SPhO 2025 Solutions (UNOFFICIAL)

FIZIKA

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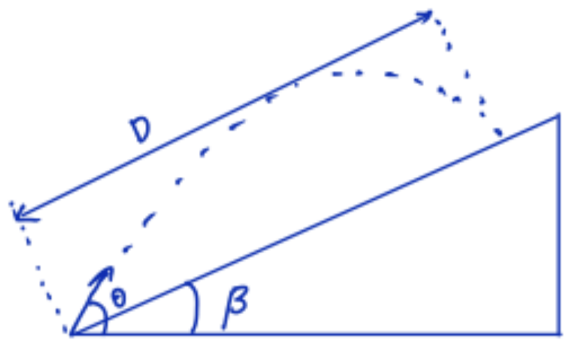
We also thank FIZIKA students who have sat for SPhO 2025 for compiling the questions for us.

1 Question 1 [7]

1.1 Question

A projectile is shot up a slope of incline angle β to the x-axis at an initial velocity v_0 and angle θ to the x-axis, where $\theta > \beta$. The slope and the projectile both start at the origin. Express all answers in terms of v_0, β, θ and g.



(i) Find the time taken for the projectile to land back on the slope. [3]

(ii) Find the distance up the slope it has travelled. [2]

(iii) Prove that for the maximum distance, $-\cot 2\theta = \tan\beta$. [2]

1.2 Solution

(i) The most efficient way is to tilt coordinate axes such that the acceleration components parallel and perpendicular to the slope are $a_\parallel = g\sin\beta$ and $a_\perp = -g\cos\beta$. We can then apply normal kinematics in the perpendicular direction:

$$y = v_0 t\sin(\theta-\beta) - \frac{1}{2}gt^2\cos\beta = 0 \quad \implies \quad t = \frac{2v_0\sin(\theta-\beta)}{g\cos\beta}.$$

(ii) Applying normal kinematics in the parallel direction and the trigonometric addition identities,

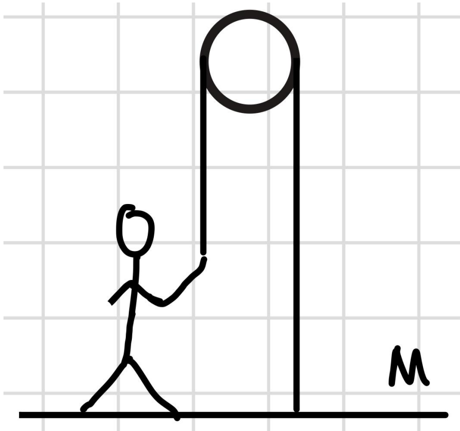
$$d = v_0 t\cos(\theta-\beta) - \frac{1}{2}gt^2\sin\beta = \frac{2v_0^2\sin(\theta-\beta)\cos\theta}{g\cos^2\beta}.$$

(iii) If we fix β, then the maximum distance occurs when $\sin(\theta - \beta)\cos\theta = \frac{1}{2}\left(\sin(2\theta - \beta) - \sin\beta\right)$ is maximised. Clearly this occurs when $2\theta - \beta = \frac{\pi}{2}$, so $\tan\beta = \tan\left(2\theta - \frac{\pi}{2}\right) = -\cot 2\theta$.

2 Question 2 [14]

2.1 Question

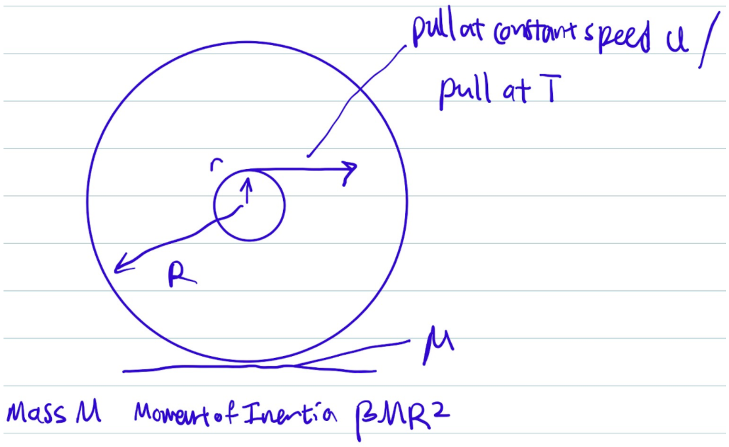
A man is on an elevator held up by a rope on a pulley. The other end of the rope is getting tugged on by the man in the elevator. The man and the elevator have a total mass of M. Refer to the figure below.



(i) Find the minimum force the man needs to apply to move upward. [2]

(ii) Say the elevator moves a height h upwards. By considering the change in GPE, comment on the conservation of energy. [2]

I have a yo-yo (see figure) of moment of inertia $I = bMR^2$ about its central axis. The coefficient of friction is μ.



(iii) If I pull the string at constant speed u, find the CM velocity, assuming no slipping. [2]

(iv) Now, I pull the string at constant tension T. Show that the condition for friction to point left is $r < bR$, still assuming no slipping. [3]

(v) Consider that there is slipping now. What is the minimum T required for friction to point left? [3]

(vi) Consider the limit $r \to bR$, and explain why your answer makes sense. [2]

2.2 Solution

(i) The total upward pulling force has to support a force of Mg. We assume the rope is light and inextensible, and hence has a constant tension T throughout. The total upward force supplied by the rope's tension on the man + elevator system is $2T$. Hence,

$$2T = Mg \quad \implies \quad T = \frac{Mg}{2}$$

(ii) When the elevator rises by h, the increase in GPE is

$$\Delta \text{GPE} = Mgh$$

It may seem like a contradiction at first – the work done by the man is naively $Th = \frac{Mgh}{2}$, which implies that energy $\frac{Mgh}{2}$ was lost.

However, there is a subtlety here – for the elevator to rise by h, the man has to pull a length $2h$ of rope! This can be understood via conservation of the rope's total length. As such, the work done by the man is actually

$$W_{\text{man}} = T\,(2h) = Mgh = \Delta \text{GPE}$$

which verifies energy is conserved.

(iii) Clearly, the yo-yo rolls to the right. Suppose it does so with an angular velocity ω.

The key idea here is that the point on the yo-yo connected to the string has its velocity constrained to be u to the right. Hence, by adding up the velocity vectors contributed by translation and rotation, the velocity vector at that point must point right and have magnitude u. Hence,

$$u = v_{\text{CM}} + \omega r$$

Coupling this with the non-slip condition $v_{\text{CM}} = R\omega$, we have

$$v_{\text{CM}} = \frac{uR}{R + r}$$

(iv) Let f be the frictional force, and we define f to be positive if it points left. Let a be the CM acceleration, and we define a to be positive if it points right.

Then, by N2L,

$$T - f = Ma$$

and by the rotational form of N2L,

$$Tr + fR = I\alpha = bMR^2\alpha = bMRa$$

where we used the non-slip condition $a = R\alpha$. (Be very careful of the signs here!)

We can solve these simultaneous equations to obtain

$$a = \frac{f\,(R + r)}{M\,(bR - r)}$$

This allows us to express f in terms of T (which we know is positive):

$$f = \frac{bR - r}{R\,(b + 1)} T$$

We require $f > 0$ for it to point left. Hence, this requires $r < bR$, as desired.

(v) If there is slipping now, friction is kinetic and is given by $f = \mu N = \mu Mg$.

Then, by N2L,

$$T - \mu Mg = Ma$$

and by the rotational form of N2L,

$$Tr + \mu MgR = bMR^2\alpha$$

where we take note that now, $a \neq R\alpha$ since the yo-yo is slipping.

For friction to the point to the left, the contact point at the bottom of the yo-yo must move to the right. This means

$$v_{\text{CM}} > R\omega \quad \Longrightarrow \quad a > R\alpha$$

by differentiating w.r.t time.

Hence, we have

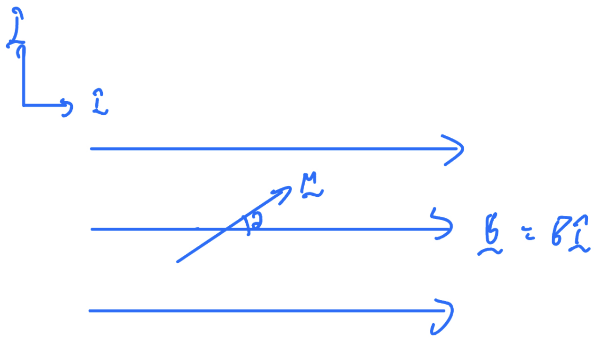
$$\frac{T - \mu Mg}{M} > R\left(\frac{Tr + \mu MgR}{bMR^2}\right) \quad \Longrightarrow \quad T_{\text{min}} = \frac{\mu Mg\left(1 + \frac{1}{b}\right)}{1 - \frac{r}{bR}}$$

(vi) When $r \to bR$, we see that $T_{\text{min}} \to \infty$. This means that at this point, the minimum tension required to make friction point left is infinite, meaning it is impossible for friction to point left (i.e. it always points right). In other words, the tension produces no preference in the direction of rotation and hence friction.

3 Question 3 [10]

3.1 Question

A magnetic dipole moment of μ makes an angle θ with the x-axis, and is placed in a uniform, constant magnetic field $\mathbf{B} = B\hat{\mathbf{i}}$. Suppose that it has a moment of inertia I about the z-axis.



(i) State the axis of rotation of the dipole. [1]

(ii) Find the torque on the magnetic dipole, as a vector in Cartesian coordinates. [2]

(iii) Find the work done by the torque as the dipole rotates from $\theta_{\text{i}} = \frac{\pi}{4}$ rad to $\theta_{\text{f}} = 0$ rad, and find the change in the potential energy. [2]

(iv) Show that for small θ, the motion of the dipole is simple harmonic, and obtain an expression for the period of oscillations. [3]

(iv) Suppose $\mathbf{B}$ now varies with y-position (increasing with increasing $\hat{\mathbf{j}}$ component). State whether there will be a net force and a net torque on the dipole. [2]

3.2 Solution

(i) By $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$, the torque points into the page, so the axis of rotation passes through the centre of the magnetic dipole pointing into the page. The axis of rotation is the z-axis.

(ii) Directly applying the formula for torque,

$$\boldsymbol{\tau} = \left((\mu\cos\theta)\hat{\mathbf{i}} + (\mu\sin\theta)\hat{\mathbf{j}}\right) \times B\hat{\mathbf{i}} = (-\mu B\sin\theta)\hat{\mathbf{k}}.$$

(iii) Integrating torque over the angle, the work done is

$$W = \int \boldsymbol{\tau} \cdot \mathrm{d}\boldsymbol{\theta} = \int_{\pi/4}^{0} (-\mu B\sin\theta)\hat{\mathbf{k}} \cdot \mathrm{d}\theta\,\hat{\mathbf{k}} = \int_{0}^{\pi/4} \mu B\sin\theta\,\mathrm{d}\theta = \left(1 - \frac{1}{\sqrt{2}}\right)\mu B.$$

Directly applying the formula for potential energy,

$$U = -\boldsymbol{\mu}\cdot\mathbf{B} \quad \Longrightarrow \quad \Delta U = -\Delta\boldsymbol{\mu}\cdot\mathbf{B} = -\mu\left(\left(1 - \frac{1}{\sqrt{2}}\right)\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right)\cdot B\hat{\mathbf{i}} = \left(\frac{1}{\sqrt{2}} - 1\right)\mu B.$$

The two values are negatives of each other.

(iv) By applying the formula for torque to the rotational form of N2L, we have

$$-\mu B\sin\theta = I\ddot{\theta} \quad \Longrightarrow \quad \ddot{\theta} + \frac{\mu B}{I}\theta = 0$$

where we used the small angle approximation $\sin\theta \approx \theta$.

This is an equation of motion for SHM, where we have

$$\omega = \sqrt{\frac{\mu B}{I}} \quad \Longrightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\mu B}}$$

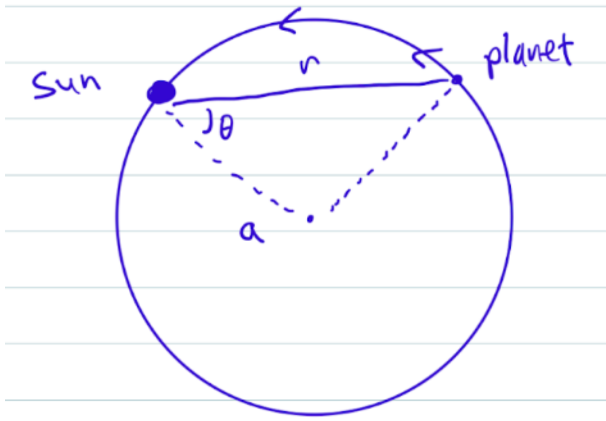
(v) If the magnetic dipole has non-zero size, then the dipole will experience a net force since the magnetic field changes across the dipole, resulting in an unbalanced magnetic force along its length. The dipole will still experience a net torque as expected.

4 Question 4 [10]

4.1 Question

(i) Given a central force is such that $F(r) = m\left(\ddot{r} - r\dot{\theta}^2\right)$, by using the substitution $u = \frac{1}{r}$, prove the Binet equation $F(u) = -\frac{L^2u^2}{m}\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right)$, where L is the angular momentum. [5]

(ii) A planet experiences a central force from the Sun and moves around in a circular orbit of radius a, as per the figure below:



If $F(r) = -Cr^n$, where C is a constant, find the value of n. [5]

4.2 Solution

(i) Angular momentum is conserved under a central force. Hence,

$$L = mr^2\dot{\theta} = \text{constant} \quad \Longrightarrow \quad \dot{\theta} = \frac{L}{mr^2}$$

Substituting this into the given equation,

$$F(r) = m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right)$$

Using the given substitution,

$$u = \frac{1}{r} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r} = -u^2\frac{\mathrm{d}}{\mathrm{d}u}$$

Additionally, from COAM,

$$L = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{Lu^2}{m}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

We can hence work out the time derivatives of r:

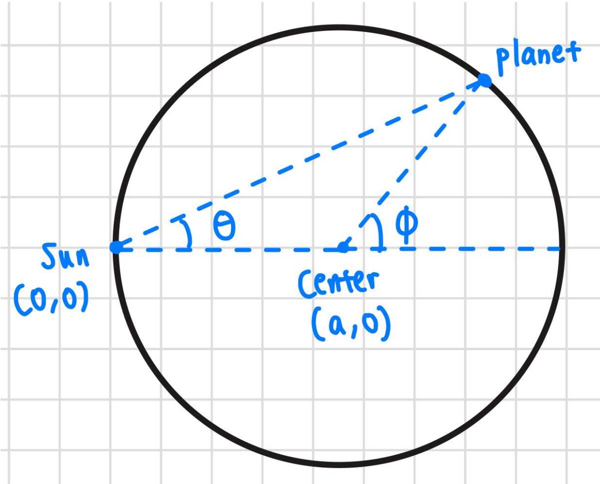
$$\dot{r} = \frac{Lu^2}{m}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{u}\right) = -\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

$$\ddot{r} = \frac{Lu^2}{m}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -\frac{L^2u^2}{m^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$$

Hence, we can rewrite the equation for $F(r)$ (which we now change to $F(u)$ to be consistent with our substitution) as such:

$$F(u) = m\left(-\frac{L^2u^2}{m^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - \frac{L^2u^3}{m^2}\right) = -\frac{L^2u^2}{m}\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right) \quad \text{(proven)}$$

(ii) Let's place the Sun at the origin $(0, 0)$ and the center of the circle at $(a, 0)$ for simplicity. Let ϕ be the angle which the planet makes as it goes around the circle's center.



The planet's Cartesian coordinates are hence $x = a(1+\cos\phi)$ and $y = a\sin\phi$.

The distance between the planet and the Sun is hence

$$r^2 = x^2 + y^2 = a^2(1+\cos\phi)^2 + a^2\sin^2\phi = 2a^2(1+\cos\phi) = 4a^2\cos^2\left(\frac{\phi}{2}\right)$$

$$\implies \quad r = 2a\cos\left(\frac{\phi}{2}\right)$$

By some circle geometry, $\theta = \frac{\phi}{2}$. Remember that θ is the angle used in the Binet equation, since it is measured with respect to the Sun. Hence,

$$r = 2a\cos\theta \quad \implies \quad u(\theta) = \frac{1}{r} = \frac{1}{2a\cos\theta} = \frac{1}{2a}\sec\theta$$

We can find the derivatives w.r.t θ:

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{1}{2a}\sec\theta\tan\theta, \quad \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{1}{2a}\left(\sec\theta\tan^2\theta + \sec^3\theta\right) = \frac{1}{2a}\left(2\sec^3\theta - \sec\theta\right)$$

We can also write the force in terms of u, as $F(u) = -\frac{C}{u^n}$. Substituting all of this into the Binet equation found in part (i), we have:

$$-\frac{C}{u^n} = -\frac{L^2u^2}{m}\left(\frac{\sec^3\theta}{a}\right) \quad \implies \quad \frac{amC}{L^2} = u^{n+2}\sec^3\theta = \left(\frac{1}{2a}\right)^{n+2}\sec^{n+5}\theta$$

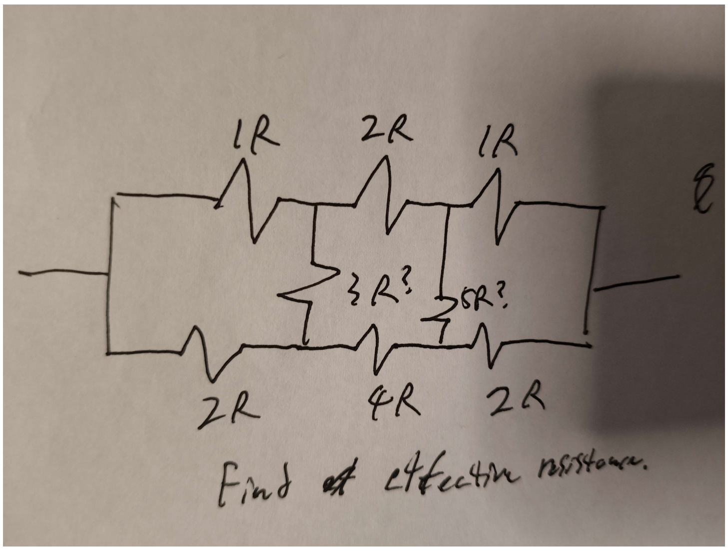
$$\implies \quad \sec^{n+5}\theta = \frac{2^{n+2}a^{n+3}mC}{L^2} = \text{constant}$$

The last line can only be satisfied if $n = -5$, which is the desired value of n.

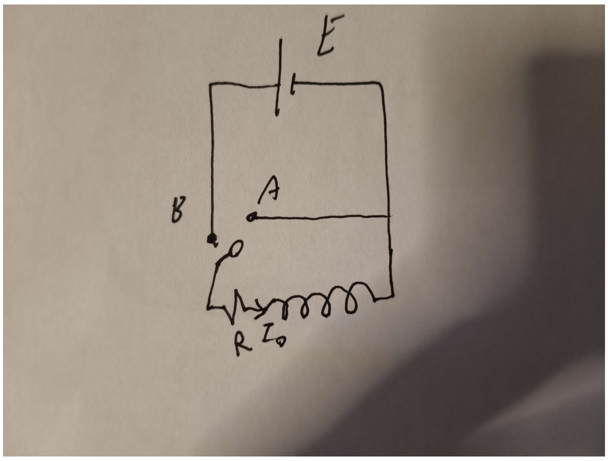
5 Question 5 [10]

5.1 Question

(i) Find the effective resistance across the terminals of the circuit below. [2]



(ii) Consider the RL circuit below with resistance R, inductance L and a battery of emf E. The switch is initially at position A. I_0 is the initial current through the circuit, and $I_0 < \frac{E}{R}$. At time $t = 0$, the switch is flipped to position B. Show that the new current in the circuit at time Δt after flipping the switch is given by $\left(I_0 - \frac{E}{R}\right) \exp\left(-\frac{R}{L}\Delta t\right) + \frac{E}{R}$. [3]



(iii) The switch is then flipped back to position A. Find the new current in the circuit at another time Δt after flipping the switch back. [2]

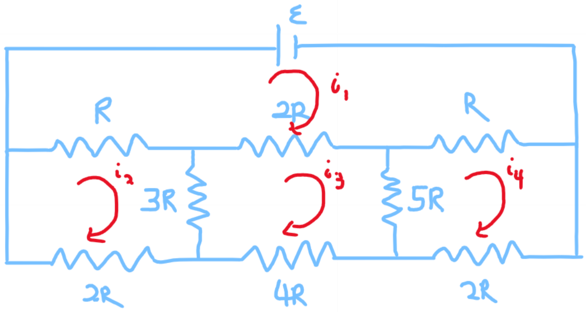
(iv) The switch is flipped back and forth A and B at a regular time intervals of Δt. Find the steady-state current at $t \to \infty$. [3]

5.2 Solution

(i) Consider removing the $3R$ and $5R$ resistors so that the setup consists of two branches and suppose some potential difference is applied across the two terminals. Notice that since both branches have the same potential difference (as they are in parallel) and the ratios between the corresponding resistors on each branch are the same, the voltage of each corresponding resistor is the same across the branches by the potential divider rule. This implies that corresponding points are equipotential across the branches, so when a resistor is connected across them, no current flows through. Hence, we can effectively return the $3R$ and $5R$ resistors to the setup and the voltages of each resistor will not change, so the effective resistance of the setup does not change. Therefore, we can just calculate the effective resistance of the setup without the $3R$ and $5R$ resistors, which is

$$R_{\text{eff}} = \left(\frac{1}{R+2R+R} + \frac{1}{2R+4R+2R}\right)^{-1} = \frac{8}{3}R.$$

Alternative Solution. We can bash this using mesh analysis. Consider connecting the ends to the terminals of a battery of emf ε.



Considering the loop currents as shown in the diagram above, we have the four simultaneous equations

$$\begin{aligned}
-4i_1 + i_2 + 2i_3 + i_4 &= \frac{\varepsilon}{R} \\
i_1 - 6i_2 + 3i_3 &= 0 \\
2i_1 + 3i_2 - 14i_3 + 5i_4 &= 0 \\
i_1 + 5i_3 - 8i_4 &= 0
\end{aligned}$$

This would be painful to solve by hand, but thankfully we have a scientific calculator with the capability to solve simultaneous equations (which is why this method is good in this context). By taking all currents to be in units of $\frac{\varepsilon}{R}$ and plugging the equations into the calculator, we get

$$i_1 = -\frac{3}{8}\frac{\varepsilon}{R}$$

where the negative comes from the current actually flowing anticlockwise rather than clockwise as defined. The magnitude of i_1 is the current flowing through the battery, so the effective resistance is

$$R_{\text{eff}} = \frac{8}{3}R.$$

Alternative Solution. You can also bash this using the $\Delta - \text{Y}$ transformations.

(ii) The inductor governs that

$$V_{\text{L}} = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

so applying Kirchhoff's voltage law on the loop gives

$$E - IR - V_{\text{L}} = 0 \qquad \Longrightarrow \qquad \frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} + I = \frac{E}{R}$$

This is a first-order linear differential equation which can be solved easily to get

$$I(t) = \left(I_0 - \frac{E}{R}\right)\exp\left(-\frac{R}{L}t\right) + \frac{E}{R}$$

where we used the initial condition $I(0) = I_0$. Substituting $t = \Delta t$, the new current in the circuit is then

$$I_1 = \left(I_0 - \frac{E}{R}\right)\exp\left(-\frac{R}{L}\Delta t\right) + \frac{E}{R}.$$

(iii) We can proceed by a similar method to obtain the differential equation

$$\frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} + I = 0$$

which can be solved to get

$$I(t) = I_1 \exp\left(-\frac{R}{L}t\right)$$

Substituting $t = \Delta t$ and our obtained expression for I_1, we have

$$\begin{aligned} I_2 &= \left(\left(I_0 - \frac{E}{R}\right)\exp\left(-\frac{R}{L}\Delta t\right) + \frac{E}{R}\right)\exp\left(-\frac{R}{L}\Delta t\right) \\ &= \left(I_0 - \frac{E}{R}\right)\exp\left(-\frac{2R}{L}\Delta t\right) + \frac{E}{R}\exp\left(-\frac{R}{L}\Delta t\right). \end{aligned}$$

(iv) As we flip the switch back to position A, the previous I_2 becomes the new I_0 and the cycle repeats. At steady state, I_0 and I_2 must then be equal, so

$$I_0 = \left(I_0 - \frac{E}{R}\right)\exp\left(-\frac{2R}{L}\Delta t\right) + \frac{E}{R}\exp\left(-\frac{R}{L}\Delta t\right)$$

If we let $x = \exp\left(-\frac{R}{L}\Delta t\right)$, we have

$$\left(I_0 - \frac{E}{R}\right)x^2 + \frac{E}{R}x - I_0 = 0 \quad\implies\quad x = \frac{I_0}{\frac{E}{R} - I_0}$$

which gives the value for I_0 at steady state as

$$I_0 = \frac{x}{1+x}\frac{E}{R}$$

This also gives our corresponding value for I_1 at steady state as

$$I_1 = \frac{1}{1+x}\frac{E}{R}$$

Neither of these are quite the steady-state current yet – this is just the values at the beginning of the charging and discharging cycles. The average of the current over the charging cycle is

$$I_{\infty,+} = \frac{1}{\Delta t}\int_0^{\Delta t}\left(\left(I_0 - \frac{E}{R}\right)\exp\left(-\frac{R}{L}t\right) + \frac{E}{R}\right)\mathrm{d}t = \frac{E}{R}\left(-\frac{1-x}{1+x}\frac{L}{R\Delta t} + 1\right)$$

and the average of the current over the discharging cycle is

$$I_{\infty,-} = \frac{1}{\Delta t}\int_0^{\Delta t} I_1 \exp\left(-\frac{R}{L}t\right)\mathrm{d}t = \frac{E}{R}\left(\frac{1-x}{1+x}\frac{L}{R\Delta t}\right)$$

Since both cycles last for the same amount of time, the overall average current is then

$$I_\infty = \frac{1}{2}\left(I_{\infty,+} + I_{\infty,-}\right) = \frac{E}{2R}$$

As it turns out, the average steady-state current is the same as if there were no inductor at all. This could be unsurprising if understood that the charging and discharging curves have the same time constant, so they are "congruent".

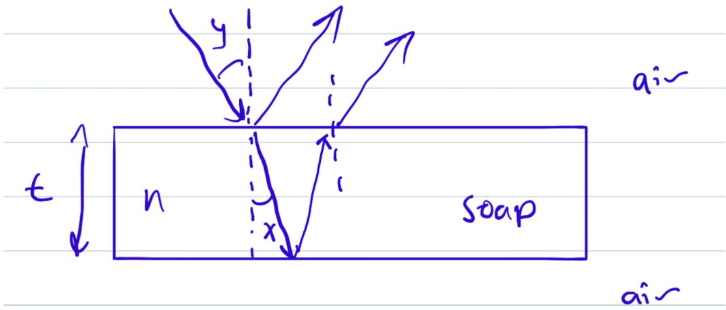
6 Question 6 [10]

6.1 Question

(i) Consider a cube of refractive index $n = 1.33$. Light enters from the top surface of the cube at its centre, at an angle of 45°. Determine whether light will exit through the side faces, and if so, determine the angle that it exits at. [2]

(ii) Consider a flat plate with n increasing linearly with thickness, from $n = 1$ at the top to $n = 2$ at the bottom. The plate is surrounded by a vacuum. If light enters at an angle of 45°, sketch the path of light in the medium. What angle does it exit at? [2]

(iii) Prove that the condition for constructive interference of this system in the figure below is $\frac{2t}{\cos x}(n - \sin x \sin y) = \left(m + \frac{1}{2}\right)\lambda$. [4]



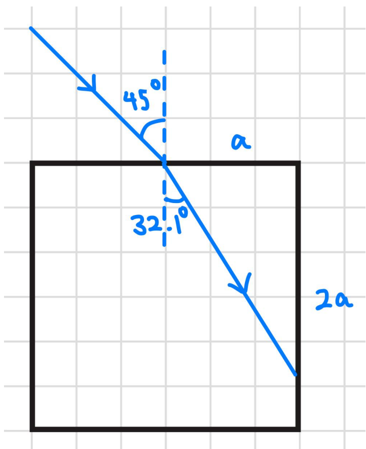
(iv) Using the numerical values $n = 1.33, t = 875\,\mathrm{nm}$ and $y = 45°$, which wavelengths in the visible light spectrum will have constructive interference? [2]

6.2 Solution

(i) By Snell's Law,

$$n_{\mathrm{air}} \sin\theta_{\mathrm{i}} = n \sin\theta_{\mathrm{r}} \quad\Longrightarrow\quad \theta_{\mathrm{r}} = 32.1°$$

Now, we need to determine which face the light ray will strike. It will either strike the right face or the bottom face, and the angle that separates these two cases is $\tan^{-1}\left(\frac{1}{2}\right) = 26.6°$. Since $32.1° > 26.6°$, the light ray will strike the right face, as seen in the diagram below.

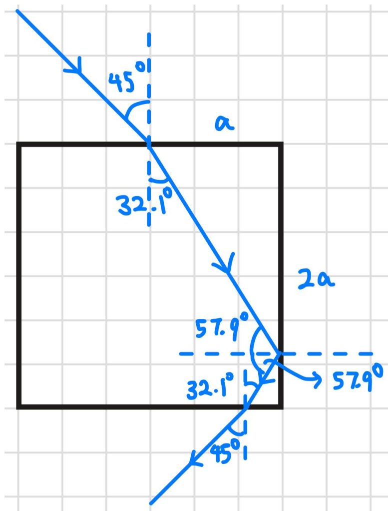


The critical angle for the cube-air interface is

$$\theta_c = \sin^{-1}\left(\frac{n_{\text{air}}}{n}\right) = \sin^{-1}\left(\frac{1}{1.33}\right) = 48.8^\circ$$

The angle which the light ray strikes the normal on the right face of the cube is $90^\circ - 32.1^\circ = 57.9^\circ$, which is larger than θ_c. Hence, the ray undergoes total internal reflection (TIR), and doesn't exit through the right face.

This means it will exit through the bottom face at an angle of 45° to the normal, as shown in the figure below:



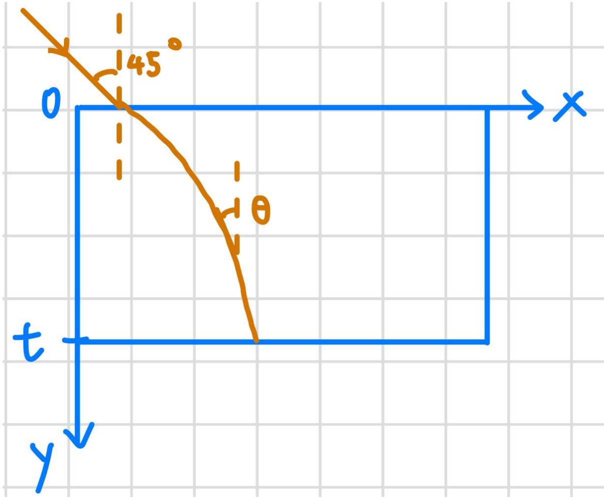
Remark. The answer depends on the interpretation of "side faces". If "side faces" are taken to be the 4 side faces that exclude the top and bottom faces, then the answer is no. If "side faces" are taken to be any face, then the answer is yes, and at 45°. I think it is more likely that the former is what they intended though.

(ii) Suppose the thickness of the plate is t. Let the top of the plate be $y = 0$ and the bottom of the plate be $y = t$. Then,

$$n(y) = 1 + \frac{y}{t},\ 0 \leq y \leq t$$

The key here is that we can split the medium into many infinitesimal layers and apply Snell's Law on each of the boundaries. Hence, we can say

$$n \sin\theta = \text{constant} = 1 \sin 45^\circ = \frac{\sqrt{2}}{2}$$



Let $k = \frac{\sqrt{2}}{2}$ for simplicity. At the same time, $y' = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$. Hence, we have

$$y' = \frac{\frac{k}{n}}{\sqrt{1-\left(\frac{k}{n}\right)^2}} = \frac{k}{\sqrt{n^2-k^2}} = \frac{k}{\sqrt{(1+\frac{y}{t})^2-k^2}}$$

This is a separable differential equation:

$$\left(\sqrt{\left(1+\frac{y}{t}\right)^2-k^2}\right)\mathrm{d}y = k\,\mathrm{d}x$$

You can try solving this, but given the mark allocation of the question, it is highly unlikely that you are required to find the exact trajectory of the light ray. A simple qualitative sketch (like the one in the figure above) should suffice.

The angle which the light ray exits is given by

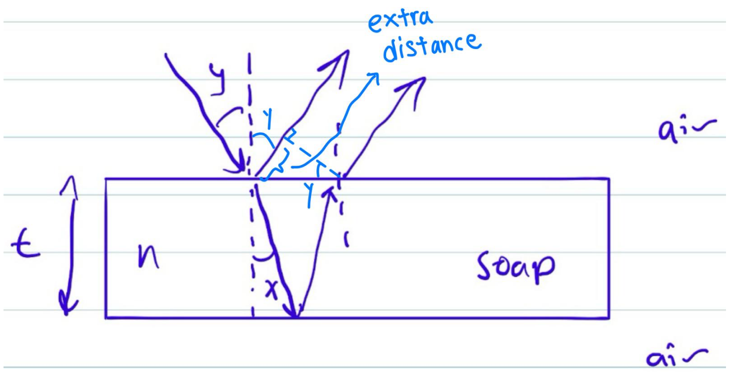
$$1\sin 45^\circ = 2\sin\theta_\mathrm{e} \quad\Longrightarrow\quad \theta_\mathrm{e} = 20.7^\circ$$

Remark. There is so ambiguity here with the definition of "exit". You could have interpreted "exit" as the angle of the light ray *after* it has left the plate (when it is in the air), which would be 45°. While this is a possible interpretation based on the phrasing of the question, it is likely that the intended solution is still the one described above (otherwise the question is rather meaningless). I guess if you wrote 45° but still have 20.7° somewhere in your working, you can hope for benefit of doubt/lenient marking.

(iii) As with all interference problems, we need to find the difference in optical path length/phase.

The physical extra distance in the soap travelled by the ray which enters the soap is $\Delta d_1 = \frac{2t}{\cos x}$ by some simple trigonometry. This leads to an optical path length difference of $n\Delta d_1 = \frac{2nt}{\cos x}$.

However, that is not all! The ray that was purely in the air (and reflected at the air-soap boundary) also travels some extra distance in the air, as indicated below:



This extra distance can be found as $\Delta d_2 = 2t \tan x \sin y$, again by some simple trigonometry. As this is in air, $n_{\text{air}} = 1$ and hence the optical path length difference is the same.

As such, the total optical path length difference is

$$\Delta\text{OPL} = n\Delta d_1 - \Delta d_2 = \frac{2nt}{\cos x} - 2t \tan x \sin y = \frac{2t}{\cos x}(n - \sin x \sin y)$$

There is one final subtlety here. The ray that reflects off the air-soap boundary experiences a π rad phase shift, since it is reflecting off a surface with higher refractive index. In contrast, the other ray doesn't experience any phase shift from reflection. As such, the condition for constructive interference is

$$\Delta\text{OPL} = \left(m + \frac{1}{2}\right)\lambda \quad \Longrightarrow \quad \frac{2t}{\cos x}(n - \sin x \sin y) = \left(m + \frac{1}{2}\right)\lambda$$

where $m \in \mathbb{Z}$, and the extra $+\frac{1}{2}$ comes from accounting for the π rad phase shift.

(iv) First, we can find x using Snell's Law:

$$1 \sin y = n \sin x \quad \Longrightarrow \quad x = \sin^{-1}\left(\frac{\sin y}{n}\right) = 32.1°$$

Recall that the visible light spectrum ranges from $380 - 750$ nm (you should memorise this). By trying out a few values of m, you should obtain that

$$\lambda_{m=3} = 563\,\text{nm}, \ \lambda_{m=4} = 438\,\text{nm}$$

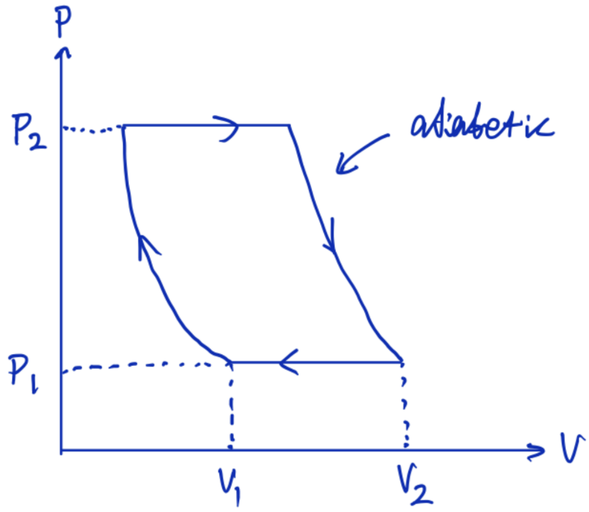
are the only two wavelengths in this range, and hence are the desired wavelengths.

Remark. Of course, there is some discrepancy in the endpoints for the range of wavelengths for the visible light spectrum. $m = 2$ will give you $\lambda = 789$ nm, while $m = 5$ will give you $\lambda = 358$ nm, both of which are terribly close to the endpoints that I have indicated. My best guess is that the marking may depend on what you have written on your paper as what *you* think constitutes the visible light spectrum, and as long as it isn't too far from $380 - 750$ nm (taken from Wikipedia), then you should be fine.

7 Question 7 [10]

7.1 Question

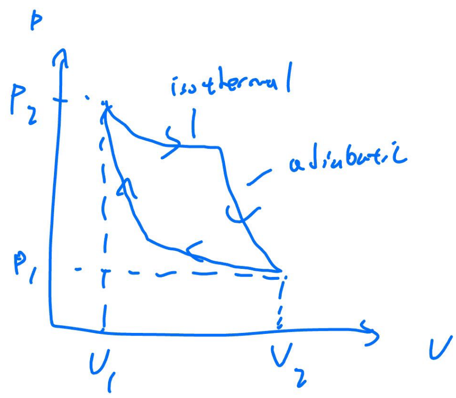
Consider the following $P - V$ diagram of the Brayton cycle, comprised of two isobaric processes and two adiabatic processes:



It is given that $P_2 = 3P_1$, $V_2 = 2V_1$, and the gas is monoatomic (hence $\gamma = \frac{5}{3}$).

(i) Show that the thermodynamic efficiency of the Brayton cycle is $\eta = 1 - \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}-1}$, and calculate its numerical value in this scenario. [5]

(ii) A Carnot cycle as shown in the figure below runs with the same pressure and volume values.



Find the thermodynamic efficiency of this Carnot cycle. [3]

(iii) You should have found that the efficiency of the Brayton cycle was higher than that of the Carnot cycle. Explain why the Brayton cycle can be more efficient than the Carnot cycle in this case. [2]

7.2 Solution

(i) The adiabatic steps contribute $Q = 0$, hence we can ignore them for efficiency calculations.

The isobaric step at P_1 contributes to Q_{out}, hence

$$Q_{\text{out}} = nc_{\text{p}}\Delta T = \frac{n\gamma R}{\gamma - 1}\Delta T = \frac{n\gamma R}{\gamma - 1}\left(\frac{P\Delta V}{nR}\right) = \frac{\gamma}{\gamma - 1}P_1\left(V_1 - V_2\right)$$

Using the adiabatic relation $PV^\gamma =$ constant, the volume endpoints for the isobaric step at P_2

are $\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_1$ and $\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_2$. The isobaric step at P_2 contributes to Q_{in}, hence

$$Q_{\text{in}} = \frac{\gamma}{\gamma - 1} P_2 \left(\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_2 - \left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_1 \right) = \frac{\gamma}{\gamma - 1} \frac{P_1^{\frac{1}{\gamma}}}{P_2^{\frac{1}{\gamma}-1}} (V_2 - V_1)$$

By definition, the thermodynamic efficiency is

$$\eta = 1 - \frac{|Q_{\text{out}}|}{|Q_{\text{in}}|} = 1 - \frac{P_1}{\frac{P_1^{\frac{1}{\gamma}}}{P_2^{\frac{1}{\gamma}-1}}} = 1 - \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}-1}$$

The numerical value is $\eta = 0.356$.

(ii) The minimum temperature here is

$$T_{\text{c}} = \frac{P_1 V_2}{nR}$$

while the maximum temperature here is

$$T_{\text{h}} = \frac{P_2 V_1}{nR}$$

By definition, the Carnot efficiency is

$$\eta_{\text{carnot}} = 1 - \frac{T_{\text{c}}}{T_{\text{h}}} = 1 - \frac{P_1 V_2}{P_2 V_1}$$

The numerical value is $\eta_{\text{carnot}} = 0.333$.

(iii) At first glance, it might seem like this violates the 2nd Law of Thermodynamics, which mandates the maximum efficiency to be achieved by the Carnot cycle. However, upon closer inspection, these 2 cycles aren't even running between the same hot and cold reservoirs! In fact, they have no points in common, except for (P_1, V_2). As such, it is possible for the Brayton efficiency to be higher than the Carnot efficiency when the reservoirs are different.

8 Question 8 [8]

8.1 Question

(i) Two radioactive species A and B are of the same initial amount. After 10 minutes, $N_A = 100N_B$. Given that twice the mean lifetime of A is equal to thrice the mean lifetime of B, find the mean lifetime of B. [4]

(ii) Prove that the de Broglie wavelength of a particle of rest mass m and kinetic energy K can be written as $\lambda = \frac{hc}{\sqrt{K(K+qmc^2)}}$, where q is a constant to be determined. [3]

(iii) The Compton wavelength is defined as $\lambda_{\text{c}} = \frac{h}{mc}$. For a particle with its kinetic energy being 3 times of its rest energy, find the value of $\frac{\lambda}{\lambda_{\text{c}}}$. [2]

8.2 Solution

(i) There is a trap here! **The mean lifetime is not the same as the half-life!** In fact, the mean lifetime is defined as $\tau = \frac{1}{\lambda}$, where λ is the decay constant.

As such, the amount of A and B at time t are given by

$$N_A(t) = N_0 e^{-\frac{t}{\tau_A}},\ N_B(t) = N_0 e^{-\frac{t}{\tau_B}}$$

At $t = 10\,\text{min}$, we have

$$N_0 e^{-\frac{10}{\tau_A}} = 100 N_0 e^{-\frac{10}{\tau_B}} \quad \Longrightarrow \quad \frac{10}{\tau_B} - \frac{10}{\tau_A} = \ln 100$$

We can substitute in $\tau_A = \frac{3}{2}\tau_B$ and solve the simultaneous equations, to get

$$\tau_B = \frac{10}{3 \ln 100}\,\text{min} = 43.4\,\text{s}$$

(ii) Using the relativistic dispersion relation and the definition of total energy,

$$E^2 = p^2c^2 + m^2c^4 \quad \Longrightarrow \quad \left(K + mc^2\right)^2 = p^2c^2 + m^2c^4 \quad \Longrightarrow \quad K^2 + 2Kmc^2 = p^2c^2$$

$$\Longrightarrow \quad p = \frac{\sqrt{K(K + 2mc^2)}}{c}$$

Hence, the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K + 2mc^2)}} \quad \Longrightarrow \quad q = 2$$

(iii) The particle has $K = 3mc^2$. Hence, substituting this into the expression for λ, we have $\lambda = \frac{h}{\sqrt{15}mc}$. As such, the desired ratio is

$$\frac{\lambda}{\lambda_c} = \frac{1}{\sqrt{15}}$$

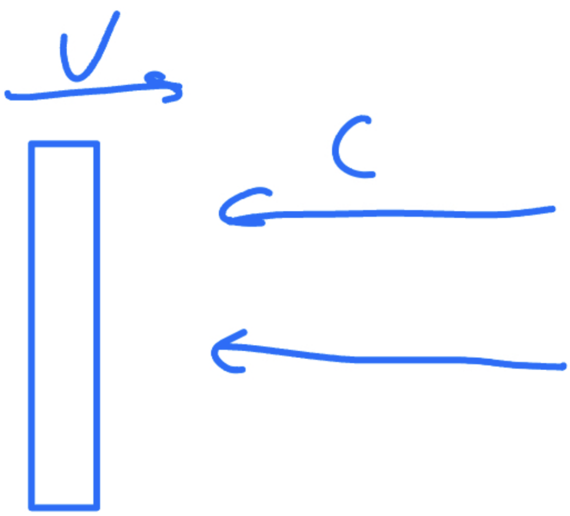
9 Question 9 [11]

9.1 Question

(i) State the Lorentz transformation of the 4-vector $(\omega, c\mathbf{k})$, where $|\mathbf{k}| = \frac{\omega}{c}$ is the wavevector/vector wavenumber of the wave. [2]

(ii) Prove that the quantity $\mathbf{k} \cdot \mathbf{r} - \omega t$ is Lorentz invariant, where $\mathbf{r}$ is position. [2]

A thin mirror moves at relativistic speed v to the right, and is bombarded by light rays going to the left. All light rays are incident normally on the mirror.



(iii) The frequency of the incoming photons is ω_{i} when measured in the lab frame. Find the angular frequency of the reflected photons ω_{r} in the lab frame. You may use $\beta = \frac{v}{c}$ to simplify your answer. [4] *Yes, ω here is defined as frequency, not angular frequency. Strange notation.*

(iv) Find the energy of one reflected photon. [1]

(v) The average incoming energy flux of photons is P_{i}. Find the average reflected energy flux P_{r}. [2]

9.2 Solution

(i) All 4-vectors transform in the same way as (ct, x, y, z). Hence, $(\omega, ck_x, ck_y, ck_z)$ transforms in the same way. Written in equation form (not sure if SPhO will accept matrix form), the LT is

$$\omega' = \gamma\left(\omega - \beta c k_x\right), \; k'_x = \gamma\left(k_x - \frac{\beta\omega}{c}\right), \; k'_y = k_y, \; k'_z = k_z$$

(ii) Recall that the inner product of two 4-vectors is Lorentz invariant. Consider the following inner product between the 4-displacement and the 4-wavevector:

$$k^\mu x_\mu = (\omega, c\mathbf{k}) \cdot (ct, \mathbf{r}) = -\omega ct + c\mathbf{k} \cdot \mathbf{r} = c\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right)$$

This quantity is Lorentz invariant, hence implying $\mathbf{k} \cdot \mathbf{r} - \omega t$ is Lorentz invariant.

(iii) This is a classic question that requires you to apply the relativistic longitudinal Doppler effect twice. The mirror receives a frequency

$$\omega_{\text{mirror,rec}} = \omega_{\mathrm{i}}\sqrt{\frac{1+\beta}{1-\beta}}$$

which is greater than ω_{i} since the mirror is moving towards the source.

The mirror then becomes the source now, and is moving towards the "observer" at speed v. Hence, the reflected photons have frequency

$$\omega_{\mathrm{r}} = \omega_{\text{mirror,rec}}\sqrt{\frac{1+\beta}{1-\beta}} = \omega_{\mathrm{i}}\left(\frac{1+\beta}{1-\beta}\right)$$

Alternative Solution. It is possible that given the high mark weightage of this part of the question, the markers may have anticipated a solution starting from first principles, using the previous parts rather than just citing the longitudinal Doppler effect. Fortunately, we can also do this part using the LT.

Let the mirror move in the $+x$ direction (hence the photons move in the $-x$ direction). Then, $k_x = -\frac{\omega_{\mathrm{i}}}{c}$, $k_y = 0$, $k_z = 0$. As such, the 4-wavevector for the incident photon in the lab frame is $k^\mu = (\omega_{\mathrm{i}}, ck_x, ck_y, ck_z) = (\omega_{\mathrm{i}}, -\omega_{\mathrm{i}}, 0, 0)$.

First, we transform from the lab frame to the mirror frame. This gives us

$$\omega' = \gamma\left(\omega_{\mathrm{i}} + \beta\omega_{\mathrm{i}}\right) = \gamma\omega_{\mathrm{i}}\left(1+\beta\right)$$

$$k'_x = \gamma\left(-\frac{\omega_{\mathrm{i}}}{c} - \frac{\beta\omega_{\mathrm{i}}}{c}\right) = -\frac{\gamma\omega_{\mathrm{i}}}{c}\left(1+\beta\right)$$

In the mirror frame, when reflection happens, the direction of propagation is flipped, but the frequency remains the same. As such, for the reflected photons in the mirror frame, we now have $\omega' = \gamma\omega_{\mathrm{i}}\left(1+\beta\right)$ and $k'_x = \frac{\gamma\omega_{\mathrm{i}}}{c}\left(1+\beta\right)$.

Now, we transform from the mirror frame back to the lab frame. Be careful when doing this – you need to use the inverse LT! This gives us the reflected frequency in the lab frame:

$$\omega_{\mathrm{r}} = \gamma\left(\omega' + \beta c k'_x\right) = \gamma\left(\gamma\omega_{\mathrm{i}}\left(1+\beta\right) + \beta\gamma\omega_{\mathrm{i}}\left(1+\beta\right)\right) = \gamma^2\left(1+\beta\right)^2\omega_{\mathrm{i}} = \omega_{\mathrm{i}}\left(\frac{1+\beta}{1-\beta}\right)$$

Remark. Since the question did not explicitly disallow the usage of the longitudinal Doppler effect, you should obtain full credit if you used it directly. However, starting from first principles is probably safer (and gives a use to the previous two parts).

(iv) Recall from quantum that the photon energy is given by $E = h\omega$. (Also recall the strange notation – ω here is frequency, not angular frequency.) Hence, the energy of a reflected photon is

$$E_{\mathrm{r}} = h\omega_{\mathrm{i}}\left(\frac{1+\beta}{1-\beta}\right)$$

(v) For the reflected photons, the energy per photon scales with an additional factor of $\frac{1+\beta}{1-\beta}$.

However, this is not complete! Due to the relative motion of the photons and the mirror, the rate of emission and rate of receiving of photons is not the same!

Let n_{i} be the rate of emission of photons in the lab frame. As the mirror is moving towards the photons, the rate of receiving is

$$n_{\text{mirror,rec}} = \left(1+\beta\right)n_{\mathrm{i}}$$

By similar logic, when the photons are reflected, the rate of reflecting is

$$n_{\text{mirror,ref}} = \left(1-\beta\right)n_{\mathrm{r}}$$

By symmetry of reflection, $n_{\text{mirror,rec}} = n_{\text{mirror,ref}}$. Hence, this means

$$n_{\mathrm{r}} = \frac{1+\beta}{1-\beta}n_{\mathrm{i}}$$

Combining these two effects (scaling of photon energy per photon and scaling of photon flux), we have

$$P_{\mathrm{r}} = \left(\frac{1+\beta}{1-\beta}\right)^2 P_{\mathrm{i}}$$

10 General Comments

Difficulty-wise, this tops the list for the hardest post-2017 SPhO. For context, SPhO 2017 and before was set by NUS, and SPhO 2018 and after are supposedly set by NTU. Amongst all the SPhOs supposedly set by NTU, SPhO 2025 definitely is the hardest so far.

Don't be too alarmed or worried if you weren't able to score anywhere as close to how you scored for any of the previous SPhOs (in your practices). The cut-offs are expected to drop by a lot due to the difficulty spike. Nobody knows how partials are awarded as well, so there is a good chance you would have underestimated your performance relative to others.

Again, these solutions are **unofficial,** and were originally produced by FIZIKA. If you spot any errors or any misinterpreted questions, please contact us. Thank you!