International Mathematical Olympiad
Hong Kong Preliminary Selection Contest 2024

Outline of solutions

Answers:

1.	65	**2.**	200	**3.**	-281	**4.**	4051
5.	84°	**6.**	$\frac{7\sqrt{2}}{8}$	**7.**	$2+\sqrt{7}$	**8.**	131
9.	233000	**10.**	32	**11.**	$\frac{3}{32}$	**12.**	$2\sqrt{2}$
13.	$\frac{136}{3}$	**14.**	90	**15.**	$\frac{40001}{400}$	**16.**	424
17.	$\frac{661}{5544}$	**18.**	$8\sqrt{5}$	**19.**	792	**20.**	4045

Solutions:

1. Let $m = 8^8 - 1$ and $n = 4^9 + 1$. Note that

$$m = 2^{24} - 1 = 2^6(2^{18}+1) - (2^6+1) = 64n - 65,$$

so any common factor of m and n would also be a factor of 65. On the other hand, since

$$n = 4^9 + 1 = (4^3+1)(4^6 - 4^3 + 1) = 65(4^6 - 4^3 + 1)$$

is a multiple of 65, we can see that $m = 64n - 65$ is also a multiple of 65. From this we conclude that the highest common factor of m and n is 65.

Remark. The highest common factor of two positive integers can also be found using the *Euclidean algorithm*. We divide the larger number by the smaller number, and subsequently we divide the divisor by the remainder in the next step, until we get a remainder of 0. The last divisor would be the highest common factor. Applying the algorithm on this case, we would get

$$\begin{array}{lclcll} (2^{24}-1) & \div & (2^{18}+1) & = & 2^6 - 1 & \ldots\ 2^{18} - 2^6 \\ (2^{18}+1) & \div & (2^{18}-2^6) & = & 1 & \ldots\ 2^6 + 1 \\ (2^{18}-2^6) & \div & (2^6+1) & = & 2^{12} - 2^6 & \ldots\ 0 \end{array}$$

and hence the highest common factor would be $2^6 + 1 = 65$.

2. Note that the unit digit of a square number can only be 0, 1, 4, 5, 6, 9 (just consider the unit digits of 1^2 to 10^2). When the tens digit is 7, only the unit digit 6 is possible because a square number is either congruent to 0 or 1 modulo 4 (to see this, note that

an even square number must be divisible by 4 and an odd square number is of the form $(2k+1)^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 4$).

Hence n^2 ends with 76. Then $n^2 - 1$ ends with 75 and is divisible by 25. It follows that $n+1$ or $n-1$ is divisible by 25 (note that $n+1$ or $n-1$ cannot be both divisible by 5 as they differ by 2). As n is even, the only possible values of n are 24, 26, 74 and 76. Indeed, we check that $24^2 = 576$, $26^2 = 676$, $74^2 = 5476$ and $76^2 = 5776$ all have tens digit 7. The answer is thus $24 + 26 + 74 + 76 = 200$.

Remark. It is not hard to get the answer by brute force. For instance, first consider the case where n has unit digit 1, and we note that the last two digits of 1^2, 11^2, 21^2, ... are respectively 01, 21, 41, ..., and similar patterns also exist when n has other unit digits.

3. Without loss of generality assume $a \geq b \geq c$. Multiplying both sides of the equation by $2^6 = 64$, we have
$$2^{a+6} + 2^{b+6} + 2^{c+6} = 37.$$
Clearly $a+6$ must be positive (otherwise the left hand side is at most $1+1+1=3$), while $c+6$ must be non-positive (otherwise the left hand side is even). Also, if $b+6$ is not positive, then the only way to make the left hand side an odd integer would be to have $b+6 = c+6 = -1$, but that gives $2^{a+6} = 36$, which has no integer solution. Hence $b+6$ is also positive, and that forces $c+6 = 0$ in order for the left hand side to be odd. The equation thus becomes
$$36 = 2^{a+6} + 2^{b+6} = 2^{b+6}(2^{a-b} + 1).$$
Since $36 = 2^2 \times 9$, we must have $b+6 = 2$ and $a-b = 3$. It follows that $(a,b,c) = (-1,-4,-6)$ and so the answer is $(-1)^3 + (-4)^3 + (-6)^3 = -281$.

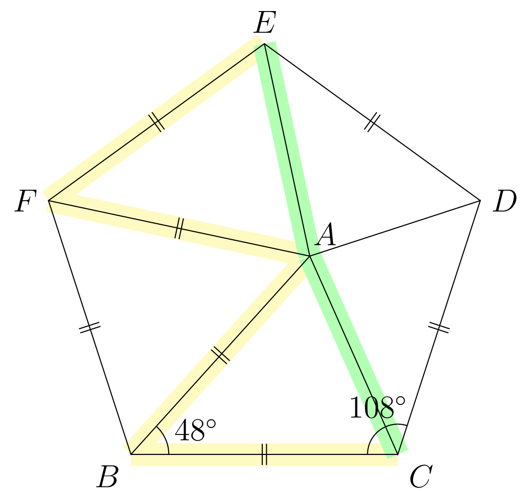
Remark. We can guess the answer intuitively by noting that $37_{10} = 100101_2$, so
$$\frac{37}{64} = \frac{2^5 + 2^2 + 2^0}{2^6} = 2^{-1} + 2^{-4} + 2^{-6}.$$
The original (long) solution merely ensures that the answer is unique.

4. For Mike, the average of the integers he wrote is 2. Hence he has written $\frac{1}{2}(m-1)$ integers smaller than 2 and $\frac{1}{2}(m-1)$ integers greater than 2. In particular, the largest integer he has written is $2 + \frac{1}{2}(m-1)$.

For Kent, the average of the integers he wrote is $\frac{1}{2}$. That means if the integers he wrote are split into the smaller half and the larger half (of m integers each), then the largest integer in the smaller half is 0 and the smallest integer in the larger half is 1. It follows that the largest integer he has written is m.

Clearly $m > 3$ as the difference between the two largest integers is 2024, so m is greater than $2 + \frac{1}{2}(m-1)$. It follows that $m - [2 + \frac{1}{2}(m-1)] = 2024$, which gives $m = 4051$.

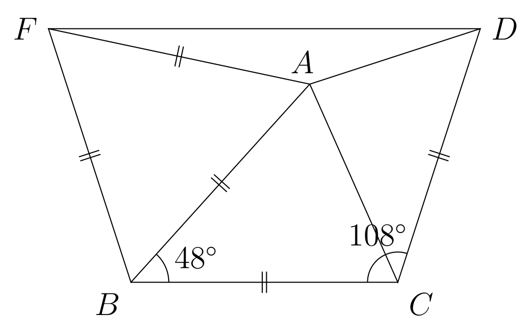
5. Note that $108°$ is equal to an interior angle of a regular pentagon. Hence we may construct a regular pentagon $BCDEF$ with A being an interior point and with AB being equal to the side length of the pentagon. Since $\angle ABF = 108° - 48° = 60°$, $\triangle ABF$ is equilateral and so AF is also equal to the side length of the pentagon.



Now it follows that $\angle EFA = \angle EFB - \angle AFB = 108° - 60° = 48°$, so $\triangle FEA$ is congruent to $\triangle BCA$, with $\angle FAE = \angle BAC = \frac{1}{2}(180° - 48°) = 66°$. It also follows that $\triangle EAD \cong \triangle CAD$, so

$$\angle CAD = \angle EAD = \frac{360° - 66° - 60° - 66°}{2} = 84°.$$

Remark. We can also work out the answer without involving the point E. Just consider the point F for which $\triangle ABF$ is equilateral and C, F are on different sides of AB. Since $\angle FBC = 60° + 48° = 108° = \angle DCB$ and $FB = BC = CD$, we see that $FBCD$ is an isosceles trapezium.



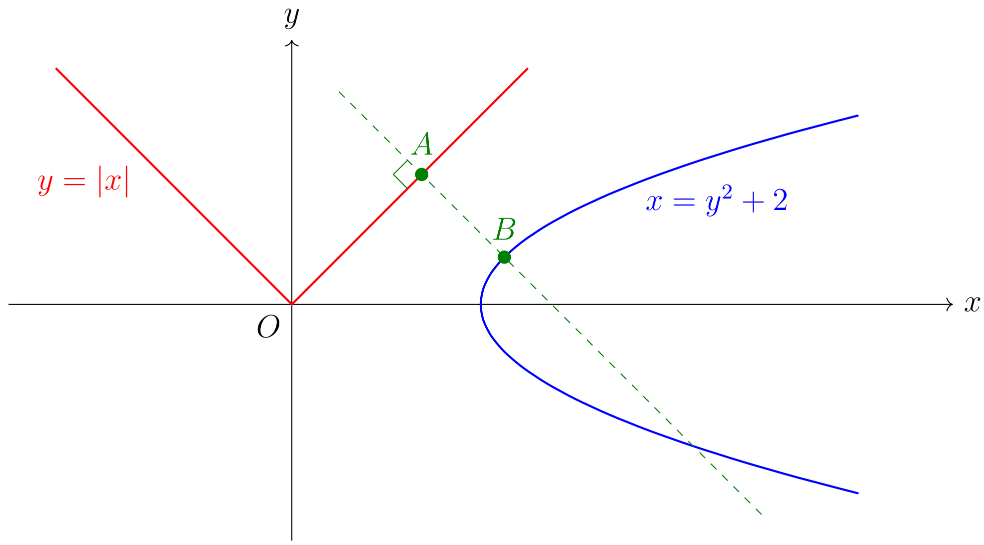
We now have $\angle DFB = 180° - \angle FBC = 72°$ and

$$\angle DBF = 108° - \angle DBC = 108° - \frac{180° - 108°}{2} = 72°,$$

so $DF = DB$. Since we also have $AF = AB$, we have $\angle DAF = \angle DAB = \frac{1}{2}(360° - 60°) = 150°$. Hence

$$\angle CAD = 150° - \angle BAC = 150° - \frac{180° - 48°}{2} = 84°.$$

6. Consider such A and B which minimise the distance between them. Then A should be in the first quadrant. Also, the line through A with slope -1 (i.e. perpendicular to the graph of $y = |x|$ in the first quadrant) intersects the parabola $x = y^2 + 2$ at two points, and the one with smaller x-coordinate (and larger y-coordinate) is B.



Let $A = (a, a)$. The line through A with slope -1 has equation $x + y = 2a$. Solving this equation together with $x = y^2 + 2$, we get $y = \frac{1}{2}(-1 \pm \sqrt{8a-7})$ and $x = \frac{1}{2}(4a + 1 \mp \sqrt{8a-7})$. It follows that

$$B = \left(\frac{4a + 1 - \sqrt{8a-7}}{2}, \frac{-1 + \sqrt{8a-7}}{2}\right).$$

From this we get

$$AB = \sqrt{\left(a - \frac{4a+1-\sqrt{8a-7}}{2}\right)^2 + \left(a - \frac{-1+\sqrt{8a-7}}{2}\right)^2} = \frac{2a + 1 - \sqrt{8a-7}}{\sqrt{2}}.$$

Setting $u = \sqrt{8a-7}$, we have

$$AB = \frac{\frac{1}{4}(u^2+7) + 1 - u}{\sqrt{2}} = \frac{\sqrt{2}}{8}[(u-2)^2 + 7],$$

which has minimum value $\dfrac{7\sqrt{2}}{8}$ when $u = 2$, i.e. $A = \left(\dfrac{11}{8}, \dfrac{11}{8}\right)$ and $B = \left(\dfrac{9}{4}, \dfrac{1}{2}\right)$.

Remark. There are many alternative ways of solving the problem, especially if one knows calculus. The following lists some of them.

(1) Using (one-variable) calculus

At points A and B which minimise the distance between them, the tangent to the parabola at B should have slope 1 (so that it is parallel to the graph of $y = |x|$ in the first quadrant). With $y = \sqrt{x-2}$, we have $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x-2}}$, and we find that $x = \frac{9}{4}$

(and $y = \frac{1}{2}$) in order for the derivative to be equal to 1. Hence the answer is simply the distance from the point $(\frac{9}{4}, \frac{1}{2})$ to the line $y = x$, which is

$$\left| \frac{\frac{9}{4} - \frac{1}{2}}{\sqrt{1^2 + 1^2}} \right| = \frac{7\sqrt{2}}{8}.$$

(2) <u>Using the theory of quadratic functions</u>

Solution (1) can be turned into a non-calculus solution by considering the slope of the tangent in a different way. Suppose the tangent to $x = y^2 + 2$ at the point $(b^2 + 2, b)$ has slope m. Then the equation of the tangent is $y - b = m(x - b^2 - 2)$. Setting $x = y^2 + 2$ in the equation thus gives $y - b = m(y^2 - b^2)$, or

$$(y - b)[1 - m(y + b)] = 0.$$

This quadratic equation in y should have a double root (which must be b in view of the first factor) because the tangent meets the parabola at just one point. Hence we have $1 - m(b + b) = 0$, so that $m = \frac{1}{2b}$. Thus we should take $b = \frac{1}{2}$ for the tangent to be parallel to the graph of $y = |x|$ in the first quadrant, and then we can proceed as in Solution (1).

(3) <u>Using multivariable caclulus</u>

Let $A = (a, a)$ and $B = (b^2 + 2, b)$. Then the square of the distance between A and B is given by

$$f(a, b) = (b^2 + 2 - a)^2 + (b - a)^2.$$

We have

$$f_a = -2(b^2 + 2 - a) - 2(b - a) \quad \text{and} \quad f_b = 2(b^2 + 2 - a)(2b) + 2(b - a).$$

To minimise f we set $f_a = f_b = 0$. From $f_a = 0$ we get $2a = b^2 + b + 2$. Putting this into $f_b = 0$, we get

$$2(b^2 + 2)(2b) - 2b(b^2 + b + 2) + 2b - (b^2 + b + 2) = 0,$$

which simplifies to $(2b - 1)(b^2 - b + 2) = 0$. As $b^2 - b + 2 = 0$ has no real solution, we have $b = \frac{1}{2}$, and this corresponds to $a = \frac{11}{8}$. From this we can compute directly to get the minimum distance between A and B to be $\frac{7}{8}\sqrt{2}$.

(4) <u>Completing square</u>

With notations in Solution (3), we may write

$$f(a, b) = 2a^2 - 2(b^2 + b + 2)a + (b^4 + 5b^2 + 4).$$

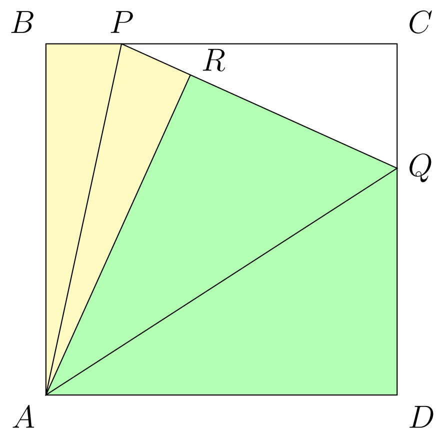
For each fixed b, this is minimised when $a = \frac{1}{2}(b^2 + b + 2)$, in which case $f(a, b)$ becomes

$$f\left(\frac{b^2 + b + 2}{2}, b\right) = \frac{1}{2}(b^4 - 2b^3 + 5b^2 - 4b + 4) = \frac{1}{2}(b^2 - b + 2)^2,$$

which is minimised when $b = \frac{1}{2}$. We can then proceed as in Solution (3).

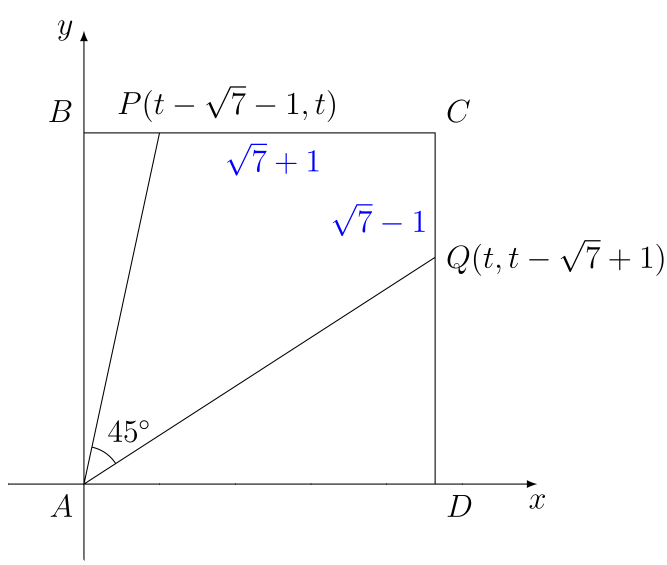
7. (**Idea for solution.** Since $\angle PAB + \angle QAD = 45^\circ = \angle PAQ$, one may split $\angle PAQ$ into two parts, whose sizes are equal to $\angle PAB$ and $\angle QAD$ respectively.)

Reflect $\triangle PAB$ across PA and $\triangle QAD$ across QA. Since $AB = AD$ and $\angle PAB + \angle QAD = \angle PAQ$, the image of AB across PA and the image of AD across QA coincide. Let this common image be AR.



Note that $\angle PRQ$ is a straight line as $\angle ARP = \angle ARQ = 90^\circ$. Let t be the side length of the square. Then $AR = t$ and so the area of $\triangle APQ$ is $\frac{1}{2} \cdot PQ \cdot AR = 2t$. It follows that the total shaded area in the above figure is $4t$ and so the area of the square is $4t+3$. This gives the equation $t^2 = 4t + 3$, solving which gives $t = 2 + \sqrt{7}$ (the negative root $2 - \sqrt{7}$ is of course rejected).

Remark. Using $CP^2 + CQ^2 = 16$ and $CP \times CQ = 6$, one can solve (without loss of generality assuming $CP > CQ$) to get $CP = \sqrt{7} + 1$ and $CQ = \sqrt{7} - 1$. This leads to several reasonably easy ways of direct algebraic computations of t. For instance one could apply the cosine formula in $\triangle PAQ$, or use coordinate geometry as follows. Set A at the origin. Then the coordinates of P and Q are respectively $(t-\sqrt{7}-1, t)$ and $(t, t-\sqrt{7}+1)$.

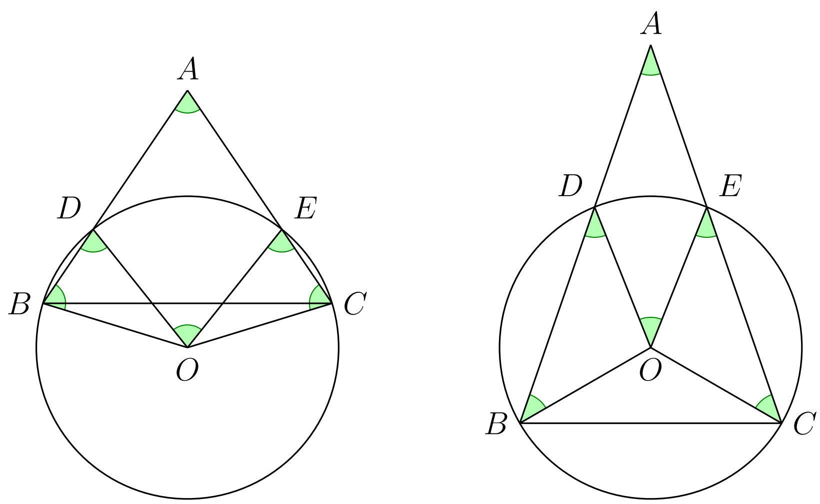


We have $\tan \angle PAD = \dfrac{t}{t-\sqrt{7}-1}$ and $\tan \angle QAD = \dfrac{t-\sqrt{7}+1}{t}$. As $\angle PAD - \angle QAD = 45^\circ$, we have

$$1 = \tan(\angle PAD - \angle QAD) = \frac{\dfrac{t}{t-\sqrt{7}-1} - \dfrac{t-\sqrt{7}+1}{t}}{1 + \dfrac{t}{t-\sqrt{7}-1} \cdot \dfrac{t-\sqrt{7}+1}{t}},$$

which simplifies to the quadratic equation $t^2 - 2\sqrt{7}t + 3 = 0$. The roots are $\sqrt{7} \pm 2$, and the positive sign is taken as t must be greater than $CP = \sqrt{7}+1$.

8. By the properties of parallelograms and isosceles triangles, $\angle DOE$, $\angle ODB$, $\angle OBD$, $\angle OEC$ and $\angle OCE$ are all equal to $\angle BAC$, i.e. x°. There are two possibilities, according to whether O lies outside $BCED$ (left) or inside $BCED$ (right). (Theoretically O may also lie on BC; it could be included into either case below, and the following computations show that O actually does not lie on BC in this question.)



- If O lies outside $BCED$, then $\angle BOC = \angle BOD + \angle DOE + \angle EOC$, so

$$y = (180 - 2x) + x + (180 - 2x) = 360 - 3x.$$

  Hence $x + (360 - 3x) = 196$, which gives $x = 82$.

- If O lies inside $BCED$, then $\angle BOC = 360^\circ - \angle BOD - \angle DOE - \angle EOC$, so

$$y = 360 - (180 - 2x) - x - (180 - 2x) = 3x.$$

  Hence $x + 3x = 196$, which gives $x = 49$.

It follows that the answer is $82 + 49 = 131$.

Remark. Since $OD = OE$, the parallelogram $ADOE$ is actually a rhombus. It is also easy to show that we must have $AB = AC$.

9. Let v be the number of vowels and c be the number of consonants. Then (v, c) must be equal to $(2, 3)$, $(3, 2)$ or $(4, 1)$.

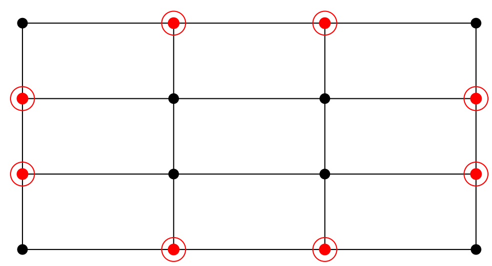
   If $(v, c) = (2, 3)$, there are C_2^5 ways of fixing 2 places in a word to be vowels (and the remaining 3 places will be consonants). There are 5 choices for each vowel and 8 choices for each consonant. Hence there are at most $C_2^5 \times 5^2 \times 8^3 = 128000$ such words.

   Similarly, the case $(v, c) = (3, 2)$ gives at most $C_3^5 \times 5^3 \times 8^2 = 80000$ words while the case $(v, c) = (4, 1)$ gives at most $C_4^5 \times 5^4 \times 8^1 = 25000$ words. The answer is thus $128000 + 80000 + 25000 = 233000$.

10. It is well-known that a figure consisting of (straight and curved) lines can be drawn in a single penstroke if and only if the figure (when considered as a graph with vertices being the intersection of the lines) has at most two *odd vertices* (i.e. vertices of odd degree).

    As an easy generalisation, if a graph has $2n$ odd vertices (note that the number of odd vertices must be even since the sum of the degrees of all vertices is equal to twice the number of edges), then at least n penstrokes are needed to trace the graph. This can be proved by induction — the key observation being that the number of odd vertices will be reduced by 2 if we add an edge joining two odd vertices.

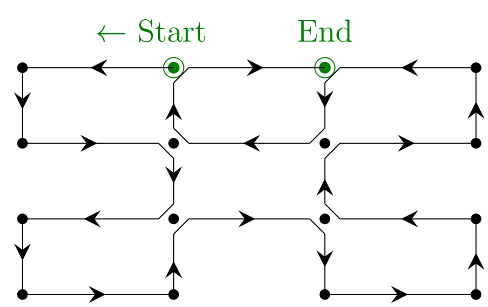
    Now the grid described in the question corresponds to a graph with 8 odd vertices as shown below:



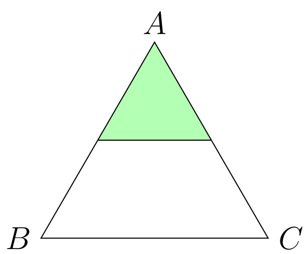
    This means tracing the whole graph requires at least 4 penstrokes, and hence Mandy would miss at least 3 edges. The total lengths of the edges missed is at least 4 (she cannot miss three edges of length 1 since there are only 2 edges of length 1 whose both endpoints are odd vertices). As the original figure consists of 12 edges of length 1 and 12 edges of length 2, the length of the figure drawn by Mandy is at most

$$1 \times 12 + 2 \times 12 - 4 = 32.$$

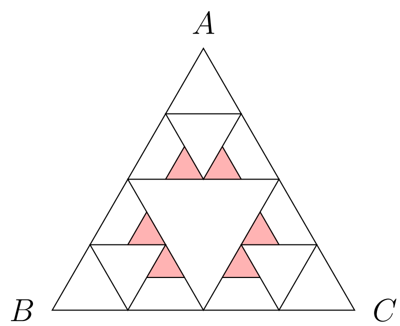
    This is attainable as shown by the example below (the path is slightly altered at some vertices for easy viewing).



11. Call the equilateral triangle ABC and suppose the vertices A, B, C are chosen in order. The first jump of the frog can be considered as a homothety centred at A with factor $\frac{1}{2}$. As the initial position of the frog is anywhere inside $\triangle ABC$, the set of possible positions of the frog after the first jump is precisely the triangle formed by A, the mid-point of AB and the mid-point of AC (the green shaded area below):



Similarly, the set of possible positions of the frog after the second and third jumps are indicated by the yellow and red shaded areas in the figure respectively. The green area is $\frac{1}{4}$ the area of $\triangle ABC$; similarly the yellow and red areas are $\frac{1}{4^2}$ and $\frac{1}{4^3}$ respectively. Now there are $3! = 6$ possible orders for choosing the vertices, and the red regions resulting from different orders of choosing the vertices will not overlap — in the figure below all the possible final positions of the frog are shaded in red:



The answer is thus $\dfrac{1}{4^3} \times 6 = \dfrac{3}{32}$.

12. Rewrite the equation as $x^4 - 4x = 1$. Adding $2x^2 + 1$ to both sides and moving the term $-4x$ to the right hand side, we get $x^4 + 2x^2 + 1 = 2x^2 + 4x + 2$, or

$$(x^2 + 1)^2 = 2(x + 1)^2.$$

It follows that $x^2 + 1 = \pm\sqrt{2}(x + 1)$. We take the positive sign in order for a real solution to exist. The resulting equation becomes $x^2 - \sqrt{2}x + 1 - \sqrt{2} = 0$. Its two roots a and b satisfy $a + b = \sqrt{2}$ and $ab = 1 - \sqrt{2}$, and so we have

$$a^2 + b^2 = (a + b)^2 - 2ab = 2\sqrt{2}.$$

Remark. We can also work out the answer by brute force. Rewrite the equation as $x^4 - 4x - 1 = 0$. We seek a factorisation of the form $(x^2 + px + q)(x^2 + rx + s) = 0$. Comparing coefficients, we have

$$p + r = 0, \quad pr + q + s = 0, \quad ps + qr = -4 \quad \text{and} \quad qs = -1.$$

The first and last equations enable us to express r in terms of p and s in terms of q. We thus get

$$-p^2 + q - \frac{1}{q} = 0 \quad \text{and} \quad -\frac{p}{q} - pq = -4.$$

It follows that

$$4 = \left(q + \frac{1}{q}\right)^2 - \left(q - \frac{1}{q}\right)^2 = \left(\frac{4}{p}\right)^2 - (p^2)^2,$$

or $p^6 + 4p^2 - 16 = 0$. Setting $m = p^2$ gives $m^3 + 4m - 16 = 0$, from which we readily see that $m = 2$ is one possibility. This allows us to get $(p, q, r, s) = (\sqrt{2}, \sqrt{2}+1, -\sqrt{2}, 1-\sqrt{2})$ as one possibility, and so the equation becomes

$$(x^2 + \sqrt{2}x + \sqrt{2} + 1)(x^2 - \sqrt{2}x + 1 - \sqrt{2}) = 0.$$

Noting that $x^2+\sqrt{2}x+\sqrt{2}+1 = 0$ has no real root, we thus consider $x^2-\sqrt{2}x+1-\sqrt{2} = 0$ to obtain the answer as in the original solution.

13. (**Idea for solution.** Note that after rounding off, the difference between consecutive terms is 6 or 7, the difference between a term and its second subsequent term is 13 or 14, and so on. This allows us to work out some bounds. For instance, since the difference between a term and its second subsequent term is 13 and 14 after rounding off, the actual difference (which is $2d$) should also be between these bounds. On the other hand, since the difference between a term and its fourth subsequent term is always 27 after rounding off ($39 - 12 = 45 - 18 = 52 - 25 = 27$), the actual difference (which is $4d$) should be between 26 and 28. By constructing bounds like this, we can eventually work out the range of values of d.)

Let a_n denote the n-th term of the sequence. Note that $17.5 \leq a_2 < 18.5$ and $38.5 \leq a_5 < 39.5$. Hence $3d = a_5 - a_2 > 38.5 - 18.5 = 20$, so $d > \frac{20}{3}$. Similarly $5d = a_6 - a_1 < 45.5 - 11.5 = 34$, so $d < \frac{34}{5}$. All intermediate values are attainable (see remark below). It follows that

$$ab = \frac{20}{3} \times \frac{34}{5} = \frac{136}{3}.$$

Remark. Suppose $\frac{20}{3} < d < \frac{34}{5}$. We shall prove that there exists a_1 so that the arithmetic sequence with first term a_1 and common difference d have its first seven terms being 12, 18, 25, 32, 39, 45, 52 after rounding off. It suffices to show that the intersection of the following intervals

$$\begin{aligned}
I_1 &= [11.5, 12.5) \\
I_2 &= [17.5 - d, 18.5 - d) \\
I_3 &= [24.5 - 2d, 25.5 - 2d) \\
I_4 &= [31.5 - 3d, 32.5 - 3d) \\
I_5 &= [38.5 - 4d, 39.5 - 4d) \\
I_6 &= [44.5 - 5d, 45.5 - 5d) \\
I_7 &= [51.5 - 6d, 52.5 - 6d)
\end{aligned}$$

is non-empty. Indeed, it can be checked that

- if $\frac{20}{3} < d \le \frac{27}{4}$, the largest of the 7 lower bounds would be $38.5 - 4d$ while the smallest of the 7 upper bounds would be $18.5 - d$, and the latter is larger since $d > \frac{20}{3}$;
- if $\frac{27}{4} < d < \frac{34}{5}$, the largest of the 7 lower bounds would be 11.5 while the smallest of the 7 upper bounds would be $45.5 - 5d$, and the latter is larger since $d < \frac{34}{5}$.

Hence in either case we have $\displaystyle\bigcap_{k=1}^{7} I_k \neq \varnothing$.

14. Note that $2024 = 2^3 \times 11 \times 23$ and $2024^{2024} = 2^{6072} \times 11^{2024} \times 23^{2024}$, so $n = 2^a 11^b 23^c$ for some non-negative integers a, b, c with $a \le 6072$ and $b, c \le 2024$. Then n has exactly $(a+1)(b+1)(c+1)$ positive factors (because each positive factor of n is of the form $2^p 11^q 23^r$ where there are $a+1$ choices for p since p can be any one of 0, 1, 2, ..., a, and similarly there are $b+1$ choices for q and $c+1$ choices for r). It follows that we have

$$(a+1)(b+1)(c+1) = 2024 = 2^3 \times 11 \times 23,$$

and we just count the number of non-negative integer solutions to the above equation, without worrying the upper bounds on a, b, c, since any violation of the upper bounds will render the left hand side too big for the equation to hold. We can distribute the factor 11 to one of the three brackets on the left hand side, so there are 3 choices for that. Similarly there are 3 choices for the factor 23. Finally, we can distribute three factors of 2 among the three brackets on the left hand side, and this can be done in $H_3^3 = C_3^5 = 10$ ways (see remark below). Hence the answer is $3 \times 3 \times 10 = 90$.

Remark. We are to distribute three identical objects (factors of 2) into three distinct boxes (brackets on the left hand side). This is equivalent to permuting three O's and two I's (which can be done in C_3^5 ways) since each such permutation corresponds to a way of distribution if we view the O's as the objects and the I's as the separators for the boxes. Some examples that illustrate the ides are shown below:

$$\begin{array}{lcccccc}
\text{OIOIO} & \longleftrightarrow & \bigcirc & | & \bigcirc & | & \bigcirc \\
\text{OOIIO} & \longleftrightarrow & \bigcirc\bigcirc & | & & | & \bigcirc \\
\text{IIOOO} & \longleftrightarrow & & | & & | & \bigcirc\bigcirc\bigcirc
\end{array}$$

This shows $H_3^3 = C_3^5$ (here H_r^n refers to the number of ways of choosing r objects out of n distinct objects, without order and with repetition allowed; by contrast C_r^n counts the same quantity with repetition not allowed). In the same way we can show that $H_r^n = C_r^{n+r-1}$.

15. Direct computation yields

$$a_2 = \frac{5}{4}, \quad a_3 = \frac{5}{3}, \quad a_4 = \frac{17}{8} \quad \text{and} \quad a_5 = \frac{13}{5}$$

and we observe that a_n is slightly larger than $\frac{n}{2}$. In fact, a more careful check reveals that

$$a_2 = 1 + \frac{1}{4}, \quad a_3 = \frac{3}{2} + \frac{1}{6}, \quad a_4 = 2 + \frac{1}{8} \quad \text{and} \quad a_5 = \frac{5}{2} + \frac{1}{10}.$$

Hence we naturally expect $a_n = \frac{n}{2} + \frac{1}{2n}$ (see Remark for a proof), and the smallest possible value of a is therefore

$$a_{200} = 100 + \frac{1}{400} = \frac{40001}{400}.$$

Remark. Since only answers are required during the contest, it is reasonable to expect that the pattern observed is true in general, and therefore one would not need to work out a proof during the contest. Nevertheless, we give the proof below for the completeness of a rigorous solution.

Let $b_n = 2a_n - n$ (so our goal is to prove $b_n = \frac{1}{n}$). Then the first recurrence relation in the question can be rewritten as

$$\frac{b_{2n} + 2n}{2} = \frac{b_n + n}{4} + \frac{3n}{4},$$

or $2b_{2n} = b_n$. The second recurrence relation can be rewritten as

$$\begin{aligned}&\frac{b_{2n+1} + (2n+1)}{2} \\ = &\frac{(b_n + n)(b_{n+1} + n + 1) + n(b_n + n) + (n+1)(b_{n+1} + n + 1) - 3n^2 - 3n - 1}{2[(b_n + n) + (b_{n+1} + n + 1) - 2n - 1]},\end{aligned}$$

which, after careful simplification, becomes

$$\frac{1}{b_{2n+1}} = \frac{1}{b_n} + \frac{1}{b_{n+1}}.$$

We complete the proof by induction. We have $b_1 = 2a_1 - 1 = 1$. Suppose $n > 1$ and $b_n = \frac{1}{n}$ holds for smaller n. Then we consider two cases:

- If n is even, say $n = 2m$, then

$$b_n = \frac{b_m}{2} = \frac{1}{2} \cdot \frac{1}{m} = \frac{1}{n}.$$

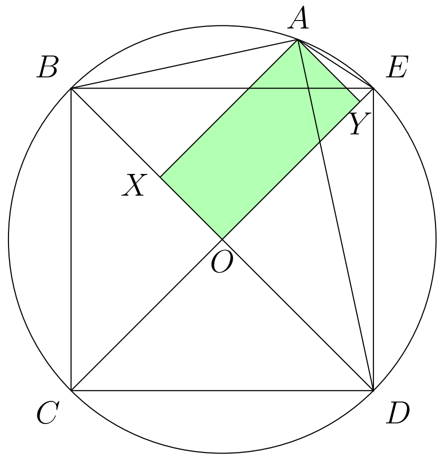
- If n is odd, say $n = 2m + 1$, then

$$\frac{1}{b_n} = \frac{1}{b_m} + \frac{1}{b_{m+1}} = \frac{1}{\frac{1}{m}} + \frac{1}{\frac{1}{m+1}} = 2m + 1 = n,$$

  so that we have $b_n = \frac{1}{n}$ as well.

It thus follows by the principle of mathematical induction that $b_n = \frac{1}{n}$ for all positive integers n.

16. Let O be the centre and r be the radius of the circle. Since $BCDE$ is a square, BD and CE meet at O. Let X and Y be the feet of perpendiculars from A to BD and CE respectively. Then $AXOY$ is a rectangle.



Note that $\triangle ABD$ is right-angled at A. Considering its area gives $BD \cdot AX = AB \cdot AD$, so that $2r \cdot AX = 224$. Similarly, we have $2r \cdot AY = 360$. On the other hand, since

$$AX^2 + AY^2 = XY^2 = OA^2 = r^2,$$

we have the equation

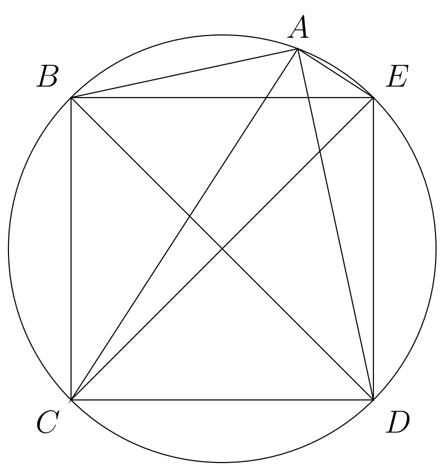
$$\left(\frac{112}{r}\right)^2 + \left(\frac{180}{r}\right)^2 = r^2.$$

It follows that $r^2 = \sqrt{112^2 + 180^2} = 212$ and so the area of the square $BCDE$ is $2r^2 = 424$.

Remark. *Ptolemy's theorem* asserts that for a cyclic quadrilateral, the product of the lengths of the diagonals is equal to the sum of the products of the lengths of the opposite sides, i.e. if $PQRS$ is a cyclic quadrilateral, then

$$PR \cdot QS = PQ \cdot RS + QR \cdot PS.$$

With knowledge of this theorem, we can solve the problem without constructing the points X and Y, as follows. Let $AB = b$, $AC = c$, $AD = d$, $AE = e$ and the side length of the square be x.



Applying Ptolemy's theorem in $ABCE$, we have $AC \cdot BE = AB \cdot CE + AE \cdot BC$, or $cx = b(\sqrt{2}x) + ex$. It follows that $c = \sqrt{2}b + e$. Similarly, Applying Ptolemy's theorem in

$ACDE$ gives $AD \cdot CE = AC \cdot DE + AE \cdot CD$, which simplifies to $\sqrt{2}d = c + e$. It follows that

$$c^2 - e^2 = (c+e)(c-e) = (\sqrt{2}d)(\sqrt{2}b) = 2bd.$$

As it is given that $bd = 224$ and $ce = 360$, we can solve to get $c = 18\sqrt{2}$ and $e = 10\sqrt{2}$. Now it follows that $2x^2 = CE^2 = c^2 + e^2 = 848$, and so the area of $BCDE$ is $x^2 = 424$.

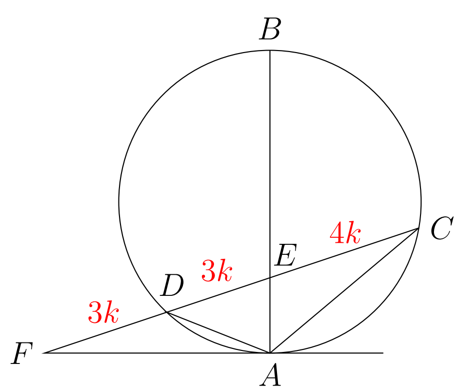
17. Let's assume all balls were drawn and returned to the respective players. Denote by B the event that Betty gets back all her balls before Emily, and C, D be the corresponding event (getting back all balls before Emily) for Cathy and Daisy respectively. Note that $P(B) = \frac{5}{7}$ (essentially we are permuting 2 B's and 5 E's and we are looking at the probability that the last letter is E). Similarly $P(C) = \frac{5}{8}$ and $P(D) = \frac{5}{9}$. On the other hand, we have $P(B \cap C) = \frac{5}{10}$ (essentially we are permuting 2 B's, 3 C's and 5 E's and we are looking at the probability that the last letter is E) and similarly

$$P(B \cap D) = \frac{5}{11}, \quad P(C \cap D) = \frac{5}{12} \quad \text{and} \quad P(B \cap C \cap D) = \frac{5}{14}.$$

Emily wins if and only if none of B, C, D happens, the probability of which is

$$\begin{aligned}
P(\overline{B} \cap \overline{C} \cap \overline{D}) &= 1 - P(B \cup C \cup D) \\
&= 1 - P(B) - P(C) - P(D) \\
&\quad + P(B \cap C) + P(B \cap D) + P(C \cap D) - P(B \cap C \cap D) \\
&= 1 - \frac{5}{7} - \frac{5}{8} - \frac{5}{9} + \frac{5}{10} + \frac{5}{11} + \frac{5}{12} - \frac{5}{14} \\
&= \frac{661}{5544}
\end{aligned}$$

18. Let $CE = 4k$ and $ED = DF = 3k$.



By the power chord theorem (see Remark), we have $FA^2 = FD \cdot FC$, so we have $FA = \sqrt{30}k$. As BA is a diameter, we have $\angle BAF = 90°$ and so $EA = \sqrt{FE^2 - FA^2} = \sqrt{6}k$. Using power chord theorem again, we have $EA \cdot EB = EC \cdot ED$, from which we get $EB = 2\sqrt{6}k$. Hence

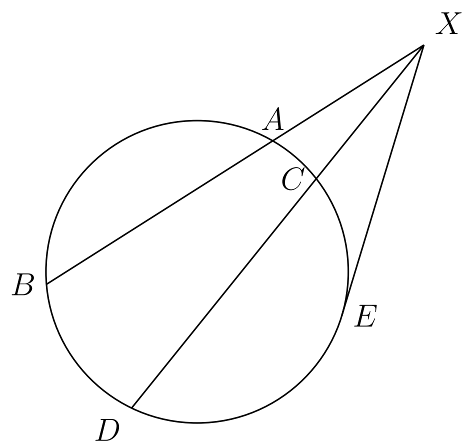
$$k = \frac{AB}{3\sqrt{6}} = \frac{24}{3\sqrt{6}} = \frac{8}{\sqrt{6}}.$$

Finally, note that the circle with EF as diameter passes through A and is centred at D. That means

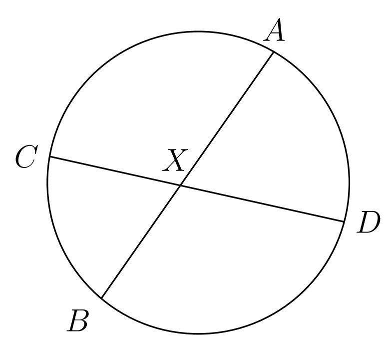
$$\angle AFD = \angle DAF = \angle ACF.$$

It follows that $AC = AF = \sqrt{30}k = 8\sqrt{5}$.

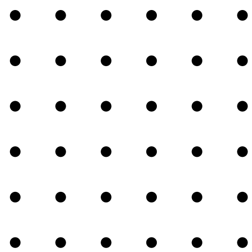
Remark. Let X be a point outside a circle. The *power chord theorem* asserts that for any straight line passing through X and intersecting the circle at A and B (possibly $A = B$ if the line is tangent to the circle), the value of $XA \cdot XB$ is a constant. Equivalently, we have $XA \cdot XB = XC \cdot XD = XE^2$ in the figure below. The theorem can be easily proved using the fact that $\triangle XAC \sim \triangle XDB$ and $\triangle XEA \sim \triangle XBE$.



The same is true if X is inside the circle, except that $A = B$ is no longer a possibility. In other words we have $XA \cdot XB = XC \cdot XD$ in the figure below, and the proof is essentially the same.

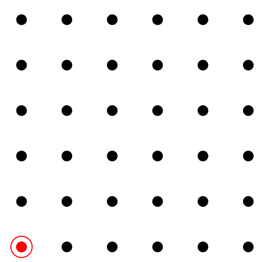


19. For each of the 36 points, we count the number of triangles with that point as circumcentre. In view of symmetry, we can classify the 36 points into types A to F as follows, so that we only need to consider one point of each type:



A	B	C	C	B	A
B	D	E	E	D	B
C	E	F	F	E	C
C	E	F	F	E	C
B	D	E	E	D	B
A	B	C	C	B	A

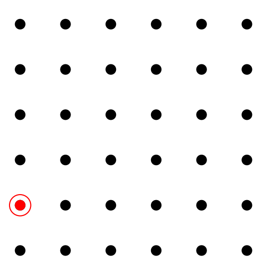
(A) We pick the Type A point at the lower left hand corner and consider the square of the distances of the remaining points from the chosen point:



25	26	29	34	41	50
16	17	20	25	32	41
9	10	13	18	25	34
4	5	8	13	20	29
1	2	5	10	17	26
0	1	4	9	16	25

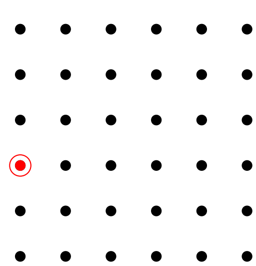
In order for the chosen point to be the circumcentre of such a triangle, we must look for three other points with the same number (square of distance from the chosen point) in the above figure as vertices of the triangle. It turns out that 25 is the only number that occurs at least three times. It occurs 4 times, so $C_3^4 = 4$ triangles can be formed with the chosen point as circumcentre.

(B) For a chosen Type B point, each of the numbers 1, 5, 10, 17, 25 occurs 3 times. Hence $C_3^3 \times 5 = 5$ such triangles can be formed with the chosen point as circumcentre.



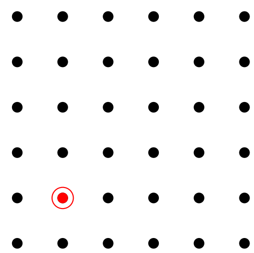
16	17	20	25	32	41
9	10	13	18	25	34
4	5	8	13	20	29
1	2	5	10	17	26
0	1	4	9	16	25
1	2	5	10	17	26

(C) For a chosen Type C point, each of 1, 4, 10, 13 occurs 3 times while 5 occurs 4 times. Hence $C_3^3 \times 4 + C_3^4 = 8$ such triangles can be formed with the chosen point as circumcentre.



9	10	13	18	25	34
4	5	8	13	20	29
1	2	5	10	17	26
0	1	4	9	16	25
1	2	5	10	17	26
4	5	8	13	20	29

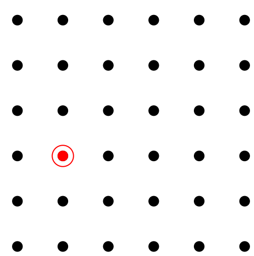
(D) For a chosen Type D point, each of 1, 2, 5, 10, 17 occurs 4 times. Hence $C_3^4 \times 5 = 20$ such triangles can be formed with the chosen point as circumcentre.



$\longleftrightarrow$

17	16	17	20	25	32
10	9	10	13	18	25
5	4	5	8	13	20
2	1	2	5	10	17
1	0	1	4	9	16
2	1	2	5	10	17

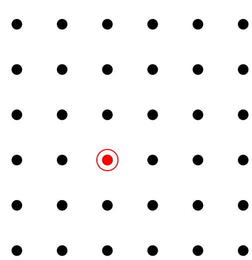
(E) For a chosen Type E point, each of 1, 2, 10 occurs 4 times, each of 4, 13 occurs 3 times, while 5 occurs 6 times. Hence $C_3^4 \times 3 + C_3^3 \times 2 + C_3^6 = 34$ such triangles can be formed with the chosen point as circumcentre.



$\longleftrightarrow$

10	9	10	13	18	25
5	4	5	8	13	20
2	1	2	5	10	17
1	0	1	4	9	16
2	1	2	5	10	17
5	4	5	8	13	20

(F) For a chosen Type F point, each of 1, 2, 4, 8, 10, 13 occurs 4 times, while 5 occurs 8 times. Hence $C_3^4 \times 6 + C_3^8 = 80$ such triangles can be formed with the chosen point as circumcentre.



$\longleftrightarrow$

13	10	9	10	13	18
8	5	4	5	8	13
5	2	1	2	5	10
4	1	0	1	4	9
5	2	1	2	5	10
8	5	4	5	8	13

As the number of points of Type A, B, C, D, E, F are 4, 8, 8, 4, 8, 4 respectively, the answer is

$$4 \times 4 + 5 \times 8 + 8 \times 8 + 20 \times 4 + 34 \times 8 + 80 \times 4 = 792.$$

Remark. Instead of computing the square of distance from the chosen point to every other point as in the above solution, basically all we need is to identify the existence of three or more points with the same distance from the chosen point. With this in mind the computation could have been simplified. (For example in looking for points with squared distance 5 is like asking which points can be reached from the chosen point with a 'knight's move', and other squared distances can be considered in a similar manner.) One *tricky thing* to pay attention to would be that since $0^2 + 5^2 = 3^2 + 4^2$, two different types of 'moves' could lead to the same squared distance. (It can also be shown that there is no other *tricky thing*, i.e. the equation $a^2+b^2=c^2+d^2$ with $a,b,c,d\in\{0,1,2,3,4,5\}$ has no solution other than $(a,b,c,d)=(0,5,4,3)$ except for its trivial permutations (e.g. $(a,b,c,d)=(3,4,5,0)$) and the trivial solutions (such as $(a,b,c,d)=(3,0,0,3)$).

20. Let $f(k)=n^{k-1}+k^2$ so that the expression in the question is equal to $f(2)f(3)\cdots f(2024)$. Note that $f(2)$, $f(3)$, ..., $f(2024)$ alternate in parity, with $f(2)$ having the same parity as n. Let z be the number of ending zeros in the base 2 representation of $f(2)f(3)\cdots f(2024)$.

    For the purpose of maximising z, we may assume n is even. This is because if n is odd, then each of $f(3)$, $f(5)$, ..., $f(2023)$ can only contribute one ending zero (each of them is the sum of two odd squares, which, as mentioned in the solution to Problem 2, must be congruent to 1 modulo 4, i.e. each of them is not divisible by 4).

    Now suppose n is even. Let $v(k)$ be the highest power of 2 dividing k. Then $z=v(f(2))+v(f(4))+\cdots+v(f(2024))$. Note that for even number k greater than 4, we have $v(f(k))=v(k^2)=2v(k)$ (see Remark 1). Hence

    $$z=2[v(2)+v(4)+\cdots+v(2024)]+v(f(2))+v(f(4))-2v(2)-2v(4).$$

    Note that

    - $v(2)+v(4)+\cdots+v(2024)$ is simply $v(2024!)$, which is equal to
    $$\left\lfloor\frac{2024}{2}\right\rfloor+\left\lfloor\frac{2024}{2^2}\right\rfloor+\left\lfloor\frac{2024}{2^3}\right\rfloor+\cdots=2017;$$
    - $v(f(2))+v(f(4))$ can be maximised if we make $f(2)=8192=2^{13}$ (see Remark 2), i.e. $n=8188$, which leads to $v(f(2))=13$ and $v(f(4))=4$ as $f(4)=8188^3+4^2$ is divisible by 16 but not 32;
    - $v(2)=1$; and
    - $v(4)=2$.

    It follows that the maximum possible value of z is $2(2017)+13+4-2(1)-2(2)=4045$.

    Remarks.

    (1) Let k be an even number greater than 4. Since $f(k)=n^{k-1}+k^2$, to prove $v(f(k))=v(k^2)$ it suffices to show that $v(n^{k-1})>v(k^2)$, or equivalently, $(k-1)v(n)>2v(k)$. As $v(n)\geq 1$, if the last inequality is to fail then we must have $v(k)\geq\frac{1}{2}(k-1)$, which requires $k\geq 2^{\frac{1}{2}(k-1)}$. However, it is easy to prove by induction that we must have $k<2^{\frac{1}{2}(k-1)}$.

    (2) To rigorously prove that $v(f(2))+v(f(4))\leq 17$ (equality attained when $n=8188$), we consider three cases:

    - If $v(n)=1$, then we have $v(f(2))=1$ and $v(f(4))=3$.
    - If $v(n)=2$, then we have $v(f(4))=4$, whereas $v(f(2))\leq 13$ since $f(2)<2^{14}$.
    - If $v(n)\geq 3$, then we have $v(f(2))=2$ and $v(f(4))=4$.