International Mathematical Olympiad
Hong Kong Preliminary Selection Contest 2025

Outline of solutions

Answers:

1.	998875541	**2.**	4501	**3.**	16900	**4.**	$\frac{\sqrt{3}-\sqrt{195}}{2}$
5.	$\sqrt{\frac{1+\sqrt{5}}{2}}$	**6.**	833	**7.**	$11+11\sqrt{5}$	**8.**	$24°$
9.	1035	**10.**	$\frac{10}{3}$	**11.**	$26°$	**12.**	$3\sqrt{5}$
13.	2100100	**14.**	30	**15.**	541	**16.**	823856
17.	1205	**18.**	$\frac{21}{5}$	**19.**	$12+6\sqrt{3}$	**20.**	21

Solutions:

1. Note that
$$10! = 2^8 \times 3^4 \times 5^2 \times 7.$$
Hence at most two of the digits of n can be equal to 9 (which is 3^2). Similarly, at most two of the digits of n can be 8 (which is 2^3), and at most one can be 7. We thus seek to find an n starting with 99887, and the product of the remaining four digits of n is
$$\frac{2^8 \times 3^4 \times 5^2 \times 7}{9 \times 9 \times 8 \times 8 \times 7} = 2^2 \times 5^2.$$
We could have up to two digits of 5 and one digit of 4, giving the answer 998875541.

2. Note that $2025 = 45^2$. Since $450^2 = 202500$ and $451^2 - 450^2 = 451 + 450 > 100$, there is no such n for which n^2 has 6 digits. On the other hand, we have $4500^2 = 20250000$ and $4501^2 = 4500^2 + 2(4500) + 1 = 20259001$, which shows that 4501 is a possible value of n, and is the smallest such n for which n^2 has an even number of digits.
It remains to check that there is no such n for which n^2 has 5 or 7 digits. Indeed, we have
$$142^2 < 20250 < 20260 < 143^2$$
and
$$1423^2 < 2025000 < 2026000 < 1424^2,$$
which show that the answer is indeed 4501.

3. Let $q = m - n$. Then $m = (m-n)^2$ implies

$$m = q^2 \quad \text{and} \quad n = q^2 - q$$

so that the H.C.F. of m and n is q. It is given that n is equal to 25 times q. Hence we have $q^2 - q = 25q$, which gives $q = 26$. It follows that $m = 26^2$ and $n = 26^2 - 26 = 26 \times 25$, and so the L.C.M. of m and n is $26 \times 26 \times 25 = 16900$.

Remark. Here is an alternative solution. Let d be the H.C.F. of m and n. Then $n = 25d$, and $m = kd$ for some integer k which is greater than 25 and not divisible by 5. The equation $m = (m-n)^2$ thus becomes $kd = (kd - 25d)^2$, or

$$k = (k-25)^2 d.$$

Hence k must be equal to 26, because k and $k - 25$ cannot contain any common factor greater than 1 (for the two numbers differ by 25 and are not divisible by 5). This gives $k = d = 26$, so the L.C.M. of m and n is $25kd = 25 \times 26^2 = 16900$.

4. Note that

$$\begin{aligned}(x^2 - 2\sqrt{3}x)(x^2 - 3) &= x(x - 2\sqrt{3})(x - \sqrt{3})(x + \sqrt{3}) \\ &= x(x - \sqrt{3}) \cdot (x - 2\sqrt{3})(x + \sqrt{3}) \\ &= (x^2 - \sqrt{3}x)(x^2 - \sqrt{3}x - 6)\end{aligned}$$

Hence, by setting $u = x^2 - \sqrt{3}x$, the equation becomes

$$u(u-6) = 2016,$$

which has solutions $u = 48$ and $u = -42$. The latter gives no real solution for x, so we must have

$$x^2 - \sqrt{3}x = 48.$$

The solutions are

$$x = \frac{\sqrt{3} \pm \sqrt{3 - 4(-48)}}{2} = \frac{\sqrt{3} \pm \sqrt{195}}{2},$$

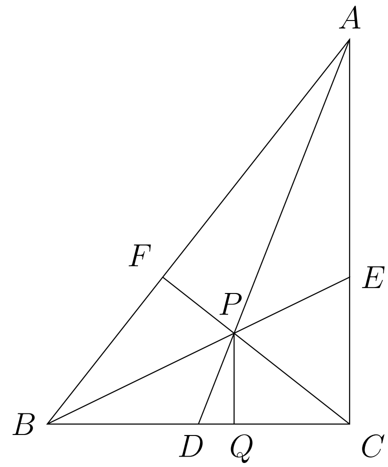
and we take the negative root.

Remark. Alternatively, one can expand the left hand side and add 9 to both sides to obtain

$$x^4 - 2\sqrt{3}x^3 - 3x^2 + 6\sqrt{3}x + 9 = 2025,$$

which gives $(x^2 - \sqrt{3}x - 3)^2 = 45^2$, and proceed in essentially the same way as above.

5. Suppose the median through A meets BC at D, the internal bisector of $\angle ABC$ meets CA at E and the altitude from C meets AB at F. Let P be the point of concurrency of AD, BE and CF, and let Q be the foot of perpendicular from P to BC. Using standard notations in trigonometry, we have $a = 1$ and we want to find $AC = b$.



We compute the lengths of QD and PQ in terms of b as follows.

- We have $BF = a\cos B$ and $\triangle BFP \cong \triangle BQP$. Hence

$$QD = BQ - BD = BF - BD = a\cos B - \frac{1}{2} = \frac{1}{\sqrt{b^2+1}} - \frac{1}{2}.$$

- We have $PQ = PF = BF\tan\frac{B}{2}$. Let $t = \tan\frac{B}{2}$. Then $b = \tan B = \frac{2t}{1-t^2}$, from which we solve to get

$$t = \frac{-1+\sqrt{b^2+1}}{b}.$$

  Hence $PQ = \dfrac{1}{\sqrt{b^2+1}} \cdot \dfrac{-1+\sqrt{b^2+1}}{b}$.

Since $\triangle ACD \sim \triangle PQD$, we have $\dfrac{AC}{CD} = \dfrac{PQ}{QD}$ and so

$$\frac{b}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{b^2+1}} \cdot \frac{-1+\sqrt{b^2+1}}{b}}{\frac{1}{\sqrt{b^2+1}} - \frac{1}{2}}.$$

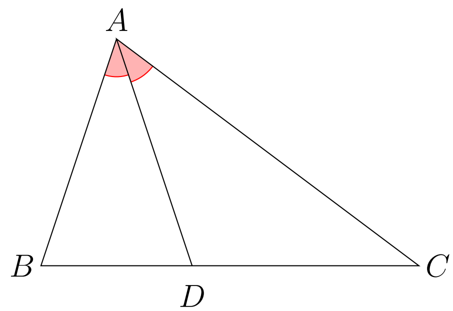
To solve for b, it would be convenient to introduce the substitution $u = \sqrt{b^2+1}$. The equation becomes

$$2b = \frac{\frac{u-1}{bu}}{\frac{2-u}{2u}} = \frac{2(u-1)}{b(2-u)} = \frac{2b}{(u+1)(2-u)},$$

where in the last step we made use of the fact that $(u+1)(u-1) = b^2$. Hence we have $(u+1)(2-u) = 1$, which gives $u = \frac{1}{2}(1+\sqrt{5})$ and hence

$$b = \sqrt{u^2-1} = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

Remark. There are many ways to simplify the solution. For instance one could avoid the computations of half angles by using the *angle bisector theorem*, which asserts that the internal bisector of an angle of a triangle divides the opposite side in a ratio proportional to the lengths of the other two sides, i.e. in the figure below we have $BD : DC = AB : AC$.



With this theorem, we could make use of the ratio $FP : PC = BF : BC$ to find PQ more easily. Furthermore, the solution will be much simpler if one knows *Ceva's theorem*, which asserts that if AD, BE and CF are concurrent, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

It is easy to work out $AF = \dfrac{b^2}{\sqrt{b^2+1}}$ and $FB = \dfrac{1}{\sqrt{b^2+1}}$. Also

$$\frac{CE}{EA} = \frac{BC}{BA} = \frac{1}{\sqrt{b^2+1}}$$

by the angle bisector theorem. Hence by Ceva's theorem we have

$$\frac{\frac{b^2}{\sqrt{b^2+1}}}{\frac{1}{\sqrt{b^2+1}}} \cdot 1 \cdot \frac{1}{\sqrt{b^2+1}} = 1,$$

from which we can solve for b easily.

6. Let $b = pa$ and $c = qa$ for some positive integers p and q. Then the given equality becomes

$$a(1+p+q) = 300.$$

Clearly, $1+p+q$ must be a positive factor of 300. Conversely, for each positive factor k of 300, if we set $k = 1+p+q$, there corresponds a unique value of a, and the number of choices for (p,q) satisfying $k = 1+p+q$ is $k-2$ (except when $k=1$ since in that case the number of choices is 0 but $k-2=-1$). Each choice of (p,q) under a fixed choice of a gives rise to a unique choice of (b,c). Hence the answer is

$$1 + \sum_{k|300}(k-2) = 1 + \sigma(300) - 2d(300)$$

where

- the term 1 accounts for the correction needed for $k = 1$ as mentioned above;
- $\sigma(n)$ refers to the sum of the positive factors of n; and
- $d(n)$ refers to the number of positive factors of n.

Since $300 = 2^2 \times 3 \times 5^2$, we have (see Remark) $d(300) = (2+1)(1+1)(2+1) = 18$ and

$$\sigma(300) = (1 + 2 + 2^2)(1 + 3)(1 + 5 + 5^2) = 868.$$

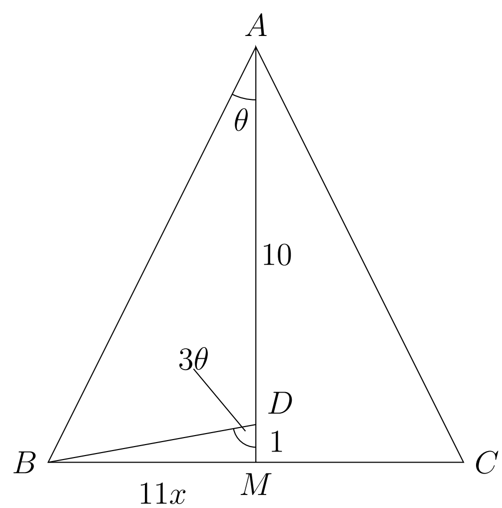
The answer is thus $1 + 868 - 2(18) = 833$.

Remark. If a positive integer n has prime factorisation $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then $d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1)$ and

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{a_1})(1 + p_2 + p_2^2 + \cdots + p_2^{a_2}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{a_r}).$$

One can of course also compute $d(300)$ and $\sigma(300)$ directly without using these formulas.

7. Let $\angle BAM = \theta$ and $\angle BDM = 3\theta$. Let also $BM = 11x$ so that $\tan\theta = x$.

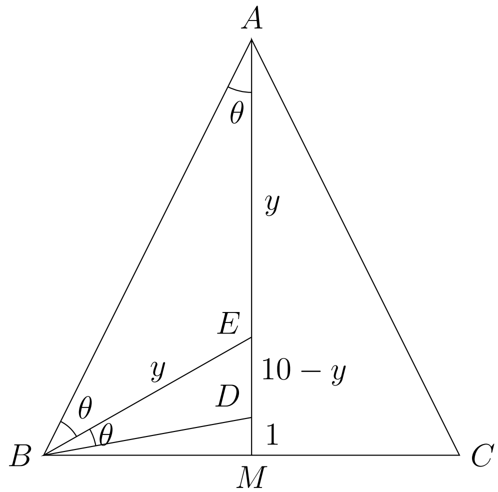


Then we have

$$11x = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{3x - x^3}{1 - 3x^2}.$$

Solving gives $x = \frac{1}{2}$. It follows that $BC = 22x = 11$ and $AC = AB = \sqrt{(11x)^2 + 11^2} = \frac{11}{2}\sqrt{5}$. The perimeter of $\triangle ABC$ is thus equal to $11 + 11\sqrt{5}$.

Remark. We can also solve the problem by pure geometry. Consider a point E on AM such that BE bisects $\angle ABD$. Since $\angle ABD = \angle BDM - \angle BAM = 2\theta$, we have $\angle ABE = \angle DBE = \theta$. We may thus let $AE = BE = y$ and $DE = 10 - y$.



We have $BM^2 = BE^2 - EM^2 = y^2 - (11-y)^2 = 22y - 121$, so

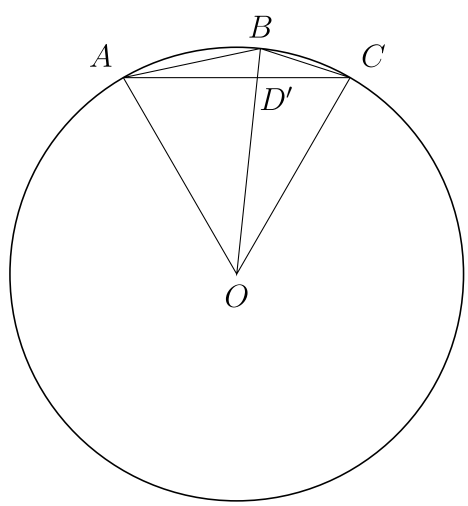
$$BD = \sqrt{BM^2 + 1^2} = \sqrt{22y - 120} \quad \text{and} \quad BA = \sqrt{BM^2 + 11^2} = \sqrt{22y}.$$

By the angle bisector theorem (see Remark after Question 5), we have

$$\frac{10-y}{y} = \frac{\sqrt{22y-120}}{\sqrt{22y}}.$$

Solving gives $y = \frac{55}{8}$. We can then work out the side lengths of $\triangle ABC$ and find its area as in the original solution.

8. Let O be the circumcentre of $\triangle ABC$. Since $\angle ABC = 180^\circ - 12^\circ - 18^\circ = 150^\circ$, the reflex $\angle AOC = 150^\circ \times 2 = 300^\circ$. Hence $\triangle OAC$ is equilateral. Suppose OB and AC meet at D'.



Note that $\angle AOB = 2\angle ACB = 36^\circ$, so $\angle OBA = \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ$. It follows that $\angle AD'B = 180 - 12^\circ - 72^\circ = 96^\circ$, which is the same as $\angle ADB$. Hence we must have

$D' = D$, and since O is a point on the extension of BD' such that $AC = AO = BO$, we must have $O = E$. It follows that

$$\angle DEC = \angle D'OC = \angle AOC - \angle AOB = 60^\circ - 36^\circ = 24^\circ.$$

9. There are
$$\binom{m}{2} + \binom{n}{2} = \frac{m(m-1) + n(n-1)}{2}$$
combinations of two balls that are of the same colour, and mn combinations of two balls that are of different colours. For Sam and Devin to have equal chance to win, we must have
$$m(m-1) + n(n-1) = 2mn,$$
which can be rearranged to
$$(m-n)^2 = m + n.$$
Let $m - n = d$. As we want to maximise m, we may assume $d \geq 0$ in view of the symmetry between m and n. Solving
$$m - n = d \quad \text{and} \quad m + n = d^2$$
gives $m = \frac{1}{2}(d^2 + d)$. Since $m + n = d^2 \leq 2025$, we have $d \leq 45$ and so $m \leq \frac{1}{2}(45^2 + 45) = 1035$. This upper bound is attainable with $d = 45$, $m = 1035$ and $n = 990$. Hence the answer is 1035.

10. Intuitively, we seek to maximise P and minimise Q, so we want each of a, b, c to have a large sum of digits and each of $a + b$, $b + c$ and $c + a$ to have a small sum of digits. Apparently, we should try to set the tens and unit digits of a, b, c to be as large as possible, so that $a + b$, $b + c$ and $c + a$ are three-digit numbers with small tens and unit digits. After some trials, we found that with $a = 46$, $b = 55$ and $c = 56$, we have
$$\frac{P}{Q} = \frac{\min\{4+6, 5+5, 5+6\}}{\max\{1+0+1, 1+1+1, 1+0+2\}} = \frac{10}{3}.$$
We prove below that this is indeed optimal. Assume on the contrary that it is not, i.e. there exist a, b, c for which $P > \frac{10}{3}Q$. We get a contradiction by considering different values of Q.

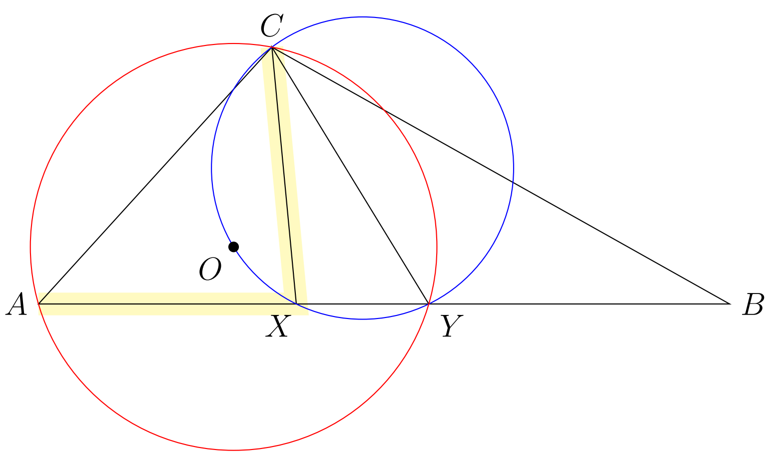
- If $Q \geq 6$, then $P > 20$, which is impossible.
- If $Q = 5$, then $P \geq 17$. The latter would force $\{a, b, c\} = \{89, 98, 99\}$, but in this case $Q \neq 5$.
- If $Q = 4$, then $P \geq 14$. This mean a, b, c are at least 59, so $a + b$, $b + c$ and $c + a$ are at least 119. For Q to be equal to 4, $a + b$, $b + c$ and $c + a$ can only be equal to 120, 121 and 130 in some order. But this is impossible, since that means $2(a + b + c) = 120 + 121 + 130$ which is odd.

- If $Q = 3$, then $P \geq 11$. This mean a, b, c are at least 29, so $a+b$, $b+c$ and $c+a$ are at least 59. For Q to be equal to 3, $a+b$, $b+c$ and $c+a$ must be among 100, 101, 102, 110, 111 and 120. We can carefully check that no three of these values could give a, b, c that are two-digit integers, all of whom have sum of digits at least 11.
- If $Q \leq 2$, then $a+b$, $b+c$ and $c+a$ must be equal to 100, 101 and 110 in some order, which is impossible since the sum of 100, 101 and 110 is odd.

11. We use A, B and $C = 103°$ to denote the interior angles of $\triangle ABC$, and O to denote the circumcentre of $\triangle ACY$ which is given to lie on the circumcircle of $\triangle XCY$. Since

$$\angle CXY = \angle COY = 2\angle CAY = 2A,$$

we have $\angle ACX = \angle CXY - \angle CAY = A$. Hence $CX = AX$.



As the perimeter of $\triangle CXY$ is equal to the length of AB, we have

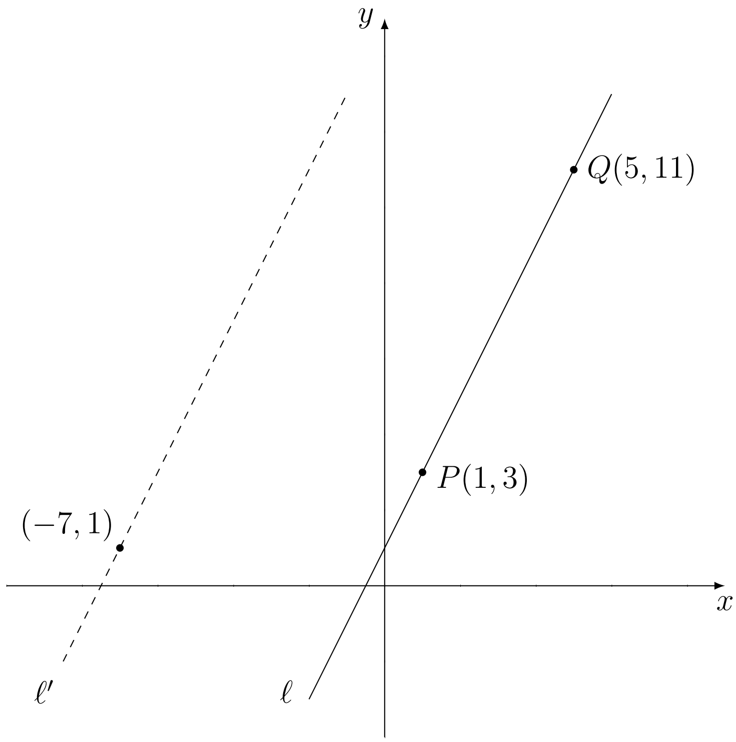
$$CY = AB - CX - XY = AB - AX - XY = YB.$$

This shows that $\angle BCY = B$. It follows that

$$\angle XCY = \angle ACB - \angle ACX - \angle BCY = C - A - B = 2C - 180° = 26°.$$

12. Let ℓ be the straight line passing through P and Q. We make the following key observations:

- Each time button $\mathcal{A}$ or button $\mathcal{C}$ is pressed, the distance of the frog from ℓ will be doubled.
- Each time button $\mathcal{B}$ or button $\mathcal{D}$ is pressed, the distance of the frog from ℓ will be halved.



Since button $\mathcal{A}$ has been pressed by the same number of times as button $\mathcal{B}$ and button $\mathcal{C}$ has been pressed by the same number of times as button $\mathcal{D}$, the final position (say, X') of the frog is at the same distance from ℓ as the initial position. In other words X' lies on the straight line passing through $(-7,1)$ and parallel to ℓ, and we denote this line by ℓ'. Since ℓ has slope 2, the equation of ℓ' is $y-1=2(x+7)$, or $2x-y+15=0$. Its distance from the origin is

$$\left|\frac{2(0)-0+15}{\sqrt{2^2+(-1)^2}}\right| = 3\sqrt{5},$$

and the point on ℓ' closest to the origin (say, X_0) is of the form $(t, -\frac{1}{2}t)$, which when put into the equation of ℓ' gives $t=-6$, i.e. $X_0=(-6,3)$. Indeed, the frog can get to this point if we press buttons $\mathcal{A}$, $\mathcal{C}$, $\mathcal{B}$, $\mathcal{D}$ in order:

$$(-7,1) \xrightarrow{\mathcal{A}} (-15,-1) \xrightarrow{\mathcal{C}} (-35,-13) \xrightarrow{\mathcal{B}} (-17,-5) \xrightarrow{\mathcal{D}} (-6,3)$$

It follows that the answer is $3\sqrt{5}$.

13. There are H_4^n terms in the sequence with at most n digits (see Remark). In particular, when $n=7$, this number is

$$H_4^7 = \binom{7+4-1}{4} = 210.$$

Hence the answer is the 11th largest 7-digit number in the sequence. Discarding 4000000 and the six terms beginning with 3 (3100000, 3010000, ..., 3000001), we need to count backward for four more terms. Carefully listing the terms in descending order: 2200000, 2110000, 2101000, 2100100, ..., we see that the answer is 2100100.

Remark. H_r^n refers to the number of ways of choosing r objects out of n distinct objects, without order and with repetition allowed; by contrast C_r^n (or $\binom{n}{r}$) counts the same quantity with repetition not allowed. In general $H_r^n = C_r^{n+r-1}$; we illustrate this with the case $n = 7$ and $r = 4$ below.

We are to distribute 4 identical objects (each contributing 1 to the sum of digits) into 7 distinct boxes (the 7 places in the number). This is equivalent to permuting 4 O's and 6 I's (which can be done in C_4^{10} ways) since each such permutation corresponds to a way of distribution (hence corresponds to a positive integer with at most 7 digits and with sum of digits 4) if we view the O's as the objects and the I's as the separators for the boxes. Some examples that illustrate the idea are shown below:

$$\begin{array}{lcccccccccccr}
\text{OIOIOIIIIO} & \longleftrightarrow & \bigcirc & | \bigcirc | & \bigcirc & | & | & | & & | \bigcirc & \longleftrightarrow & 1110001 \\
\text{OOIIOOIIII} & \longleftrightarrow & \bigcirc\bigcirc & | \quad | & \bigcirc\bigcirc & | & | & | & & | & \longleftrightarrow & 2020000 \\
\text{IIOOOIIIOI} & \longleftrightarrow & & | \quad | & \bigcirc\bigcirc\bigcirc & | & | & | \bigcirc & & | & \longleftrightarrow & 30010
\end{array}$$

One can also get the number H_4^n by considering the number of terms in the sequence with at most n digits according to the different possibilities:

- There are n terms containing the digit 4 (with all other digits being 0, same for the cases below).
- There are $n(n-1)$ terms containing a 3 and a 1.
- There are $\frac{1}{2}n(n-1)$ terms containing two 2's.
- There are $\frac{1}{2}n(n-1)(n-2)$ terms containing a 2 and two 1's.
- There are $\binom{n}{4}$ terms containing four 1's.

One can easily check that the above numbers do add up to H_4^n.

14. Let V and E denote the number of vertices and edges of $\mathcal{P}$ respectively. We work out some relationship between V, E, x and y as follows.

(1) By Euler's formula, we have

$$V - E + 32 = 2. \tag{14.1}$$

(2) Since $x + y$ edges meet at each vertex, and that each edge has two vertices, we have

$$2E = V(x + y). \tag{14.2}$$

(3) As there are x triangular vertices (i.e. vertices belonging to a triangular face) incident to each vertex, there are altogether xV triangular vertices, and hence $\frac{1}{3}xV$ triangular faces. Similarly, there are $\frac{1}{5}yV$ pentagonal faces. This gives

$$\frac{1}{3}xV + \frac{1}{5}yV = 32. \tag{14.3}$$

From (14.1), we get $E = V + 30$. Putting this into (14.2) and simplifying, we get

$$V = \frac{60}{x + y - 2}. \tag{14.4}$$

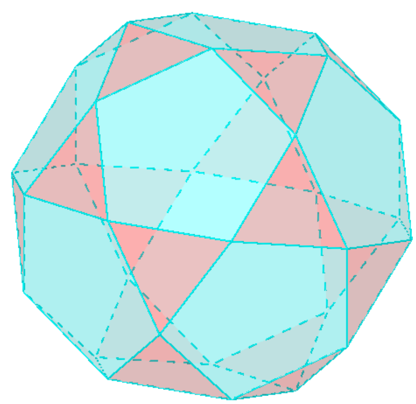
Putting (14.4) into (14.3), we get

$$\left(\frac{x}{3} + \frac{y}{5}\right)\left(\frac{60}{x + y - 2}\right) = 32.$$

This simplifies to $3x + 5y = 16$, and the only positive integer solution is $x = y = 2$. The number of vertices of $\mathcal{P}$ is thus equal to

$$V = \frac{60}{x + y - 2} = 30.$$

Remark. The *icosidodecahedron* with 30 vertices, 60 edges and 32 faces (20 triangular and 12 pentagonal) is an example of such $\mathcal{P}$.



15. Let $m = \lfloor \sqrt{n} \rfloor$. Then n satisfies the condition of the question if and only if

$$m + 0.25 \le \sqrt{n} < m + 0.26. \tag{15.1}$$

Multiplying (15.1) by 100 and then squaring, we have

$$10000m^2 + 5000m + 625 \le 10000n < 10000m^2 + 5200m + 676.$$

Let $x = 10000m^2 + 5000m + 625$ and $y = 10000m^2 + 5200m + 676$. As m and n are integers, the left-hand inequality can be strengthened to $10000m^2 + 5000m + 5000 \le 10000n$ (if m is odd, and $10000m^2 + 5000m + 10000 \le 10000n$ if m is even). Hence we have

$$200m + 51 = y - x > 10000m^2 + 5000m + 5000 - x = 4375.$$

Solving gives $m \ge 23$ (if m is odd, and much larger if m is even). When $m = 23$, we have

$$x = 5405625 \quad \text{and} \quad y = 5410276,$$

which corresponds to the smallest possible value of n being 541.

Remark. Here is an alternative solution (based on a similar idea of 'strengthening an inequality'). Let $n = m^2 + r$. Squaring (15.1) and subtracting m^2, we get

$$0.5m + 0.25^2 \leq r < 0.52m + 0.26^2.$$

Making m the subject, this is equivalent to

$$\frac{25}{13}r - 0.13 < m \leq 2r - 0.125.$$

Since m and r are both integers, the upper bound can be strengthened to $m \leq 2r - 1$. If $m = 2r - 1$, then we have

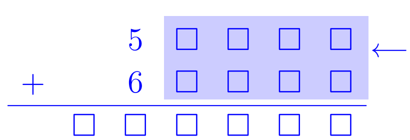
$$\frac{25}{13}r - 0.13 < 2r - 1,$$

which gives $r > 13 \times 0.87 = 11.31$. In this case the smallest possible value of $n = m^2 + r$ is $23^2 + 12 = 541$, which satisfies the condition of the question. If $m \neq 2r - 1$, then $m \leq 2r - 2$ and we have

$$\frac{25}{13}r - 0.13 < 2r - 2,$$

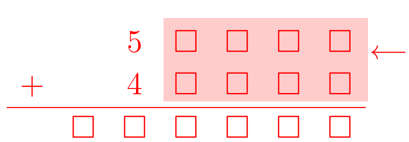
which gives $r > 13 \times 1.87 = 24.31$ and $m > \frac{25}{13}r - 0.13 > 46.62$. In this case we must have $n > 541$. It follows that the smallest possible value of n is 541.

16. Each digit of m must be 2, 3, 5 or 7, while each digit of n may be 4, 6, 8 or 9. If sum of the first digit of m and the first digit of n is at least 10, then the sum will be a six-digit number regardless of the remaining digits. As there are 4 choices for each of the remaining 8 digits, and that there are 11 such combinations for the first digits of m and n (namely, $(2,8)$, $(2,9)$, $(3,8)$, $(3,9)$, $(5,6)$, $(5,8)$, $(5,9)$, $(7,4)$, $(7,6)$, $(7,8)$ and $(7,9)$), there are 11×4^8 choices for (m,n) in this case. An example is illustrated below.



This part has no restriction if the leading digits of m and n has sum 10 or above

There is however another case which we must take into account. That is when the sum of the first digit of m and the first digit of n is equal to 9. There are two such possibilities, namely, $(3,6)$ and $(5,4)$. In this case (m,n) can still satisfy the requirement provided that $m' + n'$ is a five-digit number, where m' is the integer obtained from m by removing its first digit, and similarly for n'. An example is illustrated below.



This part must add up to a five-digit number if the leading digits of m and n has sum 9

We see that counting the number of possibilities for (m', n') is essentially the same as counting that for (m, n), except that we are having one digit fewer. This prompts us to define $f(k)$ to be the number of pairs (m, n) of k-digit numbers such that $m+n$ has $k+1$ digits, each digit of m is prime and each digit of n is composite. Then the answer to the question is simply $f(5)$.

The argument in the first and second paragraph shows that $f(5) = 11 \times 4^8 + 2f(4)$. Using essentially the same argument, we have

$$f(k) = 11 \times 4^{2(k-1)} + 2f(k-1) \quad \text{for all integers } k > 1.$$

Clearly we have $f(1) = 11$ from the list given in the first paragraph. Using this and the above recurrence relation, we obtain $f(2) = 198$, $f(3) = 3212$, $f(4) = 51480$ and $f(5) = 823856$.

17. Since the decomposition of an integer is unique (e.g. 10 can only be replaced by 3, 3, 4 and nothing else) and the game ends precisely when no integer on the board is greater than 1, there is essentially no 'strategy' in this game. All it matters is the number of decompositions it takes to kill all numbers that are greater than 1 — if this number is odd, then Amy wins; if this number is even, then Betty wins. Let $f(n)$ denote this number. For example, we have $f(3) = 1$ and $f(4) = 2$ — if $n = 4$ then it takes two steps to end the game:

$$4 \longrightarrow 1, 1, 2 \longrightarrow 1, 1, 1, 1, 0$$

We also note that, knowing $f(3)$ and $f(4)$, we can compute some other function values such as $f(10)$, since the initial decomposition of 10 must be into 3, 3, 4. We have

$$f(10) = 1 + f(3) + f(3) + f(4) = 5.$$

More generally, for positive integer k, we have the recurrence relations

$$\begin{aligned} f(3k) &= 1 + f(k) + f(k) + f(k) \\ f(3k+1) &= 1 + f(k) + f(k) + f(k+1) \\ f(3k+2) &= 1 + f(k) + f(k+1) + f(k+1) \end{aligned}$$

This enables us to easily list the values of $f(n)$ for small n:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	0	1	1	2	3	4	4	4	4	5

n	11	12	13	14	15	16	17	18	19	20
$f(n)$	6	7	8	9	10	11	12	13	13	13

We also notice that the value of $f(n)$ will stay at 13 for quite a while; more precisely until $f(27)$. After that we will have $f(28) = 14$, $f(29) = 15$ and so on. We take a more careful look at those n for which the value of $f(n)$ 'stays', i.e. $f(n) = f(n+1)$:

$$2,\ 6,\ 7,\ 8,\ 18,\ 19,\ 20,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ \ldots$$

These are precisely the numbers whose first digit in base 3 is equal to 2. More precisely, we observe that (see Remark for a proof)

- for any $n \in [2 \cdot 3^{m-1}, 3^m]$, we have $f(n) = \frac{1}{2}(3^m - 1)$; and
- for any $n \in (3^m, 2 \cdot 3^m)$, we have $f(n) = n - \frac{1}{2}(3^m + 1)$.

Recall that we want to count the number of positive integers n not exceeding 2025 for which $f(n)$ is odd. We count according to the classification above. Note that $f(1)$ is even. For any $n > 1$, either $n \in [2 \cdot 3^{m-1}, 3^m]$ or $n \in (3^m, 2 \cdot 3^m)$ for some positive integer m.

- Suppose $n \in [2 \cdot 3^{m-1}, 3^m]$. Then $f(n)$ is odd if and only if m is odd. There are $3^{m-1}+1$ such n for $m \in \{1, 3, 5\}$, whereas for $m = 7$ we only have $2025 - 2 \cdot 3^6 + 1 = 568$ such n. Hence the number of possibilities in this case is
$$(3^0 + 1) + (3^2 + 1) + (3^4 + 1) + 568 = 662.$$
- Suppose $n \in (3^m, 2 \cdot 3^m)$. For $m \in \{1, 2, 3, 4, 5, 6\}$ we can find $3^m - 1$ numbers in the interval. Since $f(n) = f(n-1) + 1$ for all such n's, exactly half of such n will have $f(n)$ being odd. Hence the number of possibilities in this case is
$$\frac{(3^1 - 1) + (3^2 - 1) + \cdots + (3^6 - 1)}{2} = 543.$$

The answer is thus $662 + 543 = 1205$.

Remark. We can prove the statements

- $P(m)$: for any $n \in [2 \cdot 3^{m-1}, 3^m]$, we have $f(n) = \frac{1}{2}(3^m - 1)$; and
- $Q(m)$: for any $n \in (3^m, 2 \cdot 3^m)$, we have $f(n) = n - \frac{1}{2}(3^m + 1)$.

by induction on m. We can check the function values in the table in the solution to verify that $P(1)$ and $Q(1)$ hold. Now suppose $m > 1$ and suppose $P(m-1)$ and $Q(m-1)$ hold. Then we prove $P(m)$ and $Q(m)$ as follows.

- Suppose $n \in [2 \cdot 3^{m-1}, 3^m]$. If $n = 3k$, then $k \in [2 \cdot 3^{m-2}, 3^{m-1}]$. By the induction hypothesis, $f(k) = \frac{1}{2}(3^{m-1} - 1)$, so $f(n) = 1 + 3f(k) = \frac{1}{2}(3^m - 1)$ as desired. The proof is exactly the same for $n = 3k + 1$ and $n = 3k + 2$.
- Suppose $n \in (3^m, 2 \cdot 3^m)$. If $n = 3k$, then $k \in (3^{m-1}, 2 \cdot 3^{m-1})$, so the induction hypothesis says $f(k) = k - \frac{1}{2}(3^{m-1} + 1)$, and we have $f(n) = 1 + 3f(k) = n - \frac{1}{2}(3^m + 1)$ as desired. There is a small difference in the proof when $n = 3k + 1$ and $n = 3k + 2$, because it may happen that $k = 3^{m-1}$ (if $n = 3^m + 1$ or $n = 3^m + 2$). Yet in that case we can use the induction hypothesis on $P(m-1)$ (rather than on $Q(m-1)$) to get $f(k) = \frac{1}{2}(3^{m-1} - 1)$, and it is indeed the same as $k - \frac{1}{2}(3^{m-1} + 1)$ when $k = 3^{m-1}$. The rest of the proof is thus the same as the $n = 3k$ case.

18. Starting with the given equality
$$1^9 + 2^9 + \cdots + n^9 = a_0 + a_1 n + a_2 n^2 + \cdots + a_{10} n^{10}, \tag{18.1}$$

we can replace n by $n+1$ to get

$$1^9 + 2^9 + \cdots + (n+1)^9 = a_0 + a_1(n+1) + a_2(n+1)^2 + \cdots + a_{10}(n+1)^{10}. \qquad (18.2)$$

Taking the difference (18.2) − (18.1), we get

$$(n+1)^9 = a_1[(n+1) - n] + a_2[(n+1)^2 - n^2] + \cdots + a_{10}[(n+1)^{10} - n^{10}]. \qquad (18.3)$$

Comparing the coefficient of n^9 on both sides of (18.3), we have $1 = 10a_{10}$. We can also compare the coefficient of n^4 to get

$$\binom{9}{4} = \binom{5}{4}a_5 + \binom{6}{4}a_6 + \binom{7}{4}a_7 + \binom{8}{4}a_8 + \binom{9}{4}a_9 + \binom{10}{4}a_{10} \qquad (18.4)$$

and the coefficient of n^5 to get

$$\binom{9}{5} = \binom{6}{5}a_6 + \binom{7}{5}a_7 + \binom{8}{5}a_8 + \binom{9}{5}a_9 + \binom{10}{5}a_{10}. \qquad (18.5)$$

Noting that $\binom{9}{4} = \binom{9}{5}$, taking the difference (18.4) − (18.5) gives

$$0 = 5a_5 + 9a_6 + 14a_7 + 14a_8 - 42a_{10}.$$

Recalling that $1 = 10a_{10}$, we get $5a_5 + 9a_6 + 14a_7 + 14a_8 = 42a_{10} = \frac{21}{5}$.

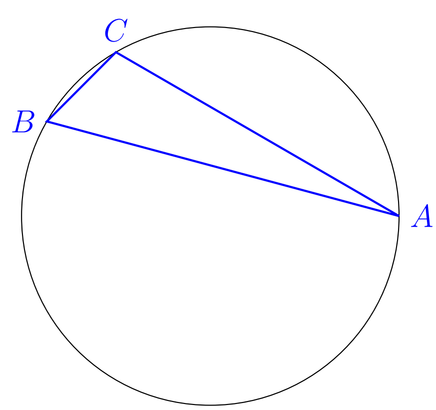
19. Suppose $a = \sqrt{2}$, $b = 3+\sqrt{3}$ and $c = 2\sqrt{2}+\sqrt{6}$ (where standard conventions in trigonometry are adopted). Then $a^2 = 2$, $b^2 = 12 + 6\sqrt{3}$ and $c^2 = 14 + 8\sqrt{3}$. Hence

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{4 + 2\sqrt{3}}{2\sqrt{2}(2\sqrt{2} + \sqrt{6})} = \frac{4 + 2\sqrt{3}}{8 + 4\sqrt{3}} = \frac{1}{2}$$

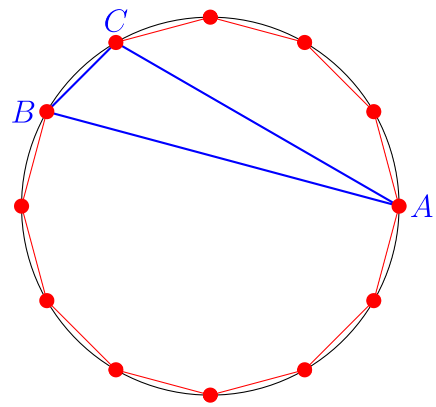
and so $B = 60°$. Also,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{-2\sqrt{3}}{2\sqrt{2}(3 + \sqrt{3})} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

and so $C = 105°$ (this can be verified using $\cos 105° = \cos 60° \cos 45° - \sin 60° \sin 45°$). Thus $A = 180° - B - C = 15°$. Consider the circumcircle of $\triangle ABC$, which is also the circumcircle of $\mathcal{P}$.



Since the chord BC subtends an angle of $15°$ at the circumference, it subtends an angle of $30°$ at the centre. This means the number of sides of $\mathcal{P}$ must be divisible by 12. Indeed, it is possible for A, B, C to be vertices of a regular 12-sided polygon. If we construct a regular 12-sided polygon on the circle with B, C as adjacent vertices, then A will be one of the vertices, with 3 vertices between C and A:



To check that this is indeed the case, simply note that if A' were the fourth vertex of the polygon clockwise from C, then CA' should subtend an angle of $360° \times \frac{4}{12} = 120°$ at the centre. Since $\angle CBA$ is equal to $60°$, CA also subtends an angle of $120°$ at the centre, from which we know that $A' = A$. Hence $\mathcal{P}$ must be 12-sided, with circumradius given by

$$R = \frac{b}{2\sin B} = \frac{3+\sqrt{3}}{2\sin 60°} = \sqrt{3}+1.$$

So we can compute the area of $\mathcal{P}$ by dividing it into 12 identical isosceles triangles at the circumcentre (with side lengths R, R, a), which has total area

$$12 \cdot \frac{1}{2}R^2 \sin 30° = 12 + 6\sqrt{3}.$$

20. Note that $2187 = 3^7$. For positive integer m, let $v(m)$ be the highest power of 3 dividing m (for instance $v(2025) = 4$ since $2025 = 3^4 \times 5^2$). Let also $v(f(n)) = w(n)$. Hence the question is equivalent to finding the smallest positive integer n such that

$$w(n) + w(n+1) + w(n+2) \geq 7.$$

Note also that $f(x) = (x^5-2)(x^5-3)(x^5-4)$, which is the product of three consecutive integers (hence divisible by 3) if x is an integer, meaning $w(n) \geq 1$ for all positive integers n. Now we compute the remainders when x^5 is divided by 9 for $x \in \{0, 1, ..., 8\}$ (essentially we only have to compute the remainders when 1^5, 2^5 and 4^5 are divided by 9, since x^5 is divisible by 9 if x is divisible by 3, and we have the 'skew-symmetry' due to $5^5 \equiv (-4)^5 = -4^5 \pmod 9$ and so on):

x	0	1	2	3	4	5	6	7	8
$x^5 \pmod 9$	0	1	5	0	7	2	0	4	8

From this we conclude that

- $w(n) \geq 2$ if $n \equiv 5 \pmod 9$ or $n \equiv 7 \pmod 9$; and
- $w(n) = 1$ otherwise.

We must further investigate when strict inequality holds in the first case, i.e. when will $n^5 \equiv 2$ or $4 \pmod{27}$. Suppose $n = 9k+5$ for some non-negative integer k. Then

$$n^5 - 2 = (9k+5)^5 - 2 \equiv 5(9k)(5^4) + 5^5 - 2 \equiv 18k + 18 \pmod{27},$$

which means $n^5 - 2$ is divisible by 27 if and only if $k \equiv 2 \pmod 3$, or equivalently $n \equiv 23 \pmod{27}$. Similarly, if $n = 9k+7$, then

$$n^5 - 4 = (9k+7)^5 - 4 \equiv 5(9k)(7^4) + 7^5 - 4 \equiv 18k + 9 \pmod{27},$$

which means $n^5 - 4$ is divisible by 27 if and only if $k \equiv 1 \pmod 3$, or equivalently $n \equiv 16 \pmod{27}$. Hence we have the following table (recall that $w(n) = 1$ for $n \not\equiv 5, 7 \pmod 9$):

n	5	7	14	16	23	25
$v(n^5-2)$	2	0	2	0	≥ 3	0
$v(n^5-3)$	0	0	0	0	0	0
$v(n^5-4)$	0	2	0	≥ 3	0	2
$w(n)$	2	2	2	≥ 3	≥ 3	2

Now we investigate further the values of $w(16)$ and $w(23)$.

- We have $w(16) = 3$ since

$$16^5 - 4 = 256 \cdot 256 \cdot 16 - 4 \equiv 13 \cdot 13 \cdot 16 - 4 \equiv 27 \pmod{81}.$$

- On the other hand, we have

$$23^5 - 2 \equiv 529 \cdot 529 \cdot 23 \equiv 43 \cdot 43 \cdot 23 - 2 \equiv 0 \pmod{243},$$

  which shows that $w(23) \geq 5$.

Hence we have $w(21) + w(22) + w(23) \geq 7$, whereas for $n \leq 20$ we have $w(n) + w(n+1) + w(n+2) \leq 6$ (equality is attained when $n = 14$). It follows that the answer is 21.