as $4m$. It follows that these numbers are equal, and so

$$n = 1 + 100 + m = 1 + 100 + \frac{100 \times 49}{4} = 1326.$$

Remark. The scenario described in the question is indeed possible. Note that the answer is actually $\binom{52}{2}$. To construct the scenario, label the contestants by 2-element subsets of $\{1, 2, ..., 52\}$, and let two contestants play a game if and only if their corresponding subsets have a common element. It is easy to check that all given conditions are satisfied.

19. For convenience we shall call 'a student whose class number is even' an *even student*. We make the following observations.

    (1) Every question is solved by at least one even student (as the teacher could choose all even students).

    (2) Similarly, every question is solved by at least one of students 5, 10, 15, 20.

    (3) Students 2, 4, 6, ..., 18 (i.e. all even students except student 20) cannot solve all questions (since any choice of the teacher must include either all even students or all students whose numbers are multiples of 5). By (1), there must be a question which was solved by student 20 but not by any other even student.

    (4) For the same reason in (3), there must be a question which was solved by student 10 but not by any other even student.

    (5) We claim that there exist at least *two* questions that were solved by student 2 but not any other even student. Using the same argument as in (3), we can show that there is at least *one* such question. Now suppose there is *exactly* one such question P. Now neither student 10 nor 20 solved P, so by (2) we know that either student 5 or 15 solved P. Without loss of generality suppose student 5 did it. Consider students 4, 5, 6, 8, 10, 12, 14, 16, 18, 20 (i.e. all even students but with student 2 replaced by student 5). Since student 2 and student 15 are missing, these students together did not solve all questions. That means there is a question Q which cannot be solved by these students (note that $P \neq Q$ since student 5 solved P). As student 2 cannot solve Q either (since P is the only question solved by student 2 but not any other even student), that means no even student could solve Q, which contradicts (1). This proves the claim.

    (6) For the same reason in (5), for each of students 4, 6, 8, 12, 14, 16, 18, there must be at least two questions that were solved by that student but not any other even student.

    From (3), (4), (5) and (6), there are at least $1 + 1 + 2 + 2 \times 7 = 18$ questions, and they are clearly distinct (by considering which even student solved each question). On the other hand, we can construct a paper with exactly 18 questions as follows:

    - Question 10 — solved by student 10 only
    - Question 20 — solved by student 20 only
    - Question x for each $x \in \{2, 4, 6, 8, 12, 14, 16, 18\}$ — solved by students 5 and x only
    - Question x' for each $x \in \{2, 4, 6, 8, 12, 14, 16, 18\}$ — solved by students 15 and x only

    It can be easily checked that all even students together solved all questions, all students whose numbers are multiples of 5 together solved all questions, and that if not all even students are present and not all students whose numbers are multiples of 5 are present, then at least one question is not solved by the students picked. The answer is thus 18.

Remark. Due to ambiguities in the wording used in the live paper, special consideration was made during the grading of the live scripts to allow an alternative answer that could arise from a different interpretation of the original problem.

20. It suffices to choose v and w since there is a unique choice for $u = v + 2w$ after fixing v and w. Note that $289 = 17^2$ and $2023 = 7 \times 17^2$. We consider two cases.

    Case 1: w is divisible by 17

    In this case v is also divisible by 17 (if not, then $u = v + 2w$ not divisible by 17, so neither $u^2 + w^2$ nor $v^2 + w^2$ can be divisible by 17). Furthermore, any choice of positive multiples of 17 not exceeding 2023 for each of v and w satisfies the given condition. Hence there are 7×17 choices for each of v and w, and thus $(7 \times 17)^2$ sets of (u, v, w) in this case.

    Case 2: w is not divisible by 17

    There are $2023 - 7 \times 17 = 7 \times 17 \times 16$ choices for w. For each w, we try to find v. First note that

$$\begin{aligned}(u^2 + w^2)(v^2 + w^2) &\equiv (u^2 - 16w^2)(v^2 - 16w^2) \\ &= (u + 4w)(u - 4w)(v + 4w)(v - 4w) \\ &= (v + 6w)(v - 2w)(v + 4w)(v - 4w) \qquad \pmod{17}\end{aligned}$$

    Hence we can choose v as follows.

    - From the above we see that v must be equal to $-6w$, $2w$, $-4w$ or $4w$ modulo 17 (and they are distinct modulo 17).
    - Each of the 4 choices of v modulo 17 corresponds to exactly choice of v modulo 289 for which $(u^2+w^2)(v^2+w^2)$ is divisible by 289 (see first remark below).
    - Each choice of v modulo 289 corresponds to 7 actual possible values of v since $2023 = 7 \times 289$.

    Hence for each w, there are 4×7 choices for v. It follows that there are altogether $(7 \times 17 \times 16) \times (4 \times 7) = 7^2 \times 17 \times 64$ sets of (u, v, w) in this case.

    It follows that the answer is $(7 \times 17)^2 + 7^2 \times 17 \times 64 = 7^2 \times 17 \times 81 = 67473$.

    Remarks.

    - To prove that each choice of v modulo 17 (among $-6w$, $2w$, $-4w$ and $4w$) gives rise to exactly one choice of v modulo 289 for which $(u^2+w^2)(v^2+w^2)$ is divisible by 289, we shall consider the case $v \equiv 2w \pmod{17}$ as the other three cases are essentially the same. Write $v = 17k + 2w$, where k can be considered modulo 17 since we are considering v modulo 289. Then we have $u = 17k + 4w$ and so

$$\begin{aligned}(u^2 + w^2)(v^2 + w^2) &= [(17k + 4w)^2 + w^2][(17k + 2w)^2 + w^2] \\ &= (17^2k^2 + 8 \cdot 17kw + 17w^2)(17^2k^2 + 4 \cdot 17kw + 5w^2) \\ &\equiv (8 \cdot 17kw + 17w^2)(4 \cdot 17kw + 5w^2) \\ &\equiv 17(8kw + w^2)(5w^2) \\ &= 17(40w^3k + 5w^4) \qquad \pmod{289}\end{aligned}$$

and so we need $40w^3k + 5w^4 \equiv 0 \pmod{17}$. As 17 is prime and $40w^3$ is not divisible by 17, we know that for each fixed w there is exactly one solution for k modulo 17.

- We could also avoid the above process of 'lifting' from modulo 17 to modulo 289 by observing that $38^2 \equiv -1 \pmod{289}$ (or simply by knowing that such a square number exists, or equivalently, that -1 is a *quadratic residue* modulo 289). In this case we could directly write

$$\begin{aligned}(u^2 + w^2)(v^2 + w^2) &\equiv (u^2 - 38^2w^2)(v^2 - 38^2w^2) \\ &= (u + 38w)(u - 38w)(v + 38w)(v - 38w) \\ &= (v + 40w)(v - 36w)(v + 38w)(v - 38w) \qquad \pmod{289}\end{aligned}$$

and so v must be equal to one of $-40w$, $36w$, $-38w$ and $38w$ modulo 289, i.e. there are exactly 4 choices for v modulo 289 for each fixed w.