香港資優教育學苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad
Preliminary Selection Contest – Hong Kong 2024

國際數學奧林匹克 — 香港選拔賽初賽 2024

25 May 2024 (Saturday)
2024 年 5 月 25 日（星期六）

Question Book 問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3-hour written test.
   比賽以筆試形式進行，限時三小時。

2. Questions are bilingual. Answer all questions.
   題目中英對照。全卷題目均須作答。

3. Put your answers on the answer sheet.
   請將答案寫在答題紙上。

4. The use of calculators and protractors is NOT allowed.
   不可使用計算機和量角器。

5. Rulers and compasses can be used.
   可使用直尺和圓規。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Find the highest common factor of 8^8-1 and 4^9+1. (1 mark)

求 8^8-1 和 4^9+1 的最大公因數。（1 分）

2. Let n be a positive integer less than 100 such that the tens digit of n^2 is 7. Find the sum of all possible values of n. (1 mark)

設 n 為小於 100 的正整數，且 n^2 的十位數字為 7。求 n 的所有可能值之和。（1 分）

3. Let a, b, c be integers such that $2^a+2^b+2^c=\frac{37}{64}$. Find the value of $a^3+b^3+c^3$. (1 mark)

設 a、b、c 為整數，使得 $2^a+2^b+2^c=\frac{37}{64}$。求 $a^3+b^3+c^3$ 的值。（1 分）

4. Mike wrote down m consecutive integers with sum $2m$, while Kent wrote down $2m$ consecutive integers with sum m. If the difference between the largest integer written by Mike and the largest integer written by Kent is 2024, find m. (1 mark)

小明寫下了 m 個連續整數且它們之和為 $2m$，小強則寫下了 $2m$ 個連續整數且它們之和為 m。若小明寫下的最大整數與小強寫下的最大整數相差 2024，求 m。（1 分）

5. $ABCD$ is a convex quadrilateral in which $AB = BC = CD$. If $\angle ABC = 48°$ and $\angle BCD = 108°$, find $\angle CAD$. (1 mark)

$ABCD$ 是凸四邊形，其中 $AB = BC = CD$。若 $\angle ABC = 48°$，且 $\angle BCD = 108°$，求 $\angle CAD$。（1 分）

6. Let A be a point on the graph of $y=|x|$, and B be a point on the graph of $x=y^2+2$. Find the smallest possible distance between A and B. (1 mark)

設 A 為圖像 $y=|x|$ 上的一點，且 B 為圖像 $x=y^2+2$ 上的一點。求 A 和 B 之間的距離的最小可能值。（1 分）

7. $ABCD$ is a square. P and Q are points on the sides BC and CD respectively such that $\angle PAQ = 45°$, $PQ = 4$ and the area of ΔPCQ is 3. Find AB. (1 mark)

$ABCD$ 是正方形。P 和 Q 分別是邊 BC 和 CD 上的點，使得 $\angle PAQ = 45°$，$PQ = 4$，且 ΔPCQ 的面積為 3。求 AB。（1 分）

8. In ΔABC, D and E are points on the sides AB and AC respectively such that B, C, D, E lie on a common circle with centre O and that $ADOE$ is a parallelogram. Suppose $\angle BAC = x°$ and $\angle BOC = y°$. If $x+y=196$, find the sum of all possible values of x. (1 mark)

在 ΔABC 中，D 和 E 分別是邊 AB 和 AC 上的點，使得 B、C、D、E 均位於一個以 O 為中心的公共圓上，且 $ADOE$ 為平行四邊形。設 $\angle BAC = x°$ 及 $\angle BOC = y°$。若 $x+y=196$，求 x 的所有可能值之和。（1 分）

9. In a language, there are 5 vowels and 8 consonants in the alphabet. Each word is made up of exactly 5 (not necessarily distinct) letters, with at least 2 vowels and at least 1 consonant. What is the maximum number of words in this language? (1 mark)

某種語言的字母有 5 個元音和 8 個輔音。每個單字均由剛好 5 個（不一定相異的）字母組成，當中包括最少 2 個元音和最少 1 個輔音。這語言中最多有多少個單字？ （1 分）

10. A 3×3 grid is printed on a card. It is made up of 4 horizontal lines and 4 vertical lines. Each cell of the grid has width 2 and height 1. Mandy drew along the gridlines in a single pen stroke such that no point is visited more than once except at the intersections of the grid lines. Find the maximum possible length of the figure drawn by Mandy. (1 mark)

咭紙上印有一個 3×3 的方格表，它由 4 條橫線和 4 條直線組成，且方格表中每格的闊均為 2、高均為 1。敏怡沿着格線以一筆描繪，過程中同一點不會經過超過一次（於格線的交點除外）。求敏怡所描繪的圖的總長度的最大可能值。 （1 分）

11. A frog is at an arbitrary point inside an equilateral triangle with area 1. Each time the frog picks one of the vertices of the triangle that has not been picked before, and jumps halfway from where it is towards the chosen vertex. After three jumps, the frog is at point X. What is the total area of the region formed by all possible positions of X? (2 marks)

一隻青蛙位於一個面積為 1 的等邊三角形內的隨意一點。每次該青蛙均會選取三角形的一個未被選過的頂點，並從它所在的位置跳向所選頂點的一半路程。如是者，經過三次跳躍後，該青蛙到達 X 點。那麼，由 X 點的所有可能位置組成的區域的總面積是多少？ （2 分）

12. The equation $x^3-\dfrac{1}{x}=4$ has two real roots a and b. Find the value of a^2+b^2. (2 marks)

方程 $x^3-\dfrac{1}{x}=4$ 有兩個實根 a 和 b。求 a^2+b^2 的值。 （2 分）

13. 'There is an arithmetic sequence with common difference d,' said the teacher. 'When each term of the sequence is rounded off to the nearest integer, the first 7 terms are 12, 18, 25, 32, 39, 45, 52. What is the range of possible values of d?' The correct answer to the teacher's question is $a<d<b$ where a and b are two real numbers. Find the value of ab. (2 marks)

老師說：「現有一個公差為 d 的等差數列。如果數列的每項皆四捨五入至最接近整數，則數列的首 7 項為 12、18、25、32、39、45、52。那麼，d 的可能值範圍是甚麼？」老師的問題的正確答案為 $a<d<b$，其中 a 和 b 為兩實數。求 ab 的值。 （2 分）

14. How many positive integers n are there such that n has exactly 2024 positive factors, and that n is also a factor of 2024^{2024}? (2 marks)

有多少個正整數 n 滿足以下條件：n 有剛好 2024 個正因數，且 n 亦是 2024^{2024} 的因數？ （2 分）

15. Let a_1, a_2, a_3, … be a sequence such that $a_1 = 1$, and that the equalities $a_{2n} = \frac{a_n}{2} + \frac{3n}{4}$ and $a_{2n+1} = \frac{4a_n a_{n+1} + 2na_n + 2(n+1)a_{n+1} - 3n^2 - 3n - 1}{2(2a_n + 2a_{n+1} - 2n - 1)}$ hold for all positive integers *n*. If *a* is a term in the sequence such that $a > 100$, find the smallest possible value of *a*. (2 marks)

設 a_1, a_2, a_3, … 為一數列，使得 $a_1 = 1$，且對任意正整數 n 皆有 $a_{2n} = \frac{a_n}{2} + \frac{3n}{4}$ 和 $a_{2n+1} = \frac{4a_n a_{n+1} + 2na_n + 2(n+1)a_{n+1} - 3n^2 - 3n - 1}{2(2a_n + 2a_{n+1} - 2n - 1)}$。若 a 為數列的其中一項，且 $a > 100$，求 a 的最小可能值。（2 分）

16. *A*, *B*, *C*, *D*, *E* are five points on a circle with $AB \times AD = 224$ and $AC \times AE = 360$. If *BCDE* is a square, find its area. (2 marks)

A、B、C、D、E 為某圓上的五點，使得 $AB \times AD = 224$ 且 $AC \times AE = 360$。若 $BCDE$ 為正方形，求它的面積。（2 分）

17. Betty, Cathy, Daisy and Emily played a game in which they were given 2, 3, 4 and 5 balls respectively. Each player wrote their name on each ball they had and all the balls were then put into a bag. The balls in the bag were then drawn one by one, and the balls drawn were returned to the respective players. The first player who got back all her balls is the winner. What is the probability for Emily to win? (2 marks)

二妹、三妹、四妹和五妹玩遊戲。她們分別有 2、3、4、5 個球，而各人均把自己的名字寫在自己所擁有的每個球上。她們把所有的球都放進一個袋子中，然後從袋子中把球逐一抽出來，並把抽出來的球發還給其所屬的人。最先取回自己所有的球者勝出遊戲。那麼，五妹勝出遊戲的概率是多少？（2 分）

18. *AB* is a diameter of a circle of radius 12. *C* and *D* are two points on the circle such that the chord *CD* meets *AB* at *E*. The tangent to the circle at *A* meets the extension of *CD* at *F*. If $CE : ED : DF = 4 : 3 : 3$, find *AC*. (2 marks)

AB 是一個半徑為 12 的圓的直徑。C 和 D 是圓上的兩點，使得弦 CD 與 AB 相交於 E。該圓於 A 的切線與 CD 的延線相交於 F。若 $CE : ED : DF = 4 : 3 : 3$，求 AC。（2 分）

19. There are 36 points on the rectangular coordinate plane whose *x*- and *y*-coordinates are both positive integers not exceeding 6. How many triangles have their circumcentres and vertices being among these 36 points? (2 marks)

直角座標平面上有 36 點的 x 座標和 y 座標均為不超過 6 的正整數。有多少個三角形的外心和頂點全在這 36 點當中？（2 分）

20. Let *n* be a 4-digit positive integer. When $(n+2^2)(n^2+3^2)(n^3+4^2)\cdots(n^{2023}+2024^2)$ is expressed in binary notation, at most how many ending zeros can there be? (2 marks)

設 n 為四位正整數。當 $(n+2^2)(n^2+3^2)(n^3+4^2)\cdots(n^{2023}+2024^2)$ 以二進制表示時，其末尾的零最多有幾個？（2 分）

End of Paper

全卷完