Решение

T Условие

S Решение

Русский

A1 0.20 Determine the pressure p_1 at the bottom of the vessel. Express your answer in terms of m, g, and S.

Ответ:

$$p = \frac{mg}{S}$$

A2 0.30 Determine the pressure $p(z)$ on the vessel walls as a function of depth z. Express the answer in terms of ρ, g, and z.

According to Pascal's law, the pressure of a liquid depends only on the depth of immersion, therefore:

Ответ:

$$p(z) = \rho g z$$

B1 0.50 Write down the equilibrium condition for this layer. The equation may include the pressures $P(z)$ and $P(z+\mathrm{d}z)$, $\mathrm{d}z$, the average pressure on the vessel walls $\sigma(z)$, the mass of the layer $\mathrm{d}m$, as well as μ, g, and R.

The equilibrium condition projected onto the vertical axis:

Ответ:

$$(P(z+\mathrm{d}z) - P(z))\pi R^2 - \mathrm{d}mg + \mu\sigma(z)\cdot 2\pi R\mathrm{d}z = 0$$

B2 1.20 Show that the bottom pressure P as a function of the fill height h of the granular column can be written as:

$$P = A + Be^{-h/\lambda}.$$

Determine the parameters A, B, and λ. Express your answers in terms of the bulk density ρ, as well as g, R, K, and μ.

Using the relations $\mathrm{d}m = \rho\pi R^2\mathrm{d}z$ and $\sigma(z) = KP(z)$, we obtain:

$$\frac{\mathrm{d}P}{\mathrm{d}z} + \frac{2K\mu P(z)}{R} - \rho g = 0$$

Solving the equation and taking into account that $P(0) = 0$, we obtain:

$$P(z) = \frac{\rho g R}{2\mu K}\left(1 - e^{-2\mu K z/R}\right)$$

Let us find the pressure at the bottom by substituting $z = h$ into the expression obtained:

$$P(h) = \frac{\rho g R}{2\mu K}\left(1 - e^{-2\mu K h/R}\right)$$

Hence:

Ответ:

$$A = \frac{\rho g R}{2\mu K},\quad B = -\frac{\rho g R}{2\mu K},\quad \lambda = \frac{R}{2\mu K}$$

B3 1.20 Determine the bulk density of the granules ρ and the coefficient K. Give the answers to two significant figures. For this part, assume the numerical value of the vessel radius is $R = 50\ \mathrm{cm}$ and the coefficient of friction is $\mu = 0.30$.

For small heights of the grain layer, the dependence of the pressure on the bottom on the height is approximately described by the expression:

$$P(h) \approx \rho g h$$

Let us draw a tangent to the graph at the point (0, 0). The slope of this tangent is:

$$k_1 = 6.5\ \mathrm{kPa/m}$$

Hence:

Ответ:

$$\rho = \frac{k_1}{g} \approx 660\ \frac{kg}{m}$$

For sufficiently large h, the pressure on the bottom reaches a steady value. The steady-state pressure is:

$$P_\infty = 26.7\ \mathrm{kPa}$$

Hence:

Ответ:

$$K = \frac{\rho g R}{2\mu P_\infty} \approx 0.20$$

C1 0.10 In the specified approximation, obtain the dependence of the vessel's cross-sectional radius on height $r(z)$. Express the answer in terms of R, α, and z.

Ответ:

$$r = R - \alpha z$$

C2 0.60 Write down the equilibrium condition for the layer. The equation may include the pressures $P(z)$ and $P(z+\mathrm{d}z)$, $\mathrm{d}z$, the average pressure on the vessel walls $\sigma(z)$, the mass of the layer $\mathrm{d}m$, as well as μ, g, $r(z)$, $r(z+\mathrm{d}z)$, α, and R.

Equilibrium condition in the projection on the vertical axis (here it is assumed that $\alpha \ll 1$, therefore $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha$):

Ответ:

$$P(z+\mathrm{d}z)\pi r^2(z+\mathrm{d}z) - P(z)\pi r^2(z) + 2\pi(\mu+\alpha)\sigma(z)r(z)\mathrm{d}z - \mathrm{d}mg = 0$$

C3 0.80 Determine the parameters A, γ, and β. Express the answer in terms of the bulk density of the granules ρ, g, $z_{\max}$, K, μ, and α.

Next, in the equation obtained in the previous step, we will neglect the term α compared to μ.
Note that near the free surface of the granular material, the pressure is zero:

$$P(0) = A(1-\gamma) = 0$$

From this:

Ответ:

$$\gamma = 1$$

The dependence $P(z)$ is described by the following differential equation:

$$\frac{\mathrm{d}(Pr^2)}{\mathrm{d}z} + 2K\mu P(z)r(z) - \rho g r^2(z) = 0$$

Let us introduce the notation $\xi = \dfrac{z}{z_{\max}}$. Substitute the expression for $P(z)$ given in the problem statement into the differential equation, taking into account that $\gamma = 1$:

$$\alpha^2 A z_{\max}\left(-1+\beta(1-\xi)^{\beta-1}\right)(1-\xi)^2 + 2(K\mu-\alpha)A\alpha z_{\max}(1-\xi)\left(1-\xi-(1-\xi)^{\beta}\right) - \rho g\alpha^2 z_{\max}^2(1-\xi)^2 = 0$$

The equation can be satisfied for all ξ only if the coefficients of $(1-\xi)^2$ and $(1-\xi)^{\beta+1}$ vanish.
Therefore:

$$-A\alpha^2 + 2(K\mu-\alpha)A\alpha - \rho g\alpha^2 z_{\max} = 0$$

Hence:

Ответ:

$$A = \frac{\rho g\alpha z_{\max}}{2K\mu - 3\alpha}$$

Also, the following holds:

$$\alpha^2 A\beta - 2(K\mu-\alpha)A\alpha = 0$$

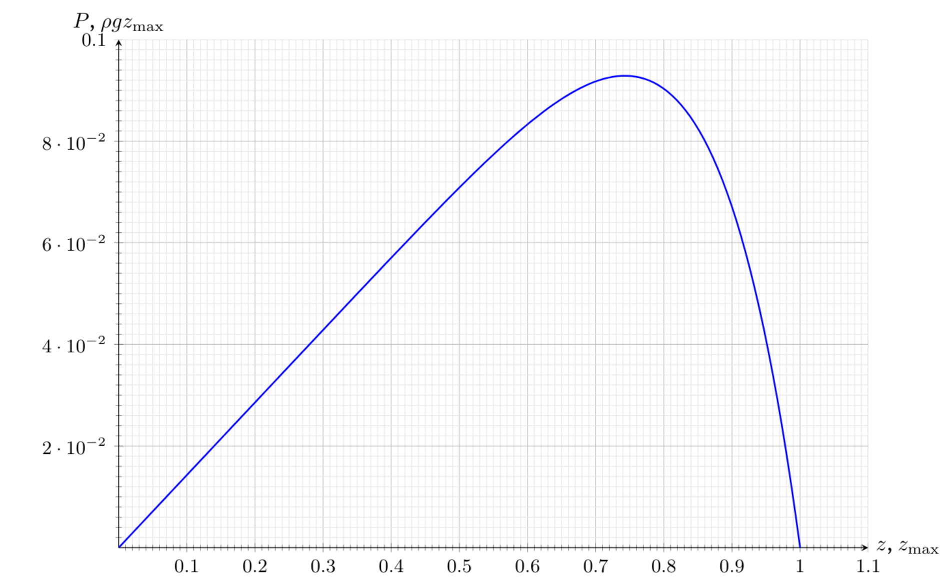
From this:

Ответ:

$$\beta = \frac{2(K\mu-\alpha)}{\alpha}$$

C4 0.80 Sketch a qualitative graph of $P(z)$ and determine the ratio of the maximum vertical pressure $P_{\max}$ to $\rho g z_{\max}$ for $\alpha = 0.2K\mu$. Provide the answer to two significant figures.

Ответ:



For $\alpha = 0.2K\mu$, let us perform the calculations:

$$A \approx 0.143\rho g z_{\max}, \quad \beta = 8$$

The dependence $P(z)$ has a maximum when $1-\xi-(1-\xi)^8 \to \max$, that is:

$$8(1-\xi)^7 = 1, \quad z = (1-8^{-\frac{1}{7}})z_{\max} \approx 0.26 z_{\max}.$$

It corresponds to the pressure:

Ответ:

$$P_{\max} \approx 0.093\rho g z_{\max}$$

D1 0.40 Write down the equilibrium condition for the layer. The equation may include the pressures $P(z)$ and $P(z+\mathrm{d}z)$, $\mathrm{d}z$, the average pressure on the rod and vessel walls $\sigma(z)$, the mass of the layer $\mathrm{d}m$, as well as μ_1, μ_2, g, a, and R.

Equilibrium condition in the projection on the vertical axis (here the friction force acting on the granules from the side of the vessel is directed downwards, since the force acting on the vessel lifts the vessel):

Ответ:

$$(P(z+\mathrm{d}z) - P(z))\pi(R^2-a^2) - \mathrm{d}mg - \mu_1\sigma(z)\cdot 2\pi R\mathrm{d}z + \mu_2\sigma(z)\cdot 2\pi a\mathrm{d}z = 0$$

D2 0.30 Show that the bottom pressure P as a function of the fill height h of the granular column can be written as:

$$P = A + Be^{-h/\lambda}.$$

Determine the parameters A, B, and λ. Express the answers in terms of the bulk density of the granules ρ, as well as g, a, R, μ_1, μ_2, and K.

The resulting equation is perfectly analogous to the equation from **B1**. Solving it, we obtain:

Ответ:

$$A = \frac{\rho g(R^2-a^2)}{2(-\mu_1 R + \mu_2 a)K}, \quad B = -\frac{\rho g(R^2-a^2)}{2(-\mu_1 R + \mu_2 a)K}, \quad \lambda = \frac{R^2-a^2}{2(-\mu_1 R + \mu_2 a)K}$$

D3 0.80 Determine the minimum level H of the granules relative to the bottom at which the cylinder can be lifted as described above. Assume that $H \ll \lambda$. Express the answer in terms of a, R, μ_2, and K.
Hint: You may use the following approximation:

$$e^x \approx 1 + x + \frac{x^2}{2}, \quad x \ll 1$$

The dependence of the pressure $P(z)$ for a fixed level H of the granules is similarly given by the formula:

$$P(z) = A + Be^{-z/\lambda}$$

Let us calculate the total force acting on the rod:

$$F = 2\pi a K\mu_2 \int_0^H P(z)\mathrm{d}z = 2\pi a K\mu_2 A(H - \lambda(1-e^{-H/\lambda})) \approx 2\pi a K\mu_2 A\frac{H^2}{2\lambda},$$

since $H \ll \lambda$. According to Newton's third law, the same buoyant force acts on the granules. In the critical case, it must balance the gravitational force acting on the granules:

$$\rho g\pi(R^2-a^2)H = \pi a K\mu_2 A\frac{H^2}{\lambda}$$

From this, we express H:

Ответ:

$$H = \frac{R^2-a^2}{Ka\mu_2}$$

E1 0.20 Derive an equation relating S_1, S_2, Δl_1, and Δl_2.

From the incompressibility condition:

Ответ:

$$S_1\Delta l_1 = S_2\Delta l_2$$

E2 0.10 Determine the total energy E (the sum of kinetic and potential energies) of the selected fluid volume $S_1\Delta l_1$. Express the answer in terms of ρ, S_1, Δl_1, v_1, g, and h_1. Take the potential energy at $h=0$ as the zero reference level.

Ответ:

$$W_1 = \rho S_1 \Delta l_1 \left(\frac{v_1^2}{2} + gh_1\right)$$

E3 0.30 Determine the work A done by pressure forces on the fluid bounded by the chosen stream tube during time $\Delta t = \dfrac{\Delta l_1}{v_1}$. Express the answer in terms of p_1, p_2, S_1, and Δl_1.

A volume V_1 of fluid is "pushed into" the stream tube at pressure p_1, and a volume $V_2 = V_1$ is "pushed out" of it at pressure p_2. Hence:

Ответ:

$$A = p_1V_1 - p_2V_2 = (p_1 - p_2)S_1\Delta l_1$$

E4 0.30 Using the law of conservation of energy, express the pressure p_2 in terms of p_1, ρ, g, h_1, h_2, v_1, and v_2. The resulting equation is known as the **Bernoulli equation**.

We write the law of conservation of energy as:

$$\rho S_1\Delta l_1 \frac{v_2^2 + 2gh_2 - v_1^2 - 2gh_1}{2} = (p_1 - p_2)S_1\Delta l_1$$

By canceling $S_1\Delta l_1$ from both sides, we obtain:

Ответ:

$$p_2 = p_1 - \frac{v_2^2 + 2gh_2 - v_1^2 - 2gh_1}{2}$$

E5 0.30 Determine the fluid outflow velocity $v(h)$ as a function of the height above the orifice. Express the answer in terms of g and h.

From the Bernoulli equation derived in the previous section, for ideal fluid flow, the following holds:

$$p + \rho g h + \rho\frac{v^2}{2} = \mathrm{const}$$

The pressure at the orifice is equal to the atmospheric pressure, as is the pressure at the upper boundary of the fluid and air. Thus, we have:

$$p_0 + \rho g h + 0 = p_0 + 0 + \rho\frac{v(h)^2}{2}$$

From this, it follows:

Ответ:

$$v(h) = \sqrt{2gh}$$

E6 0.60 Determine the time τ required for all the fluid initially above the orifice to drain through it. Express the answer in terms of g, H, R, and D.

Volumetric flow rate through the orifice:

$$\dot V = -\frac{\pi D^2}{4}\sqrt{2gh}$$

From the continuity equation, we obtain a differential equation for the water height above the orifice:

$$\pi R^2 \dot h = -\frac{\pi D^2}{4}\sqrt{2gh}$$

Note that since $R \gg D$, it follows that $\dot h \ll \sqrt{2gh}$; therefore, the assumption that the velocity far from the orifice is negligible is valid.
We can now express the drainage time:

Ответ:

$$\tau = \frac{2\sqrt{2}R^2}{D^2\sqrt{g}}\int_0^H \frac{\mathrm{d}h}{\sqrt{h}} = \frac{4\sqrt{2}R^2\sqrt{H}}{D^2\sqrt{g}}$$

F1 0.20 Due to internal friction between the granules, the maximum angle at which the free surface of the granular material can deviate from the horizontal is θ_r. What is the maximum wall thickness $\omega_{\max}$ for which the flow of granules from the vessel is possible? Express the answer in terms of D and θ_r.

In the critical case, the material is tilted from the horizontal by an angle θ_r and is positioned at the outer edge of the orifice. From this, we obtain:

Ответ:

$$\omega_{\max} = D\,\mathrm{ctg}\,\theta_r$$

F2 0.80 Determine q and n.

Estimate the granule velocity at the orifice exit using the law of conservation of energy:

$$v \propto \sqrt{D}$$

Estimate the mass flow rate of the granules:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho S v \propto D^2\sqrt{D} \propto D^{\frac{5}{2}}$$

From this, it follows:

Ответ:

$$n = \frac{5}{2}$$

$$q = 0$$