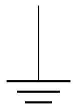
T3. Single-electron transistor

In all parts of this problem, the elementary charge $e = 1.6 \times 10^{-19}$ C may appear in the answer.

The single-electron transistor is an electronic component whose operation is based on the *Coulomb blockade* phenomenon observed at low temperatures (on the order of several kelvin). In this problem, we will investigate this effect and explain the operating principle of a single-electron transistor at zero temperature.

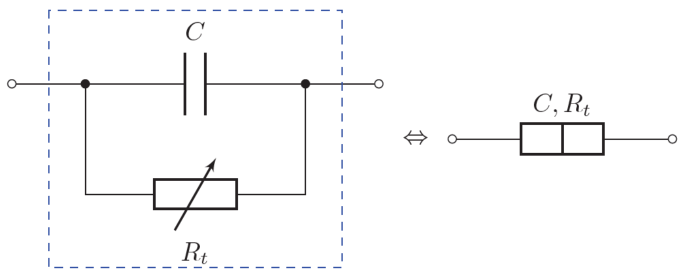
Note: In electrical circuits, the "ground" refers to the point where the potential is defined as zero. The "ground" symbol is shown in the figure.



Part A. Tunneling effect

The Coulomb blockade phenomenon is fundamentally based on *tunneling*, or *the tunnel effect*. Its essence lies in the fact that, under certain conditions, a system can overcome potential energy barriers. Tunneling is possible if the final energy does not exceed the initial energy. The excess energy releases as heat.

Tunneling constitutes the sequential transfer of individual elementary charges between the plates of a tunnel junction. In the "classical" model of this system, electron transport occurs through a resistor R_t connected in parallel with a capacitor, where energy dissipation takes place. The resistance remains finite when tunneling is energetically permissible, and becomes infinite otherwise.



Tunnel junction circuit (left) and its symbol in electrical circuit diagrams (right)

Let us consider a tunnel junction disconnected from the power supply.

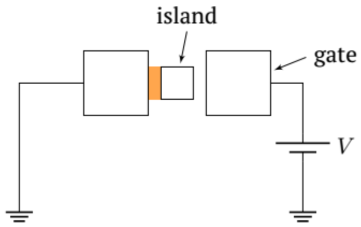
A1	Write the expression for the energy of a capacitor with capacitance C (in the equivalent circuit of a tunnel junction) when its plates hold a charge equal to n electron charges. Now let the value n increase by one. What is the change in the capacitor's energy ΔW?

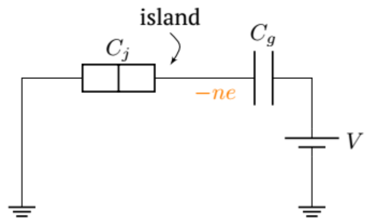
Hereafter, we consider tunnel junctions connected in an electrical circuit with other elements. In circuit diagrams, a tunnel junction is represented as two touching rectangles.

Single-electron box

The single-electron box is a system comprising three aligned electrodes. The central electrode is called the island, while the lateral electrodes are called the banks. The island contains n electrons — a quantity that may vary through tunneling events. Note that n can be **either positive or negative** (indicating an excess or deficiency of electrons on the island).

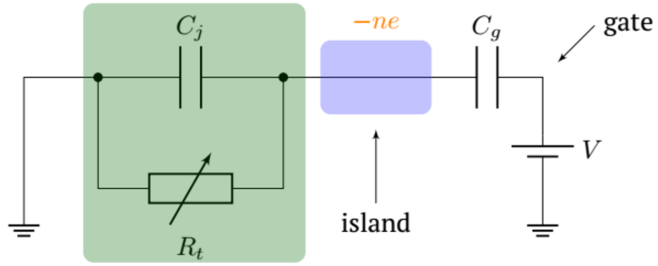
Between the island and the left bank lies a tunnel junction with capacitance C_j, while the island and right bank form a conventional capacitor with capacitance C_g. In this configuration, the right bank is designated as the *gate*. A constant potential difference V is maintained between the banks. The corresponding electrical circuit diagram, specifying all component parameters, is shown in the figure.





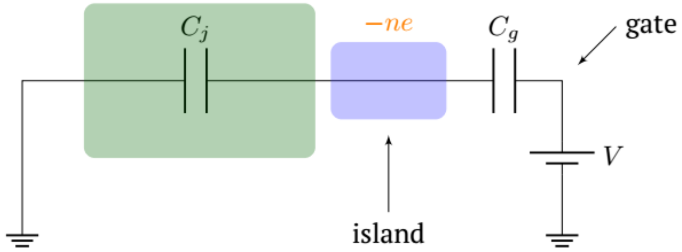
Single-electron box

The "classical" model of tunneling when it is energetically permissible.



Tunneling is permissible

The "classical" model of tunneling when it is energetically forbidden.



Forbidden tunneling

A2	Derive expressions for the charges on the capacitor and tunnel junction q_g and q_j. The capacitances of the capacitor and tunnel junction are C_g and C_j, respectively, the source voltage is V, and the number of electrons on the island is n.

A3	Derive the expression for the change in energy of the system consisting of the capacitor and tunnel junction when the number n of electrons on the island increases by one. Express the answer in terms of n, C_g, $C = C_j + C_g$.

A4	Derive expressions for the work done by the voltage source A and the heat dissipation Q in the circuit during the tunneling of one electron onto the island (corresponding to an increase by one in the number n of island electrons). Express the answers in terms of n, C_g, C, V.

Hereafter, we will consider tunneling to be allowed when the amount of heat Q dissipated in the circuit is greater than zero.

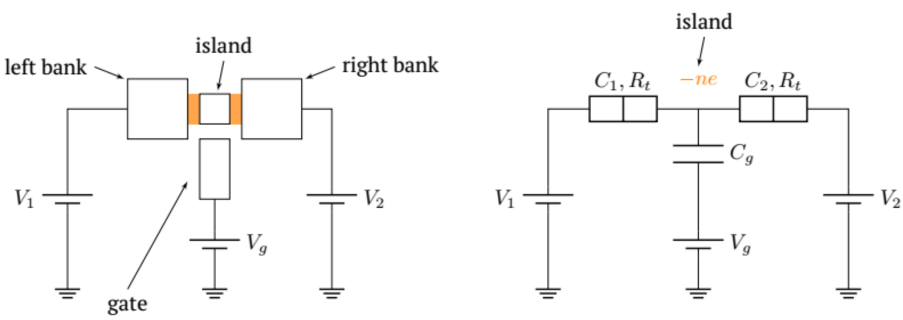
A5	Assume that the heat Q is dissipated during current flow through the tunnel resistance R_t. Derive the expression for the current I_t through R_t during tunneling of one electron onto the island. Also, specify the constraint on n for which this expression is valid (i.e., when tunneling is energetically permissible). Express the answers in terms of n, C_g, C, V, R_t.

A6	Assuming tunneling is energetically permissible, express the current I flowing through the source in terms of the tunnel current I_t. The positive current direction is toward the negative terminal of the source. The expression may also include the quantities C_g, C, V. Consider processes corresponding to both an increase and a decrease in the number of electrons on the island.

A7	Plot the $I(V)$ characteristics for $n = 0$ and $n = 1$ on the single graph. Display the dependencies over a sufficiently wide voltage range to capture all distinctive features.

Part B. Single-electron transistor

The single-electron transistor consists of an island connected to two banks via tunnel junctions and a gate that forms a capacitor with the island. The gate is purely capacitively coupled (its resistance is infinite). A voltage $V_1 = V/2$ is applied to the left bank, and a voltage $V_2 = -V/2$ is applied to the gate. The transistor's circuit diagram and equivalent circuit, with specified parameters of the electronic components, are shown in the figure.



Single-electron transistor

B1	Let the island contain n electrons. Express the island's potential φ in terms of n, V, V_g, C_1, C_2, C_g.

B2	Derive the expression for the total work done by the voltage sources during the tunneling of one electron onto the island from the left (first) bank. Express your answer in terms of V, V_g, C_1, C_2, C_g.

B3	Let the island contain n electrons. Express the total heat dissipated in the circuit during the tunneling of one electron onto the island from the left (first) bank. Express the answer in terms of n, V, V_g, C_1, C_2, C_g.

B4	Derive the condition (inequality) for V corresponding to heat dissipation positivity during the tunneling from the left bank **onto** the island. Express the answer in terms of n, V, V_g, C_g, C_2.

B5	Derive conditions for V analogous to those in question B4 in cases of tunneling **onto** the island from the right bank and tunneling **from** the island onto the both banks. Sketch a diagram of regions corresponding to the existence of n electrons on the island after sufficient time has elapsed. The diagram should be plotted in axes V vs. V_g. For each region specify the values of n on the diagram.

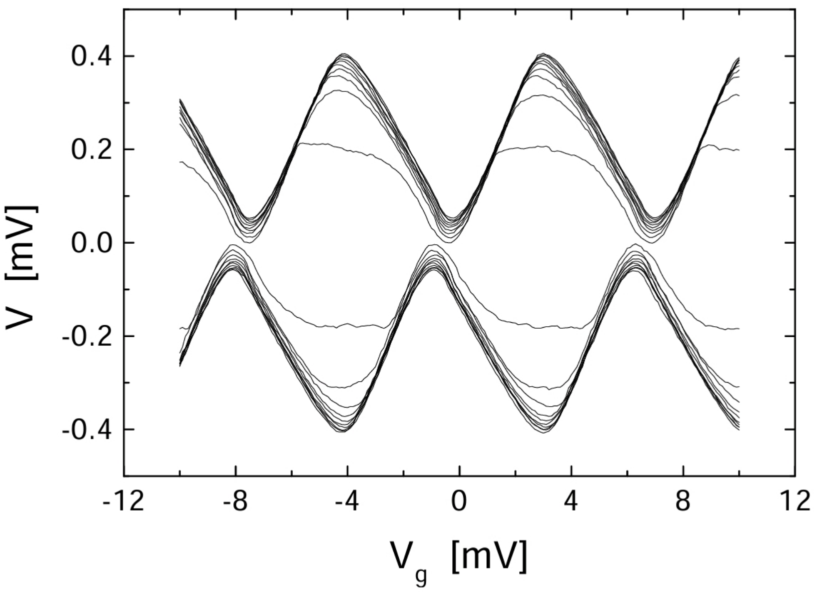
Consider regions on the (V, V_g) diagram where, after sufficient time, only two transitions are energetically permissible: $n \to n+1$ through one junction and $n+1 \to n$ through the other junction. In this case, a steady current flows through the island from one bank to the other. We now determine its magnitude.

Since the island's charge remains unchanged after a pair of tunneling events, the charges on the capacitors also remain constant. Consequently, the resulting current is determined by the sum of the tunneling times through the first and second junctions.

B6	Assume the tunnel junctions are identical: each has a capacitance C_1 and tunnel resistance R_t. Consider voltage pairs (V_g, V) for which, after sufficient time, only states with $n=0$ and $n=1$ electrons on the island are possible. Determine the magnitude of the tunnel current $I(V_g, V)$ in the circuit. The positive current direction is from the left bank to the right bank.

Part C. Tunneling at finite temperature

C1	The graph shows the experimental $V(V_g)$ dependence at low temperature $T = 10\ \mathrm{mK}$, measured for fixed current values $I \approx 1\,\mathrm{pA}$. Based on the results obtained in Part B, determine the capacitances $C_1 = C_2$ and C_g.



V. A. Krupenin et al., 2001

Consider a fixed positive temperature T. Let $\Gamma^{(i)}_{n\to n\pm1}$ denote the probability **per unit time** (rate) of tunneling to/from the island through the i-th junction, and let $p(n)$ be the probability of finding n electrons on the island.

Note: If a process has a rate Γ, then during time $\mathrm{d}t$ it occurs with probability $\mathrm{d}p = \Gamma \mathrm{d}t$.

C2	Let $$\Gamma_{n\to n\pm1} = \Gamma^{(1)}_{n\to n\pm1} + \Gamma^{(2)}_{n\to n\pm1}.$$ An electron can tunnel to/from either bank. Express the time derivative $\mathrm{d}p(n)/\mathrm{d}t$ of the probability of having n electrons on the island in terms of the probabilities $p(n)$, $p(n\pm1)$ and the rates $\Gamma_{n\to n\pm1}$, $\Gamma_{n\pm1\to n}$.

We will examine the steady-state current flow through the transistor (meaning all probabilities $p(n)$ are constant). We also assume there exists a critical number n_0 of electrons on the island, beyond which tunneling ceases: $p(n) = 0$ and $\Gamma_{n-1\to n} = 0$ for $n > n_0$.

C3	From the stated assumptions and established boundary condition, derive a relation connecting $p(n)$ and $p(n+1)$ in the form $p(n+1) = A\cdot p(n)$, where A is an expression containing only the rates $\Gamma_{m\to m\pm1}$ for various m.

We will consider low temperatures ($kT \ll e^2/C$) and small voltages: $|V| \ll e/C$, $|V_g - e/2C_g| \ll e/C_g$.

According to quantum mechanics, the general form for the tunneling rate per unit time to/from the island through the i-th junction is:

$$\Gamma^{(i)}_{n\to n\pm1} = \frac{1}{e^2 R_t}\frac{Q^{(i)}_{\pm}(n)}{1-\exp\left[-\frac{Q^{(i)}_{\pm}(n)}{kT}\right]},$$

where $Q^{(i)}_{\pm}(n)$ is the heat dissipated during the tunneling process.

C4	Using the expression derived in question B3 and analogous relations, show that $\Gamma_{n\to n+1}$ for $n > 0$ and $\Gamma_{n\to n-1}$ for $n < 1$ **decay** exponentially with changing n. Assume that $T \to 0$.

Hereafter, all $p(n)$ except $p(0)$ and $p(1)$ will be treated as zero.

C5	Express the probabilities $p(0)$ and $p(1)$ in terms of the rates $\Gamma_{0\to1}$ and $\Gamma_{1\to0}$. Take into account that $p(0) + p(1) = 1$.

C6	Write the expression for the current through the transistor. Express the answer in terms of $\Gamma^{(i)}_{0\to1}$, $\Gamma^{(i)}_{1\to0}$.

Substituting the expressions for heat into the formula for the tunneling rates $\Gamma^{(i)}_{n\to n\pm1}$ yields, after extensive calculations, the current's dependence on voltage and temperature. This pronounced dependence enables the use of a single-electron transistor as a low-temperature measurement device, as explored in the science festival problem.

Solution

A1. According to the capacitor energy formula

$$W = \frac{(ne)^2}{2C} \quad \Rightarrow \quad \Delta W = \frac{e^2}{2C} \cdot \left((n+1)^2 - n^2\right).$$

$$\boxed{\Delta W = \frac{e^2}{C}(n + 1/2)}$$

A2. The sum of charges on the capacitor plates is determined by the number n of electrons on the island. The sum of voltages across the capacitors is determined by the voltage source.

By writing the charge conservation law and the sum of voltage drops, we obtain:

$$\begin{cases} q_j - q_g = -ne, \\ V = \dfrac{q_j}{C_j} + \dfrac{q_g}{C_g}. \end{cases}$$

If $C = C_j + C_g$, then after transformations the required expressions are obtained.

$$\boxed{\begin{cases} q_g = \dfrac{C_g}{C} \cdot (VC_j + ne) \\ q_j = \dfrac{C_j}{C} \cdot (VC_g - ne) \end{cases}}$$

A3. The total energy of the system:

$$W_n = \frac{q_g^2}{2C_g} + \frac{q_j^2}{2C_j} = \frac{VC_jC_g + n^2e^2}{2C}.$$

When the number of electrons on the island changes $n \to n+1$:

$$W_{n+1} = \frac{q_g^2}{2C_g} + \frac{q_j^2}{2C_j} = \frac{VC_jC_g + (n+1)^2e^2}{2C}.$$

The energy increment is $\Delta W = W_{n+1} - W_n = \frac{e^2}{C}(n + 1/2)$.

$$\boxed{\Delta W = \frac{e^2}{C}(n + 1/2)}$$

A4. The work done by the source equals the product of its voltage and the amount of charge that has flowed through it. The amount of charge that has flowed through the source equals the change in the gate charge Δq_g, therefore the source work is:

$$A = V\Delta q_g.$$

According to task A2 $\Delta q_g = eC_g/C$, which gives the answer for the source work.

$$\boxed{A = \frac{C_g}{C} eV}$$

The heat losses are determined by the energy conservation law:

$$Q = A - \Delta W,$$

from which, according to task A2, we obtain the answer.

$$\boxed{Q = \frac{e}{C}\left[VC_g - e(n + 1/2)\right]}$$

A5. The amount of heat released by the resistor equals the product of the resistor voltage and the charge that has passed through it:

$$Q = I_t R_t e$$

Using the expression for Q from point A4, we obtain

$$I_t = \frac{VC_g - e(n + 1/2)}{CR_t}$$

Tunneling is possible when $Q = I_t R_t e > 0$, i.e. when $V > \frac{e}{C_g}(n + 1/2)$, which gives the condition for n.

$$\boxed{I_t = \frac{VC_g - e(n + 1/2)}{CR_t}}$$

$$\boxed{n < \frac{C_g V}{e} - \frac{1}{2}}$$

A6. Differentiating the expression for the gate charge obtained in point A2 with respect to time gives:

$$\dot{q}_g = \frac{C_g}{C}\frac{\Delta n}{\Delta t}e.$$

Taking into account $I = \dot{q}_g$, $I_t = \frac{\Delta n}{\Delta t}e$ we obtain the answer.

$$\boxed{I = \frac{C_g}{C} I_t}$$

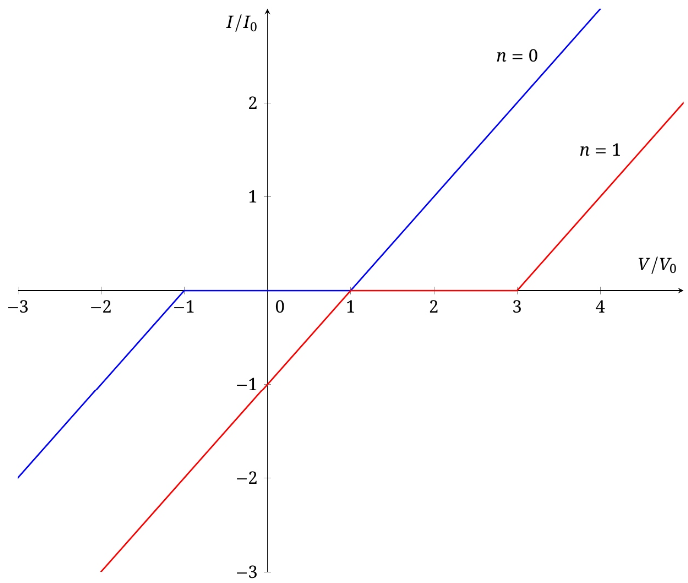
A7. From the two previous tasks:

$$I = \frac{C_g^2}{C^2 R_t}\left[V - \frac{e}{C_g}(n + 1/2)\right] \quad \text{if} \quad V > \frac{e}{C_g}(n + 1/2).$$

Performing similar calculations for the case of decreasing the number n of electrons on the island leads to the possibility of negative current flow:

$$I = \frac{C_g^2}{C^2 R_t}\left[V - \frac{e}{C_g}(n - 1/2)\right] \quad \text{if} \quad V < \frac{e}{C_g}(n - 1/2).$$

Outside the specified intervals, tunneling is impossible, meaning $I(V) = 0$. The required graphs are shown in the figure below.



The current-voltage characteristic of the single-electron box: $V_0 = \frac{e}{2C_g}$, $I_0 = \frac{eC_g}{2C^2R_t}$

B1. The sum of charges on the capacitors equals the charge on the island:

$$C_1(\varphi - V_1) + C_2(\varphi - V_2) + C_g(\varphi - V_g) = -ne.$$

Substituting the expressions for V_1, V_2, we obtain the answer.

$$\boxed{\varphi = \frac{(C_1 - C_2)V/2 + C_gV_g - ne}{C_1 + C_2 + C_g}}$$

B2. First, we express the changes in charges on the capacitor plates when the number of electrons on the island changes:

$$q_1 = C_1(\varphi - V_1) = \frac{C_1}{C}\left[(C_1 - C)V_1 + C_2V_2 + C_gV_g - ne\right]$$

$$\Delta q_1 = -\frac{eC_1}{C}\Delta n.$$

Here and hereafter in part B $C = C_1 + C_2 + C_g$. Similarly

$$\Delta q_2 = -\frac{eC_2}{C}\Delta n, \qquad \Delta q_g = -\frac{eC_g}{C}\Delta n.$$

The work done by the sources is determined by the charge flowing through them. For the branch where tunneling occurs, one must account for the transfer of the tunneling charge itself $q_t = -\Delta ne$. Thus, the total work done by the sources in our case takes the form

$$A = -V_1\Delta q_1 - V_2\Delta q_2 - V_g\Delta q_g + V_1q_t.$$

Substituting the expressions for the charges, we obtain:

$$A = \frac{e\Delta n}{C}\left[V_1(C_1 - C) + V_2C_2 + V_gC_g\right].$$

Taking into account $\Delta n = 1$ and the expressions for V_1, V_2, we write the result.

$$\boxed{A = \frac{e}{C}\left[V_gC_g - V/2\,(C_g + 2C_2)\right]}$$

B3. The total energy of the system

$$W_n = \frac{C_1}{2}(\varphi - V_1)^2 + \frac{C_2}{2}(\varphi - V_2)^2 + \frac{C_g}{2}(\varphi - V_g)^2$$

after substituting the expression for the island potential φ and simplification takes the form

$$W_n = \frac{(ne)^2}{2C} + W_0,$$

where W_0 is a term independent of the number n of electrons on the island.

Then the energy increment resulting from tunneling of one electron

$$\Delta W_n = \frac{e^2}{2C}((\Delta n)^2 + 2n\Delta n) = \frac{e^2}{C}\left(\frac{1}{2} \pm n\right),$$

where the $+$ sign corresponds to an increase in the number of electrons on the island.

The amount of heat released is determined from the energy conservation law:

$$Q = A - \Delta W_n = \pm\frac{e}{C}\left[V_1(C_1 - C) + V_2C_2 + V_gC_g - e(n \pm 1/2)\right].$$

Considering the case of increasing the number of electrons, we obtain the answer.

$$\boxed{Q = \frac{e}{C}\left[V_gC_g - V/2\,(C_g + 2C_2) - e(n + 1/2)\right]}$$

B4. Taking into account the expressions for V_1, V_2, we obtain the following for the heat released in the circuit:

$$Q = \pm\frac{e}{C}\left[V_gC_g - V/2\,(C_g + 2C_2) - e(n \pm 1/2)\right],$$

where the $+$ sign still corresponds to an increase in the number of electrons on the island.

From the inequality $Q > 0$ for the upper signs we obtain:

$$\boxed{V < \frac{V_gC_g - e(n + 1/2)}{C_2 + C_g/2}}$$

B5. For electron tunneling from the island through the first junction we obtain:

$$V > \frac{V_gC_g - e(n - 1/2)}{C_2 + C_g/2}.$$

In the case of tunneling through the second junction, we replace $V \to -V, C_1 \to C_2$ in the expression for the released heat and obtain the condition:

$$Q = \pm\frac{e}{C}\left[V_gC_g + V/2\,(C_g + 2C_2) - e(n \pm 1/2)\right] > 0.$$

From this, for tunneling onto the island we obtain the inequality

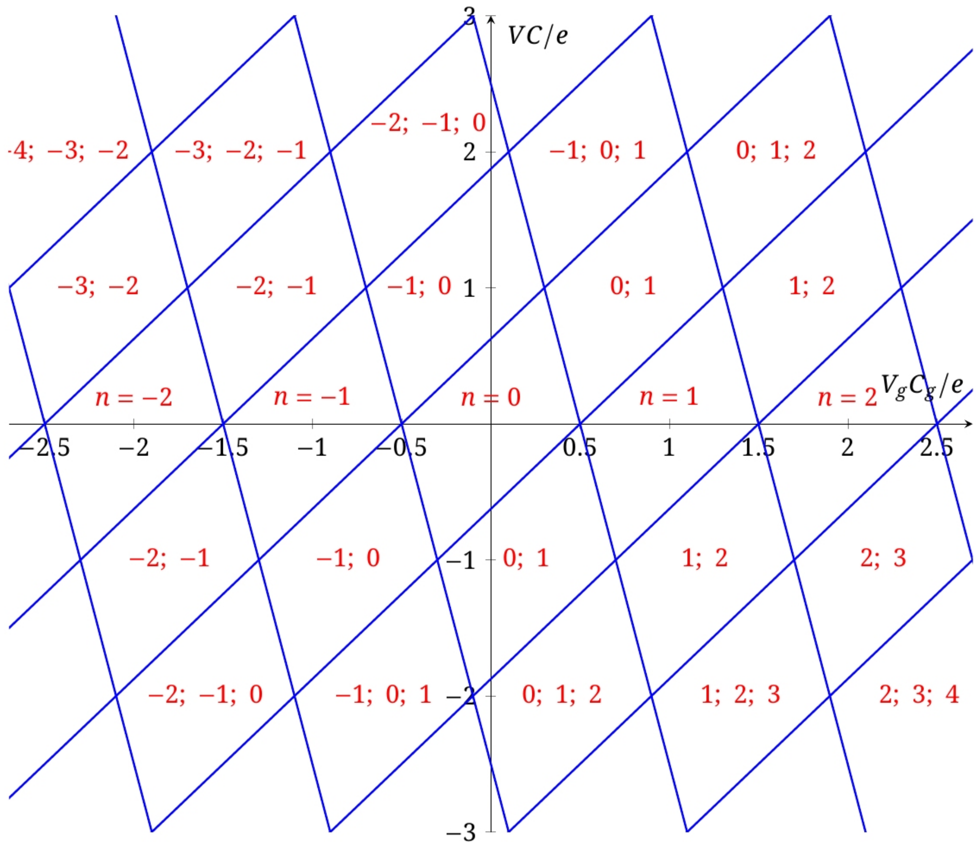
$$V > -\frac{V_gC_g - e(n + 1/2)}{C_1 + C_g/2}.$$

For tunneling from the island:

$$V < -\frac{V_gC_g - e(n - 1/2)}{C_1 + C_g/2}.$$

Thus, in the coordinates V, V_g the regions of possible tunneling are separated by two families of straight lines: with slope coefficients $k_1 = \frac{C_g}{C_2+C_g/2}$ and $k_2 = -\frac{C_g}{C_1+C_g/2}$.

Each region forms a parallelogram. The parallelograms containing the abscissa axis correspond to a single equilibrium number n. These regions represent Coulomb blockade — the impossibility of current flow through the transistor. In other regions, the resistance becomes finite, and current can flow.



The V, V_g voltage diagram of the transistor

B6. Positive current corresponds to tunneling from the second bank to the island, and then from the island to the first bank. Let us consider both these processes and use the equality $Q_\alpha = I_\alpha R_t e;\ \alpha = 1, 2$ — the tunneling process number:

$$\begin{cases} I_1 R_t e = \dfrac{e}{C}\left[V(C_1 + C_g/2) + V_g C_g - e(n + 1/2)\right], \\ I_2 R_t e = -\dfrac{e}{C}\left[-V(C_2 + C_g/2) + V_g C_g - e(n + 1 - 1/2)\right]. \end{cases}$$

For the total current flowing through the transistor, determined by the total tunneling time across both junctions, we obtain $I = (I_1^{-1} + I_2^{-1})^{-1}$. Introducing the quantity $Q_g = V_g C_g - e\bar{n}$, where $\bar{n} = (n + (n+1))/2$ is the average number of electrons on the island, then after simplification the dependence takes the form:

$$I(V, Q_g) = \frac{C^2 V^2/4 - Q_g^2}{V C^2 R_t}.$$

Particularly for $n = 0, \bar{n} = 1/2$, we obtain the required dependence.

$$\boxed{I(V, V_g) = \frac{1}{4R_t}\left[V - \frac{4}{VC^2}\left(C_g V_g - e/2\right)^2\right]}$$

C1. The graph clearly shows the "Coulomb diamonds" obtained in the task B5. The measurement of period ΔV_g allows determination of the gate capacitance, while the maximum voltage $V_{\max}$ gives the total capacitance. From the graph $\Delta V_g = 7$ mV, $V_{\max} = 0.4$ mV. Hence $C_g = e/\Delta V_g = 23$ aF, $C = e/V_{\max} = 400$ aF, $C_1 = C_2 = (C - C_g)/2 = 190$ aF.

$$\boxed{C_1 = C_2 = 190 \text{ aF}}$$

$$\boxed{C_g = 23 \text{ aF}}$$

C2. The change in probability of having n electrons on the island is related to possible processes $n+1 \to n$ and $n-1 \to n$ that increase this probability, as well as reverse processes that decrease it:

$$\boxed{\mathrm{d}p(n)/\mathrm{d}t = p(n-1) \cdot \Gamma_{n-1\to n} + p(n+1) \cdot \Gamma_{n+1\to n} - p(n) \cdot (\Gamma_{n\to n-1} + \Gamma_{n\to n+1})}$$

C3. From the previous point it is known that for arbitrary n the following holds:

$$p(n-1) \cdot \Gamma_{n-1\to n} - p(n) \cdot \Gamma_{n\to n-1} = p(n) \cdot \Gamma_{n\to n+1} - p(n+1) \cdot \Gamma_{n+1\to n}.$$

Taking into account the definition of n_0:

$$p(n_0 - 1) \cdot \Gamma_{n_0-1\to n_0} = p(n_0) \cdot \Gamma_{n_0\to n_0-1}.$$

From this it follows that for all n less than n_0, the following holds:

$$p(n-1) \cdot \Gamma_{n-1\to n} = p(n) \cdot \Gamma_{n\to n-1}.$$

Indeed, relations of this type imply vanishing of the left-hand side of the original equality for arbitrary n. Consequently, the right-hand side, corresponding to n differing by unity, also vanishes. This leads to the answer.

$$\boxed{A = \frac{\Gamma_{n\to n+1}}{\Gamma_{n+1\to n}}}$$

C4. The expressions for the released heat were obtained in part B and take the form:

$$Q^{(i)}_{n\to n\pm 1} = \pm\frac{e}{C}\left[(-1)^i\, VC/2 + (V_g C_g - e/2) - e(n - 1/2 \pm 1/2)\right].$$

Considering $|V| \ll e/C, |V_g - e/2C_g| \ll e/C_g$, for both tunneling junctions, increasing the number n of electrons when $n > 0$ leads to a negative value of the released heat $|Q_{n\to n\pm 1}| > e^2/C \gg kT$. The same inequality occurs when decreasing n for $n < 1$.

Thus, under the specified conditions, a large positive value appears in the exponent, and the unity in the denominator can be neglected. This leads to the required result.

C5. Using the result from point C3:

$$p(0) \cdot \Gamma_{0\to 1} = p(1) \cdot \Gamma_{1\to 0}$$

Taking into account the normalization condition $p(0) + p(1) = 1$, we express the probabilities $p(0)$ and $p(1)$.

$$\boxed{\begin{cases} p(0) = \dfrac{\Gamma_{1\to 0}}{\Gamma_{0\to 1} + \Gamma_{1\to 0}} \\[2ex] p(1) = \dfrac{\Gamma_{0\to 1}}{\Gamma_{0\to 1} + \Gamma_{1\to 0}} \end{cases}}$$

C6. The current through the transistor is determined by the number of tunnelings through one of the junctions (tunnelings through the other occur such that the number of electrons on the island remains unchanged):

$$I = e\left(p(1)\cdot\Gamma^{(1)}_{1\to 0} - p(0)\cdot\Gamma^{(1)}_{0\to 1}\right) = -e\left(p(1)\cdot\Gamma^{(2)}_{1\to 0} - p(0)\cdot\Gamma^{(2)}_{0\to 1}\right).$$

Note that the transition from one expression to another is easily accomplished using

$$p(0)\cdot\Gamma_{0\to 1} = p(1)\cdot\Gamma_{1\to 0}.$$

Substitution of the expressions for probabilities $p(0), p(1)$ leads to the required result.

$$\boxed{I = e\frac{\Gamma^{(2)}_{0\to 1}\Gamma^{(1)}_{1\to 0} - \Gamma^{(1)}_{0\to 1}\Gamma^{(2)}_{1\to 0}}{\Gamma^{(1)}_{0\to 1} + \Gamma^{(2)}_{0\to 1} + \Gamma^{(1)}_{1\to 0} + \Gamma^{(2)}_{1\to 0}}}$$