USA TSTST 2024 Solutions

United States of America — TST Selection Test

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Contents

§0 Problems

1. For every ordered pair of integers (i, j), not necessarily positive, we wish to select a point $P_{i,j}$ in the Cartesian plane whose coordinates lie inside the unit square defined by

$$i < x < i + 1, \qquad j < y < j + 1.$$

Find all real numbers $c > 0$ for which it's possible to choose these points such that for all integers i and j, the (possibly concave or degenerate) quadrilateral $P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$ has perimeter strictly less than c.

2. Let p be an odd prime number. Suppose P and Q are polynomials with integer coefficients such that $P(0) = Q(0) = 1$, there is no nonconstant polynomial dividing both P and Q, and

$$1 + \cfrac{x}{1 + \cfrac{2x}{\ddots \\ 1 + \cfrac{}{1 + (p-1)x}}} = \frac{P(x)}{Q(x)}.$$

Show that all coefficients of P except for the constant coefficient are divisible by p, and all coefficients of Q are *not* divisible by p.

3. Let $A = \{a_1, \ldots, a_{2024}\}$ be a set of 2024 pairwise distinct real numbers. Assume that there exist positive integers $b_1, b_2, \ldots, b_{2024}$ such that

$$a_1b_1 + a_2b_2 + \cdots + a_{2024}b_{2024} = 0.$$

Prove that one can choose $a_{2025}, a_{2026}, a_{2027}, \ldots$ such that $a_k \in A$ for all $k \geq 2025$ and, for every positive integer d, there exist infinitely many positive integers n satisfying

$$\sum_{k=1}^{n} a_k k^d = 0.$$

4. Let $ABCD$ be a quadrilateral inscribed in a circle with center O and E be the intersection of segments AC and BD. Let ω_1 be the circumcircle of ADE and ω_2 be the circumcircle of BCE. The tangent to ω_1 at A and the tangent to ω_2 at C meet at P. The tangent to ω_1 at D and the tangent to ω_2 at B meet at Q. Show that $OP = OQ$.

5. For a positive integer k, let $s(k)$ denote the number of 1s in the binary representation of k. Prove that for any positive integer n,

$$\sum_{i=1}^{n} (-1)^{s(3i)} > 0.$$

6. Determine whether there exists a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for all positive integers m and n,

$$f(m + nf(m)) = f(n)^m + 2024! \cdot m.$$

7. An infinite sequence a_1, a_2, a_3, ... of real numbers satisfies

$$a_{2n-1} + a_{2n} > a_{2n+1} + a_{2n+2} \qquad \text{and} \qquad a_{2n} + a_{2n+1} < a_{2n+2} + a_{2n+3}$$

for every positive integer n. Prove that there exists a real number C such that $a_n a_{n+1} < C$ for every positive integer n.

8. Let ABC be a scalene triangle, and let D be a point on side BC satisfying $\angle BAD = \angle DAC$. Suppose that X and Y are points inside ABC such that triangles ABX and ACY are similar and quadrilaterals $ACDX$ and $ABDY$ are cyclic. Let lines BX and CY meet at S and lines BY and CX meet at T. Prove that lines DS and AT are parallel.

9. Let $n \geq 2$ be a fixed integer. The cells of an $n \times n$ table are filled with the integers from 1 to n^2 with each number appearing exactly once. Let N be the number of unordered quadruples of cells on this board which form an axis-aligned rectangle, with the two smaller integers being on opposite vertices of this rectangle. Find the largest possible value of N.

§1 Solutions to Day 1

§1.1 TSTST 2024/1, proposed by Karthik Vedula

🔗 *Available online at* https://aops.com/community/p31006977.

Problem statement

For every ordered pair of integers (i, j), not necessarily positive, we wish to select a point $P_{i,j}$ in the Cartesian plane whose coordinates lie inside the unit square defined by

$$i < x < i + 1, \qquad j < y < j + 1.$$

Find all real numbers $c > 0$ for which it's possible to choose these points such that for all integers i and j, the (possibly concave or degenerate) quadrilateral $P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$ has perimeter strictly less than c.

¶ **Answer.** $c \geq 4$.

¶ **Proof $c < 4$ is not possible.** Let n be an arbitrary positive integer. We take an $n \times n$ subgrid of unit squares (i.e. $P_{i,j}$ for $1 \leq i, j \leq n$), and compute a lower bound on the average of all possible quadrilaterals from this subgrid.

Consider the average length of the "top side" of all possible quadrilaterals in this grid. Note that this is equal to:

$$\frac{1}{(n-1)^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{i,j}P_{i+1,j} \geq \frac{1}{(n-1)^2}\sum_{j=1}^{n-1} P_{1,j}P_{n,j} > \frac{n-2}{n-1}.$$

We can apply this bound to all four sides of the quadrilateral (the left, right, bottom, and top sides) to find that the average perimeter of all possible quadrilaterals is greater than

$$\frac{4(n-2)}{n-1} = 4 - \frac{4}{n-1}.$$

This means we can always find a quadrilateral whose perimeter is at least $4 - \frac{4}{n-1}$. By taking sufficiently large n, this lower bound will exceed c.

¶ **Proof $c = 4$ is possible.** We'll place point $P_{i,j}$ at the coordinates $(f(i), f(j))$ for some function $f\colon \mathbb{Z} \to \mathbb{R}$. The perimeter of $P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$ is then

$$2\big(|f(i+1) - f(i)| + |f(j+1) - f(j)|\big).$$

Therefore we have a valid construction for $c = 4$ if f satisfies $n < f(n) < n + 1$ and $|f(n+1) - f(n)| < 1$ for all n. This is achieved by

$$f(n) = n + 0.5 + \begin{cases} -\sum_{i=1}^{n} \frac{1}{10^i} & \text{if } n \geq 0, \\ \sum_{i=1}^{-n} \frac{1}{10^i} & \text{if } n < 0. \end{cases}$$

Let's check the conditions. The sum is bounded by $\sum_{i=1}^{\infty} \frac{1}{10^i} = \frac{1}{9}$ in magnitude, so $n < f(n) < n + 1$. Furthermore, we can verify that

$$f(n+1) - f(n) = 1 - \begin{cases} \frac{1}{10^{n+1}} & \text{if } n \geq 0, \\ \frac{1}{10^{-n}} & \text{if } n < 0 \end{cases} < 1.$$

§1.2 TSTST 2024/2, proposed by Andrew Gu

🔗 *Available online at* https://aops.com/community/p31006985.

Problem statement

Let p be an odd prime number. Suppose P and Q are polynomials with integer coefficients such that $P(0) = Q(0) = 1$, there is no nonconstant polynomial dividing both P and Q, and

$$1 + \cfrac{x}{1 + \cfrac{2x}{\ddots \\ 1 + \cfrac{}{1 + (p-1)x}}} = \frac{P(x)}{Q(x)}.$$

Show that all coefficients of P except for the constant coefficient are divisible by p, and all coefficients of Q are *not* divisible by p.

¶ Solution 1. We first make some general observations about rational functions represented through continued fractions.

Claim — Let $a_1, a_2, \ldots,$ be a sequence of nonzero integers. Define the sequence of polynomials $P_1(x) = 1$, $P_2(x) = 1 + a_1 x$, and

$$P_{k+1}(x) = P_k(x) + a_k x P_{k-1}(x)$$

for $k \geq 1$. Then the following properties hold for all $k \geq 0$:

- $P_k(0) = 1$,
- $\gcd(P_{k+1}, P_k) = 1$, and
- $\deg P_k = \lfloor k/2 \rfloor$.
- $$1 + \cfrac{a_k x}{1 + \cfrac{a_{k-1} x}{\ddots \\ 1 + \cfrac{}{1 + a_1 x}}} = \frac{P_{k+1}(x)}{P_k(x)}.$$

Proof. These all follow by induction. □

With the setup of the claim, the polynomials P and Q in the problem are exactly P_p and P_p for the sequence $a_1 = p - 1$, $a_2 = p - 2$, $\ldots$, $a_{p-1} = 1$. From here, we will define P and Q in terms of this recurrence and make two transformations on the a_i.

First, subtract p from each a_i so we now have $a_1 = -1$, $a_2 = -2$, $\ldots$, $a_{p-1} = -(p-1)$. The coefficients of P and Q only have changed by a multiple of p. The claim also shows that $P(0) = Q(0) = 1$, $\gcd(P, Q) = 1$, and the degrees of P and Q have not changed, i.e. no leading terms have been added or removed. Therefore it is equivalent to work with this sequence instead.

Similarly, we can multiply a_i by -1 to get $a_1 = 1$, $a_2 = 2$, $\ldots$, $a_{p-1} = p - 1$. This is equivalent to replacing x with $-x$, so some coefficients of P and Q are negated by this. The key conditions still remain unchanged.

Remark. For rigor, it's important to prove the facts about continued fractions first rather than going straight to the coefficient replacement as the requirements imposed by the problem statement on P and Q are not always preserved modulo p.

At this point, we are ready to directly calculate the polynomials P_n.

Claim — For all n, we have

$$P_n(x) = \sum_{k\geq 0} \frac{n!}{2^k k!(n-2k)!} x^k.$$

Here, we define $\frac{n!}{(n-2k)!} = n(n-1)\dots(n-2k+1)$ so the sum actually stops at $k = \lfloor n/2 \rfloor$.

Proof. We use induction. The base cases are clear, and

$$\begin{aligned}
P_n + nxP_{n-1} &= \sum_{k\geq 0} \frac{1}{2^k k!} \frac{n!}{(n-2k)!} x^k + nx \sum_{k\geq 0} \frac{1}{2^k k!} \frac{(n-1)!}{(n-2k-1)!} x^k \\
&= \sum_{k\geq 0} \frac{1}{2^k k!} \frac{n!}{(n-2k)!} x^k + \sum_{k\geq 0} \frac{1}{2^k k!} \frac{n!}{(n-2k-1)!} x^{k+1} \\
&= \sum_{k\geq 0} \frac{1}{2^k k!} \frac{n!}{(n-2k)!} x^k + \sum_{k\geq 1} \frac{1}{2^{k-1}(k-1)!} \frac{n!}{(n-2k+1)!} x^k \\
&= 1 + \sum_{k\geq 1} \left[\frac{1}{2^k k!} \frac{n!}{(n-2k)!} + \frac{1}{2^{k-1}(k-1)!} \frac{n!}{(n-2k+1)!} \right] x^k \\
&= 1 + \sum_{k\geq 1} \frac{1}{2^k k!} \frac{n!}{(n-2k+1)!} \left[(n-2k+1) + 2k \right] x^k \\
&= 1 + \sum_{k\geq 1} \frac{1}{2^k k!} \frac{(n+1)!}{(n-2k+1)!} x^k \\
&= \sum_{k\geq 0} \frac{1}{2^k k!} \frac{(n+1)!}{(n-2k+1)!} x^k \\
&= P_{n+1}.
\end{aligned}$$

□

At this point we can directly check the coefficients of P and Q. We have

$$P(x) = P_p(x) = \sum_{k\geq 0} \frac{p!}{2^k k!(p-2k)!} x^k.$$

For $k = 0$, we get a coefficient of 1. For $k \geq 1$, the denominator is not a multiple of p, so the remaining coefficients are multiples of p. Meanwhile, the coeffiicents for Q are $\frac{(p-1)!}{2^k k!(p-1-2k)!}$. None of these are divisible by p because $(p-1)!$ is not divisible by p.

Remark. This problem was created from the identity

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{3x}{1-\dots}}}} = \sum_{i=0}^{\infty}(2i-1)!!x^i.$$

You can use this to solve the problem. However, the proof of the identity as well as the details to convert this into the given problem are very lengthy so this is only left as a remark.

Lemma

Define a series of polynomials A_n, B_n, C_n, D_n by letting $A_1 = 1$, $B_1 = x$, $C_1 = 1$, and $D_1 = 0$, and for $n \geq 2$,

$$A_n = A_{n-1} + B_{n-1} \qquad B_n = nxA_{n-1}$$
$$C_n = C_{n-1} + D_{n-1} \qquad D_n = nxC_{n-1}.$$

Then, if we define $f_k(t) = 1 + \frac{kx}{t}$, then

$$(f_1 \circ f_2 \circ \cdots \circ f_n)(t) = \frac{A_n t + B_n}{C_n t + D_n}.$$

¶ Solution 2.

Proof. Straightforward by induction: the $n = 1$ case is true and for $n \geq 2$,

$$\frac{A_{n-1}f_n(t) + B_{n-1}}{C_{n-1}f_n(t) + D_{n-1}} = \frac{A_{n-1}(t+nx) + B_{n-1}t}{C_{n-1}(t+nx) + D_{n-1}t} = \frac{(A_{n-1}+B_{n-1})t + nxA_{n-1}}{(C_{n-1}+D_{n-1})t + nxD_{n-1}}. \qquad \square$$

It is possible to solve these recurrences algebraically, but there is in fact a nice combinatorial interpretation of these polynomials:

Lemma

Let $\mathcal{I}_n$ be the set of involutions on $\{1, 2, \dots, n\}$ and for $\pi \in \mathcal{I}_n$, let $a(\pi)$ denote the number of transpositions in π. Let $\mathcal{J}_n \subseteq \mathcal{I}_n$ be the set of involutions π where the minimal fixed point is less than the minimal x with $\pi(x) < x$. Then, for all $k \geq 1$, we have

$$A_n = \sum_{\pi \in \mathcal{I}_n} x^{a(\pi)} \qquad B_n = \sum_{\substack{\pi \in \mathcal{I}_{n+1} \\ \pi(n+1) \neq n+1}} x^{a(\pi)}$$
$$C_n = \sum_{\pi \in \mathcal{J}_n} x^{a(\pi)} \qquad D_n = \sum_{\substack{\pi \in \mathcal{J}_{n+1} \\ \pi(n+1) \neq n+1}} x^{a(\pi)}.$$

Proof. We use induction on n, with the case $n = 1$ being easily verified. For $n \geq 2$, note that

$$A_n - B_{n-1} = \sum_{\substack{\pi \in \mathcal{I}_{k+1} \\ \pi(k+1) = k+1}} x^{a(\pi)} = A_{n-1}.$$

Moreover, every $\pi \in \mathcal{I}_{n+1}$ swapping $n+1$ with something else can be characterized by one of n choices for $\pi(n+1)$ and an involution on the remaining $n-1$ elements. Therefore $B_n = nxA_{n-1}$. A similar proof holds for the C_n and D_n. □

We now note that

$$\frac{P(x)}{Q(x)} = (f_1 \circ f_2 \circ \cdots \circ f_{p-1})(1) = \frac{A_{p-1} + B_{p-1}}{C_{p-1} + D_{p-1}} = \frac{A_p}{C_p}.$$

It suffices to show that all nonconstant coefficients of A_p are divisible by p and that all coefficients of C_p are not divisible by p. If we show this, then any nonconstant factor dividing A_p must have a leading coefficient divisible by p (proof: reduce the factorization mod p), then any nonconstant factor dividing C_p cannot have a leading coefficient divisible by p. Moreover, since A_p and C_p have constant terms of 1, they are relatively prime and thus $P = A_p$ and $Q = C_p$ (up to sign).

We now evaluate the coefficients of A_p and C_p. We observe that

$$A_p = \sum_{k=0}^{\frac{p-1}{2}} \binom{p}{2k} (2k-1)!! x^k,$$

and it is easy to see that $A_p \equiv 1 \pmod{p}$. Computing the coefficients of C_n is a bit harder. Consider elements of $\mathcal{J}_p$ with $0 \le k \le \frac{p-1}{2}$ swaps and a minimal fixed point of $a+1$ (for $0 \le a \le k$). Everything in $\{1, \dots, a\}$ must be paired with something greater than $a+1$, and there are $k-a$ swaps among the remaining elements. It follows that

$$\begin{aligned}
C_p &= \sum_{k=0}^{\frac{p-1}{2}} \sum_{a=0}^{k} \frac{(p-a-1)!}{(p-2a-1)!} \binom{p-2a-1}{2k-2a} (2k-2a-1)!! \cdot x^k \\
&= \sum_{k=0}^{\frac{p-1}{2}} \sum_{a=0}^{k} \frac{(p-a-1)!}{(p-2k-1)! 2^{k-a} (k-a)!} x^k \\
&= \sum_{k=0}^{\frac{p-1}{2}} \frac{(p-1)!}{(p-2k-1)!} \left(\sum_{a=0}^{k} \frac{1}{2^{k-a}(k-a)! \prod_{j=1}^{a}(p-j)} \right) x^k \\
&\equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{(p-1)!}{(p-2k-1)!} \left(\sum_{a=0}^{k} \frac{(-1)^a}{2^{k-a}(k-a)!a!} \right) x^k \\
&= \sum_{k=0}^{\frac{p-1}{2}} \frac{(p-1)!}{(p-2k-1)!k!} \left(\sum_{a=0}^{k} \binom{k}{a} \frac{(-1)^a}{2^{k-a}} \right) x^k \\
&= \sum_{k=0}^{\frac{p-1}{2}} \frac{(p-1)!}{(p-2k-1)!k!} \left(-1 + \frac{1}{2} \right)^k x^k.
\end{aligned}$$

The result follows.

Remark. It is not hard to see that C_p simplifies further mod p to

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k (2k-1)!! x^k.$$

§1.3 TSTST 2024/3, proposed by Daniel Zhu

🔗 *Available online at* https://aops.com/community/p31006991.

Problem statement

Let $A = \{a_1, \dots, a_{2024}\}$ be a set of 2024 pairwise distinct real numbers. Assume that there exist positive integers $b_1, b_2, \dots, b_{2024}$ such that

$$a_1 b_1 + a_2 b_2 + \cdots + a_{2024} b_{2024} = 0.$$

Prove that one can choose $a_{2025}, a_{2026}, a_{2027}, \dots$ such that $a_k \in A$ for all $k \geq 2025$ and, for every positive integer d, there exist infinitely many positive integers n satisfying

$$\sum_{k=1}^{n} a_k k^d = 0.$$

It will be convenient to use 0-based indexing here, i.e. $A = \{a_0, \dots, a_{2023}\}$ and so on. Let $m = \sum_{i=0}^{2023} b_i$. By appending $b_i - 1$ copies of a_i for each i, we may extend the sequence to $a_0, \dots, a_{m-1}$ such that $a_0 + \cdots + a_{m-1} = 0$.

¶ Solution by direct construction. Let $s_m(n)$ be the sum of the base-m digits of n, reduced modulo m. We now claim that $a_k = a_{s_m(k)}$ works. It is clear that this extends the original sequence in a valid way.

> **Claim —** Fix d. Then the sum
>
> $$\sum_{k=0}^{n-1} a_k (k+1)^d$$
>
> is zero when $n = cm^{d+1}$ for any positive integer c.

If we prove this, then the problem is solved as there are infinitely many possible c. We have two proofs of this.

Proof of claim by expansion (Andrew Gu). We only need to show that each block of m^{d+1} terms sums to zero. Each block takes the form

$$S = \sum_{k=cm^{d+1}}^{(c+1)m^{d+1}-1} a_{s_m(k)} (k+1)^d.$$

We group terms based on the value of $s_m(k)$. The values of k in this sum are given by

$$\left\{ cm^{d+1} + \sum_{i=0}^{d} e_i m^i \mid (e_0, \dots, e_d) \in \{0, \dots, m-1\}^{d+1} \right\}.$$

In this form, the sum of the base-m digits is $s_m(c) + e_0 + \cdots + e_d$. Therefore we can write

$$S = \sum_{i=0}^{m-1} a_i \underbrace{\left(\sum_{\substack{(e_0,\dots,e_d) \in \{0,\dots,m-1\}^{d+1} \\ e_0 + \cdots + e_d \equiv i - s_m(c) \pmod m}} \underbrace{\left(cm^{d+1} + 1 + \sum_{j=0}^{d} e_j m^j \right)^d}_{(2)} \right)}_{(1)}.$$

Since $\sum a_i = 0$, it suffices to show that the value of the coefficient given by (1) is independent of i.

Expand the term (2) with the multinomial theorem. Each monomial in the expansion is a product of terms c, m, e_j. Since the exponent is d, no monomial is divisible by the whole product $e_0e_1 \cdots e_d$. Since the tuples $(e_0, \dots, e_d)$ are uniformly distributed over all possible tuples when one or more coordinates are removed, that means that the sum of each monomial term is independent of i. We conclude that the sum (1) is independent of i as well. □

Proof of claim by Fourier transform. Let $\omega \neq 1$ be an m^{th} root of unity. Then, define the operator

$$\Delta_a^{(\omega)} f(x) = \sum_{i=0}^{m-1} \omega^i f(x + ia).$$

By looking at the leading coefficients, one can see that $\Delta_a^{(\omega)}$ lowers the degree of a polynomial by at least 1.[1] As a consequence, by letting $f(x) = (x+1)^d$ we conclude that

$$\sum_{k \in [0, m_e)} \omega^{s_m(k)} (k+1)^d = (\Delta_{m^{e-1}}^{(\omega)} \cdots \Delta_m^{(\omega)} \Delta_1^{(\omega)} f)(0) = 0.$$

Since $a_1 + \cdots + a_m = 0$, by a discrete Fourier transform we may write $a_i = \sum_{j=1}^{m-1} c_j e^{2\pi ij/m}$ for some $c_1, \dots, c_{m-1} \in \mathbb{C}$. Therefore

$$\sum_{k=0}^{m^e-1} a_k (k+1)^d = \sum_{j=1}^{m-1} c_j \sum_{k=0}^{m^e-1} (e^{2\pi ij/m})^{s_m(k)} (k+1)^d = 0.$$ □

¶ Inductive solution. Define m and extend the sequence to a_{m-1} as before. For $d \geq 0$, let $\Sigma_d(S)$ denote $\sum_{k \in S} k^d$. Also, for $d \geq 0$ call a tuple of sets $(S_0, \dots, S_{m-1})$ to be *d-uniform* if for all $0 \leq d' \leq d$ we have $\Sigma_{d'}(S_0) = \Sigma_{d'}(S_1) = \cdots = \Sigma_{d'}(S_{m-1})$.

Lemma 1.1

Suppose $(S_0, \dots, S_{m-1})$ is d-uniform. Then for all $0 \leq d' \leq d+1$, $x \in \mathbb{R}$, and $0 \leq i, j < m$, the quantity $\Sigma_{d'}(S_i + x) - \Sigma_{d'}(S_j + x)$ is independent of x.

Proof. We may expand

$$\Sigma_{d'}(S_i + x) = \sum_{k=0}^{d'} \binom{d'}{k} x^{d'-k} \Sigma_k(S_i).$$

By d-uniformity, the terms with $k \leq d$ are independent of i and go away in the subtraction. The only term left is when $k = d' = d+1$. However, this term is independent of x, as desired. □

Lemma 1.2

Suppose $(S_0, \dots, S_{m-1})$ is a d-uniform partition of $[0, n)$. Then there exists a $(d+1)$-uniform partition $(S_0', \dots, S_{m-1}')$ of $[0, mn)$ such that $S_i \subseteq S_i'$ for all i.

[1] In fact the degree decreases by exactly 1, but this is not that important.

Proof. We let $S_i' = \bigcup_{k=0}^{m-1}(S_{(i+k) \bmod m} + kn)$. Then, for every $0 \le d' \le d+1$ and $0 \le i, j < m$, we have

$$\begin{aligned}\Sigma_{d'}(S_i') - \Sigma_{d'}(S_j') &= \sum_{k=0}^{m-1} \Sigma_{d'}(S_{(i+k) \bmod m} + kn) - \Sigma_{d'}(S_{(j+k) \bmod m} + kn) \\ &= \sum_{k=0}^{m-1} \Sigma_{d'}(S_{(i+k) \bmod m}) - \Sigma_{d'}(S_{(j+k) \bmod m}) \\ &= 0.\end{aligned}$$

□

Consider the 0-uniform partition of $[0, m)$ given by $(\{0\}, \{1\}, \dots, \{m-1\})$. By repeatedly applying Lemma 1.2, we get d-uniform partitions $(S_0^{(d)}, \dots, S_{m-1}^{(d)})$ of $[0, m^{d+1})$ such that $S_i^{(d)} \subseteq S_i^{(d+1)}$ for all d and i. For every k, let i be the unique index such that $k \in S_i^{(d)}$ for some d, and let $a_k = a_i$. (This doesn't redefine a_0, …, a_{m-1} as for $k < m$, we have $k \in S_k^{(0)}$.) Then, for each $e \ge d$, we find that

$$\sum_{k=0}^{m^{e+1}-1} a_k (k+1)^d = \sum_{i=0}^{m-1} a_i \sum_{k \in S_i^{(e)}} (k+1)^d = \sum_{i=0}^{m-1} a_i \sum_{j=1}^{d} \binom{d}{j} \Sigma_j(S_i^{(e)}) = 0.$$

Remark. The construction here is essentially the same as the sum-of-digits construction (in fact it would be exactly the same if we replace $(i+k) \bmod m$ in the proof of Lemma 1.2 with $(i-k) \bmod m$). However, it hints at some flexibility in the construction which is less apparent with other approaches. For example, if G is any transitive subgroup of permutations on $[0, m)$ and $\pi_0 = \mathrm{id}$, π_1, …, $\pi_{|G|-1}$ is an enumeration of the elements of G, one can turn a d-uniform partition $(S_0, \dots, S_{m-1})$ of $[0, n)$ into a $(d+1)$-uniform partition $(S_0', \dots, S_{m-1}')$ of $[0, |G|\, n)$ by letting $S_i' = \bigcup_{k=0}^{|G|-1}(S_{\pi_k(i)} + kn)$. Here we used the special case $\pi_k(i) = (i+k) \bmod m$; it may also be natural to let G be the set of all $m!$ permutations.

Remark. The construction in the solution has appeared a number of times in the literature; see

- D. H. Lehmer, The Tarry-Escott problem, *Scripta Math.* **13**, 37–41, 1947;
- E. Prouhet, Mémoire sur quelques relations entre les puissances des nombres, *C. R. Acad. Sci. Paris* **33**, 225, 1851;
- E. M. Wright, Equal sums of like powers, *Proc. Edinburgh Math. Soc.* 2nd series **8**, 138–142, 1949;
- E. M. Wright, Prouhet's 1851 solution of the Tarry-Escott problem of 1910, *Amer. Math. Monthly* **66**, 199–201, 1959.

§2 Solutions to Day 2

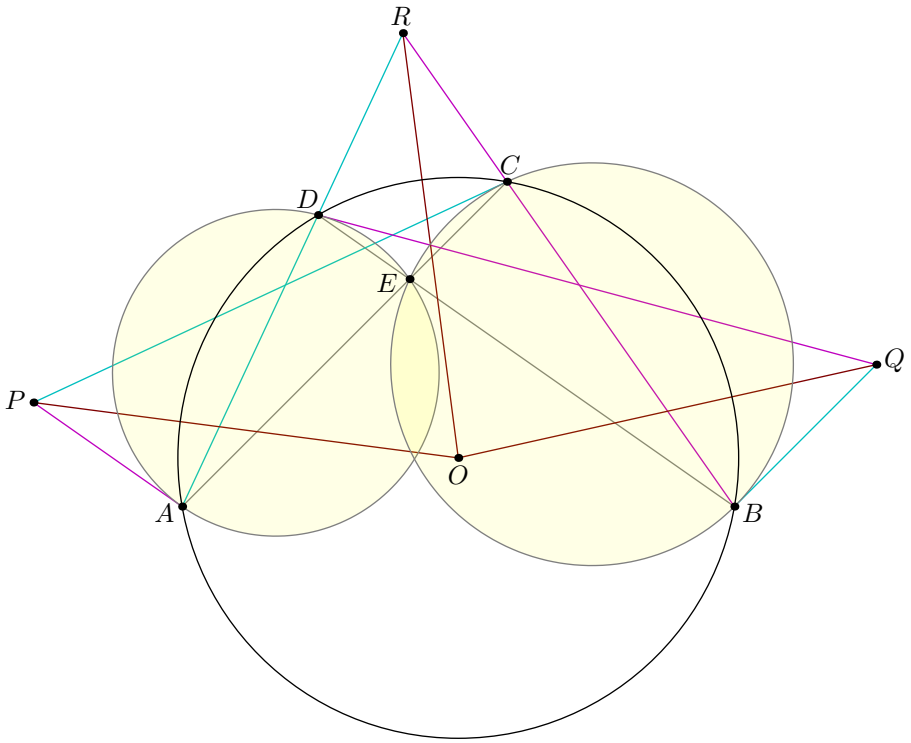
§2.1 TSTST 2024/4, proposed by Merlijn Staps

🔗 *Available online at* https://aops.com/community/p31007004.

Problem statement

Let $ABCD$ be a quadrilateral inscribed in a circle with center O and E be the intersection of segments AC and BD. Let ω_1 be the circumcircle of ADE and ω_2 be the circumcircle of BCE. The tangent to ω_1 at A and the tangent to ω_2 at C meet at P. The tangent to ω_1 at D and the tangent to ω_2 at B meet at Q. Show that $OP = OQ$.

¶ Solution 1. Let $R = \overline{AD} \cap \overline{BC}$ (possibly at infinity, but we'll see it's an Euclidean point later).



Claim — $ACRP$ is an isosceles trapezoid with $\overline{AC} \parallel \overline{PR}$. Consequently, R is an Euclidean point, and $OP = OR$.

Proof. It is equivalent to show that $\triangle PAC \mathrel{\bar{\cong}} \triangle RCA$, or without using R, that

$$\measuredangle PAC = \measuredangle ACR = \measuredangle ACB, \quad \text{and} \quad \measuredangle PCA = \measuredangle CAR = \measuredangle CAD.$$

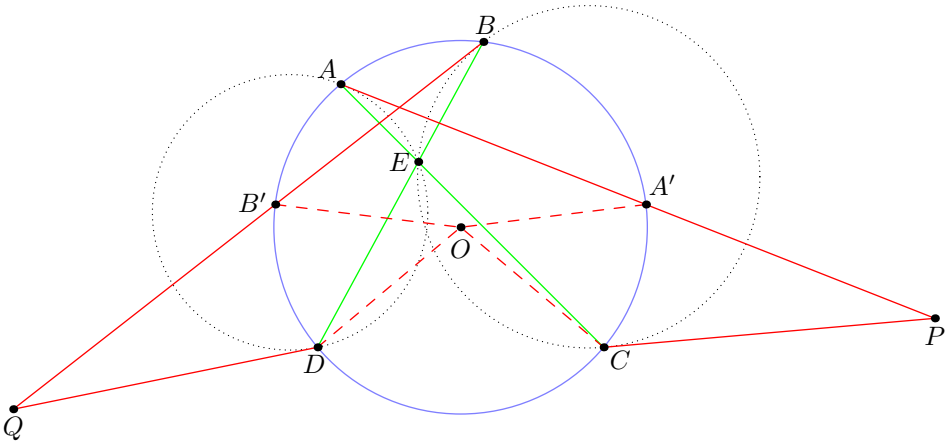
Indeed, we have

$$\measuredangle PAC = \measuredangle ADE = \measuredangle ADB = \measuredangle ACB$$

and likewise $\measuredangle PCA = \measuredangle CAD$, as requested. □

Similarly $OQ = OR$, so we're done.

¶ Solution 2.



Let PA intersect $(ABCD)$ again at A' and let QB intersect $(ABCD)$ again at B'. The problem follows from the following claim.

> **Claim —** $OA'PC \cong ODQB'$.

Proof. We use directed angles mod π. Note that

$$\measuredangle ODQ = \measuredangle ODA + \measuredangle ADQ = \measuredangle DAO + \measuredangle PAD = \measuredangle PAO = \measuredangle OA'P,$$

and similarly $\measuredangle OCP = \measuredangle OB'Q$. Furthermore,

$$\measuredangle A'AD = \measuredangle AED = \measuredangle CEB = \measuredangle CBB',$$

so $A'D = CB'$, i.e. $A'CDB'$ is an isosceles trapezoid. This means $\measuredangle A'OC = \measuredangle DOB'$. Combined with $OA' = OB' = OD = DC$, these three angle equalities imply the congruency. □

¶ Solution 3 using complex numbers (Noah Walsh). Use complex numbers with $(ABCD)$ as the unit circle. Because PA is tangent to (ADE), we have $\angle PAC = \angle ADB = \frac{1}{2}\angle AOB$, and therefore

$$\begin{aligned} \frac{\frac{p-a}{c-a}}{\frac{\overline{p}-a^{-1}}{c^{-1}-a^{-1}}} &= \frac{a}{b} \\ \implies b\frac{p-a}{c-a} &= a\frac{ac\overline{p}-c}{a-c} \\ \implies bp + a^2c\overline{p} &= ab + ac. \end{aligned}$$

The condition that PC is tangent to (BCE) can be expressed by swapping a with c and b with d, so we get

$$\begin{aligned} bp + a^2c\overline{p} &= ab + ac \\ dp + ac^2\overline{p} &= cd + ac \\ \implies p(ad-bc) &= acd + a^2c - abc - ac^2 \\ \implies p &= \frac{ac(a+d-b-c)}{ad-bc}. \end{aligned}$$

Similarly,

$$q = \frac{bd(a+d-b-c)}{ad-bc}.$$

It is now obvious that p and q have the same magnitude.

> **Remark.** One possible trick one might use to compute p is to compute the coordinates of the points $A', C' \in (ABCD)$ such that AA' and CC' are tangent to (ADE) and (BCE) respectively, and then use the chords intersection formula.
> If one does this, one gets $a' = ac/b$ and $c' = ac/d$. At this point, one can simply recognize that A' and C' are the reflection of B and D over the perpendicular bisector of AC, reflect everything over said perpendicular bisector, and suddenly realize that one has solved the problem synthetically.

¶ Solution 4 by linearity (Noah Walsh). If $AE = DE$ the problem is true by symmetry, so we assume $AE \neq DE$.

Fix A, D, and E, and move B linearly along line DE. By e.g. Reim's theorem, C moves linearly as well.

How do we compute O? It lies on the perpendicular bisector of AD, which is fixed. It also lies on the perpendicular bisector of BC, which is parallel to a fixed line, and passes through the midpoint of BC, which moves linearly. Therefore, O moves linearly as well.

Point P is the intersection of the tangent to (ADE) at A, which is fixed, and the tangent to (BCE) at C, which moves linearly, and therefore is linear. Similarly, Q moves linearly. Therefore, to verify that $\overrightarrow{OP} \times \overrightarrow{OP} = \overrightarrow{OQ} \times \overrightarrow{OQ}$, it suffices to check three cases.

The two cases where $EA = EB$ are trivial by symmetry. We can also check that $\lim_{B \to \infty_{DE}} \frac{\overrightarrow{OP} \times \overrightarrow{OP}}{\overrightarrow{OQ} \times \overrightarrow{OQ}} = 1$.

Indeed, obviously $\lim_{B \to \infty_{DE}} \frac{|OP|}{|AP|} = \lim_{B \to \infty_{DE}} \frac{|OQ|}{|DQ|} = 1$. In addition,

$$\begin{aligned}
\frac{dAP}{dDQ} &= \frac{dAP}{dAC} \times \frac{dAC}{dBD} \times \frac{dBD}{dBQ} \\
&= \frac{\sin \angle ACP}{\sin \angle APC} \times \frac{dEC}{dEB} \times \frac{\sin \angle BQD}{\sin \angle BDQ} \\
&= \frac{\sin \angle ACP}{\sin(\angle ACP + \angle CAP)} \times \frac{\sin \angle EBC}{\sin \angle ECB} \times \frac{\sin(\angle BDQ + \angle DBQ)}{\sin \angle BDQ} \\
&= \frac{\sin \angle BCE}{\sin(\angle BCE + \angle ADE)} \times \frac{\sin \angle DAE}{\sin \angle ADE} \times \frac{\sin(\angle DAE + \angle ECB)}{\sin \angle EAD} \\
&= \frac{\sin(\angle DAE + \angle ECB)}{\sin(\angle BCE + \angle ADE)} \\
&= 1.
\end{aligned}$$

It follows that $\lim_{B \to \infty_{DE}} \frac{|AP|}{|DQ|} = 1$. Therefore,

$$\begin{aligned}
\lim_{B \to \infty_{DE}} \frac{|OP|}{|OQ|} &= \lim_{B \to \infty_{DE}} \frac{|OP|}{|AP|} \times \lim_{B \to \infty_{DE}} \frac{|AP|}{|DQ|} \times \lim_{B \to \infty_{DE}} \frac{|DQ|}{|OQ|} \\
&= 1,
\end{aligned}$$

which completes the $B \to \infty_{DE}$ case, and we are done.

§2.2 TSTST 2024/5, proposed by Holden Mui

🔗 *Available online at* https://aops.com/community/p31007010.

Problem statement

For a positive integer k, let $s(k)$ denote the number of 1s in the binary representation of k. Prove that for any positive integer n,

$$\sum_{i=1}^{n}(-1)^{s(3i)} > 0.$$

¶ Solution 1. Given a set of positive integers S, define

$$f(S) = \sum_{k\in S}(-1)^{s(k)}.$$

We also define

$$S_{\text{even}} = \{k \in S \mid k \text{ is even}\}$$
$$S_{\text{odd}} = \{k \in S \mid k \text{ is odd}\}$$

and apply functions on sets pointwise, e.g.

$$\frac{S-1}{2} = \left\{\frac{k-1}{2} \mid k \in S\right\}.$$

The problem follows from the first bullet of the following claim.

Claim — For every positive integer n,

- $f(\{3, 6, \ldots, 3n\}) > 0$
- $f(\{1, 4, \ldots, 3n-2\}) < 0$
- $f(\{2, 5, \ldots, 3n-1\}) \le 0$

Proof. Induct on n. The base case $n = 1$ is easy to check.

- For $S = \{3, 6, \ldots, 3n\}$,

$$\begin{aligned} f(S) &= f(S_{\text{even}}) + f(S_{\text{odd}}) \\ &= f(S_{\text{even}}/2) - f((S_{\text{odd}} - 1)/2) > 0 \end{aligned}$$

  since the elements of $S_{\text{even}}/2$ are 0 mod 3 and the elements of $(S_{\text{odd}} - 1)/2$ are 1 mod 3.

- For $S = \{1, 4, \ldots, 3n-2\}$,

$$\begin{aligned} f(S) &= f(S_{\text{even}}) + f(S_{\text{odd}}) \\ &= f(S_{\text{even}}/2) - f((S_{\text{odd}} - 1)/2) < 0 \end{aligned}$$

  since the elements of $S_{\text{even}}/2$ are 2 mod 3 and the elements of $(S_{\text{odd}} - 1)/2$ are 0 mod 3. (Note that $(S_{\text{odd}} - 1)/2$ has multiples of 3 starting at 0 rather than 3, but this is fine because $f(\{3, \ldots, 3n\}) > 0$ clearly implies $f(\{0, \ldots, 3n\}) > 0$ as well.)

- For $S = \{2, 5, \ldots, 3n-1\}$, there are two cases.
  - Case 1: $2^{2k-1} \le 3n-1 < 2^{2k}$. Define
$$S_{\text{small}} = \{s \in S \mid s < 2^{2k-1}\}$$
$$S_{\text{big}} = \{s \in S \mid s \ge 2^{2k-1}\}.$$
Then
$$\begin{aligned} f(S) &= f(S_{\text{small}}) + f(S_{\text{big}}) \\ &= f(S_{\text{small}}) - f(S_{\text{big}} - 2^{2k-1}) < 0 \end{aligned}$$
since $S_{\text{big}} - 2^{2k-1} = \{0, \ldots, 3n-1-2^{2k-1}\}$ consists of a prefix of the $0 \pmod 3$ positive integers in addition to 0.
  - Case 2: $2^{2k} < 3n-1 < 2^{2k+1}$. Note that if $a+b = 2^{2k+1}-1$, then $f(\{a,b\}) = 0$. If we use this to cancel out all pairs of numbers in S which add to $2^{2k+1}-1$, then we find
$$f(S) = f(\{2, 5, \ldots, 2^{2k+1} - 3n - 2\}) \le 0.$$

This completes the induction. □

¶ Solution 2. Let
$$f(n) = \sum_{i=0}^{n} (-1)^{s(3i)}, \qquad g(n) = \sum_{i=0}^{n} (-1)^{s(i)}.$$
Note that the problem is equivalent to showing $f(n) \ge 2$ for all $n \ge 1$.

Lemma

$|g(n)| \le 1$ for all n.

Proof. Note that $s(2i+1) = -s(2i)$, so $g(2n+1) = 0$ and $g(2n) = (-1)^{s(2n)}$ by pairing terms. □

We will now prove that $f(n) \ge 2$ for all $n \ge 1$ by strong induction. We can manually verify $n \le 5$ as a base case. For the inductive step, let $m = \lfloor \frac{3n-3}{4} \rfloor$ so that $4m+3 \le n$. Let $S = \{0, 3, \ldots, 3n\}$. Given $0 \le k \le m$, we can append two digits to the binary representation of k (equivalently, $4k+r$ for $0 \le r < 4$) to get a number in S. If k is a multiple of 3, we can append 00 or 11. Otherwise, we can append one of 01 or 10 depending on $k \pmod 3$. Almost all elements of S can be covered uniquely in this way, except for at most one missing element. Note that appending 00 or 11 will preserve the parity of the number of 1s while appending 01 or 10 will change the parity. Therefore we can write
$$f(n) \ge 2 \sum_{\substack{0 \le i \le m \\ i \equiv 0 \pmod 3}} (-1)^{s(i)} - \sum_{\substack{0 \le i \le m \\ i \not\equiv 0 \pmod 3}} (-1)^{s(i)} - 1$$
(the -1 is to account for the possible missing element). Let
$$x = \sum_{\substack{0 \le i \le m \\ i \equiv 0 \pmod 3}} (-1)^{s(i)}, \qquad y = \sum_{\substack{0 \le i \le m \\ i \not\equiv 0 \pmod 3}} (-1)^{s(i)}.$$
According to the lemma, $|x+y| \le 1$. By the inductive hypothesis, $x \ge 2$. Therefore
$$f(n) \ge 2x - y - 1 = 3x - (x+y) - 1 \ge 6 - 1 - 1 > 2$$
as desired.

¶ Solution 3. For $r \in \{0,1,2\}$, define

$$S_r(a,b) = \sum_{\substack{i \in [a,b) \\ i \equiv r \bmod 3}} (-1)^{s(i)},$$

where $s(i)$ denotes the number of 1s in i's binary representation. The problem is equivalent to proving $S_0(0, 3n+1) > 1$ for all $n \geq 1$.

Lemma

For any integer d, $S_r(0, 2^d)$ is given by:

d	$S_0(0,2^d)$	$S_1(0,2^d)$	$S_2(0,2^d)$
d odd	$3^{\frac{d-1}{2}}$	$-3^{\frac{d-1}{2}}$	0
d even	$2 \cdot 3^{\frac{d-2}{2}}$	$-3^{\frac{d-2}{2}}$	$-3^{\frac{d-2}{2}}$

Proof. Induction with $d \leq 2$ being clear. For the inductive step, observe that

$$S_0(0,2^d) = S_0(0,2^{d-1}) + S_0(2^{d-1},2^d) = S_0(0,2^{d-1}) - S_{2^{d-1}}(0,2^{d-1})$$
$$S_1(0,2^d) = S_1(0,2^{d-1}) + S_1(2^{d-1},2^d) = S_1(0,2^{d-1}) - S_{1+2^{d-1}}(0,2^{d-1})$$
$$S_2(0,2^d) = S_2(0,2^{d-1}) + S_2(2^{d-1},2^d) = S_2(0,2^{d-1}) - S_{2+2^{d-1}}(0,2^{d-1}).$$

Using

$$2^{d-1} \equiv \begin{cases} 1 \bmod 3 & d \text{ odd} \\ 2 \bmod 3 & d \text{ even} \end{cases}$$

and applying the inductive hypothesis gives the desired result. □

This proves the problem for powers of 2. To prove the problem for general n, split $[0, 3n+1)$ into blocks

$$[0,3n+1) = [0,2^{d_1}) \sqcup [2^{d_1}, 2^{d_1}+2^{d_2}) \sqcup [2^{d_1}+2^{d_2}, 2^{d_1}+2^{d_2}+2^{d_3}), \ldots$$

whose lengths are decreasing powers of 2. By the lemma,

$$S_0(0,2^{d_1}) \geq \begin{cases} 3^{\frac{d_1-1}{2}} & d_1 \text{ odd} \\ 2 \cdot 3^{\frac{d_1-2}{2}} & d_1 \text{ even} \end{cases}$$
$$S_0(2^{d_1}, 2^{d_1}+2^{d_2}) = -S_{2^{d_1}}(0,2^{d_2}) \geq 0$$

and for all i,

$$S_0(2^{d_1}+\cdots+2^{d_{i-1}}, 2^{d_1}+\cdots+2^{d_i}) = (-1)^{i-1} S_{2^{d_1}+\cdots+2^{d_{i-1}}}(0,2^{d_i})$$
$$\geq \begin{cases} -2 \cdot 3^{\frac{d_i-2}{2}} & d_i \text{ even} \\ -1 \cdot 3^{\frac{d_i-1}{2}} & d_i \text{ odd.} \end{cases}$$

Let $D = d_1$; assume $D \geq 4$ since checking $D \in \{1,2,3\}$ is easy. Summing over all intervals gives

$$S_0(0,3n+1) \geq 2 \cdot 3^{\frac{D-2}{2}} + 0 - \left(2 \cdot 3^{\frac{D-4}{2}} + 3^{\frac{D-4}{2}} + 2 \cdot 3^{\frac{D-6}{2}} + 3^{\frac{D-6}{2}} + \cdots\right) = 3^{\frac{D-2}{2}} > 1$$

when D is even, and

$$S_0(0,3n+1) \geq 3^{\frac{D-1}{2}} + 0 - \left(3^{\frac{D-3}{2}} + 2 \cdot 3^{\frac{D-5}{2}} + 3^{\frac{D-5}{2}} + 2 \cdot 3^{\frac{D-7}{2}} + \cdots\right) = \tfrac{1}{2} \cdot 3^{\frac{D-3}{2}} > 1$$

when D is odd.

Remark. The following Python code calculates $\sum_{i=0}^{n}(-1)^{s(3i)}$ in $O(\log n)$ time using digit DP (and in $O(K \log n)$ time if 3 is replaced by K):

```python
import functools

K = 3

def sgn(n: int) -> int:
    return -1 if bin(n).count('1') % 2 else 1

@functools.lru_cache(maxsize=None)
def f(n: int) -> list[int]:
    # Returns an array of size K where arr[i] = sum of sgn(j) over 0 <= j <=
    n with j % K = i
    ans = [0] * K
    if n == 0:
        ans[0] += 1
        return ans
    x = f(n // 2)
    for i in range(K):
        ans[2 * i % K] += x[i]
        ans[(2 * i + 1) % K] -= x[i]
    if n % 2 == 0:
        ans[(n + 1) % K] -= sgn(n + 1)
    return ans
```

Remark. Define

$$f(n) = \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^{s(3i)}.$$

Then there exist constants $0 < c_1 < c_2$ such that

$$\liminf_{n\to\infty} \frac{f(n)}{n^{\log_4(3)}} = c_1, \qquad \limsup_{n\to\infty} \frac{f(n)}{n^{\log_4(3)}} = c_2.$$

The upper constant is $c_2 = \frac{2}{3}$, achieved when n is a power of 4.

§2.3 TSTST 2024/6, proposed by Jaedon Whyte

🔗 *Available online at* https://aops.com/community/p31007016.

Problem statement

Determine whether there exists a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for all positive integers m and n,

$$f(m+nf(m)) = f(n)^m + 2024! \cdot m.$$

The answer is no. Let $P(m,n)$ denote the given FE.

¶ Solution 1 (Gopal Goel). Suppose there was a function f, and let $r = f(1)$. Note that $P(1,n)$ gives

$$f(1+rn) = f(n) + 2024!.$$

Iterating this result gives

$$f(1 + r + \cdots + r^k) = r + k \cdot 2024!$$

for all $k \in \mathbb{Z}_{\geq 0}$. If $r = 1$, this implies $f(k+1) = 1 + k \cdot 2024!$, which isn't a valid solution, so $r \geq 2$.

Let $m = 1 + r + \cdots + r^{r^2(r-1)}$. Note that

$$1 + r + \cdots + r^k = m + nf(m) \iff \frac{r^{k+1} - r^{r^2(r-1)+1}}{r-1} = n[r + r^2(r-1)2024!].$$

This has a positive integer solution for n as long as k is sufficiently large in terms of r and

$$k \equiv r^2(r-1) \pmod{\phi(1 + r(r-1) \cdot 2024!)}.$$

For such k, $P(m,n)$ implies

$$r + k \cdot 2024! = f(n)^m + 2024! \cdot m.$$

In particular, $r + (k-m) \cdot 2024!$ is a perfect mth power. This has to be true for all k sufficiently large in an arithmetic progression, which is clearly impossible, as desired.

¶ Solution 2 (Carl Schildkraut). As in the previous solution, we have

$$f(1 + r + \cdots + r^k) = r + k \cdot 2024!$$

for all $k \in \mathbb{Z}_{\geq 0}$, where $r = f(1) \geq 2$.

> **Claim —** We have $f(a) \geq \frac{2^a - 1}{2024! \cdot a}$ for $a \in \mathbb{Z}_{>0}$.

Proof. Let $c = a + af(a)$, $b = f(c) + a$, and $d = f(a)$, so that

$$a + bf(a) = a + af(a) + f(a)f(c) = c + df(c).$$

Comparing $P(a,b)$ and $P(c,d)$ tells us

$$f(b)^a + 2024! \cdot a = f(d)^c + 2024! \cdot c,$$

so

$$f(b)^a - \left(f(d)^{1+f(a)}\right)^a = 2024! \cdot af(a).$$

Since the left side of the above equation is positive, it must be at least $2^a - 1$, which implies the claim. □

Plugging in $a = 1 + r + \cdots + r^k$ into the above claim immediately gives the desired contradiction, for sufficiently large k.

¶ Solution 3 (students). If f existed, then for any $k \in \mathbb{Z}_{>0}$ we would have

$$\begin{aligned} f(f(3))^{3+kf(3)} + 2024!(3 + kf(3)) &= f(3 + kf(3) + f(3)f(3 + kf(3))) \\ &= f(3 + (k + f(3 + kf(3))f(3))) \\ &= f(k + f(3 + f(3)k))^3 + 2024! \cdot 3. \end{aligned}$$

Choosing $k = 27(2024!)^2 f(3)^2$ gives

$$f\left(k + f(3 + f(3)k)\right)^3 = \left(f(f(3))^{1+9(2024!)^2 f(3)^3}\right)^3 + (3 \cdot 2024! f(3))^3$$

which contradicts Fermat's last theorem (no two perfect cubes may sum to a perfect cube). Hence f cannot exist.

§3 Solutions to Day 3

§3.1 TSTST 2024/7, proposed by Merlijn Staps

🔗 *Available online at* https://aops.com/community/p31007023.

Problem statement

An infinite sequence a_1, a_2, a_3, ... of real numbers satisfies

$$a_{2n-1} + a_{2n} > a_{2n+1} + a_{2n+2} \qquad \text{and} \qquad a_{2n} + a_{2n+1} < a_{2n+2} + a_{2n+3}$$

for every positive integer n. Prove that there exists a real number C such that $a_n a_{n+1} < C$ for every positive integer n.

It suffices to solve the problem for sufficiently large n. Let $d_n = (-1)^{n-1}(a_{n+2} - a_n)$. The assertion simply says that $d_1, d_2, \dots$ is strictly increasing.

We consider the following cases.

- Suppose that $d_k > 0$ for some k. Then,

  $$a_{2n+1} = a_1 + (d_1 + d_3 + \cdots + d_{2n-1})$$

  clearly diverges to $+\infty$, and

  $$a_{2n} = a_2 - (d_2 + d_4 + \cdots + d_{2n-2})$$

  clearly diverges to $-\infty$, so taking $C = 0$ works.

- Now assume that $d_k \le 0$ for all k. This implies that $(a_{2n+1})_{n\ge 0}$ is weakly decreasing and $(a_{2n})_{n\ge 1}$ is weakly increasing. Adding the two expressions above together, we see that

  $$d_1 + a_1 + a_2 < a_{2n} + a_{2n+1} < a_1 + a_2.$$

  Since $(a_{2n+1})_{n\ge 0}$ is weakly decreasing, it either diverges to $-\infty$ or has a finite limit. Similarly, $(a_{2n})_{n\ge 1}$ either diverges to $+\infty$ or has a finite limit.

  - If $(a_{2n+1})_{n\ge 0}$ diverges to $-\infty$, then the first inequality above implies that $(a_{2n})_{n\ge 1}$ must diverge to $+\infty$, in which case $C = 0$ works.
  - If $(a_{2n})_{n\ge 1}$ diverges to $+\infty$, then the second inequality implies that $(a_{2n+1})_{n\ge 0}$ must diverge to $-\infty$, so again $C = 0$ works.
  - Finally assume that both $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{2n}$ exist, so the limit $L = \lim_{n\to\infty} a_{2n}a_{2n+1}$ also exists. The result follows taking $C > L$.

§3.2 TSTST 2024/8, proposed by Michael Ren

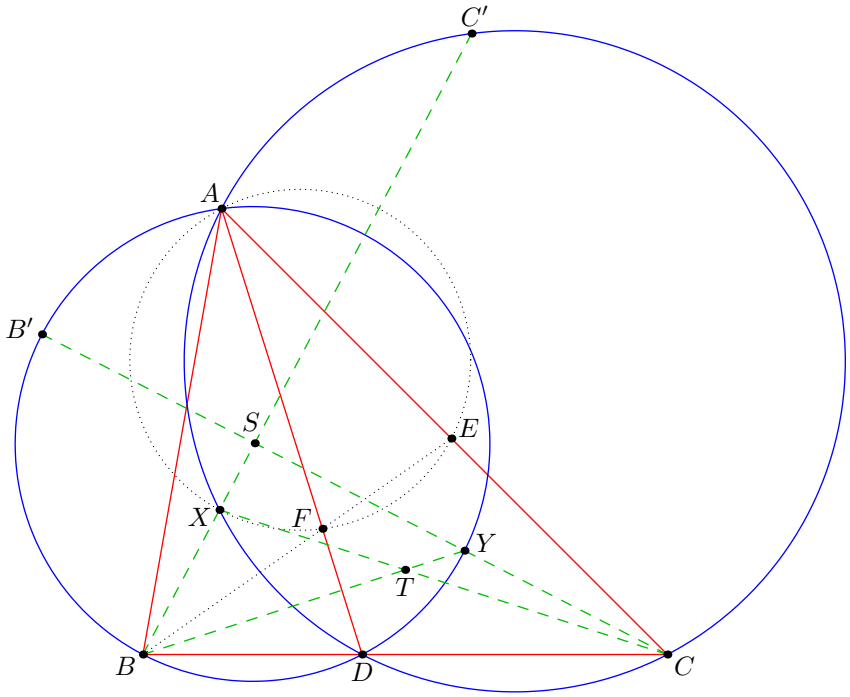
🔗 *Available online at* https://aops.com/community/p31007038.

Problem statement

Let ABC be a scalene triangle, and let D be a point on side BC satisfying $\angle BAD = \angle DAC$. Suppose that X and Y are points inside ABC such that triangles ABX and ACY are similar and quadrilaterals $ACDX$ and $ABDY$ are cyclic. Let lines BX and CY meet at S and lines BY and CX meet at T. Prove that lines DS and AT are parallel.

¶ Solution by characterizing X and Y. We first state an important property of X and Y.

Claim — Points X and Y are isogonal conjugates with respect to $\triangle ABC$.



Here are two proofs of the claim.

First proof of claim by Maxim Li. We prove an equivalent statement that S and T are isogonal conjugates with respect to $\triangle ABC$.

First, we note that $\angle YBC = \angle YAD = \angle XAD = \angle XCB$, so $BT = TC$. Now, let T' be the isogonal conjugate of S w.r.t. $\triangle ABC$. Since $\angle ABS = \angle ACS$, it follows that $T'B = T'C$. Thus, both T and T' lies on perpendicular bisector of BC.

Moreover, since AX and AY are isogonal with respect to $\angle BAC$, by DDIT on point A and $BXCY$, we get that AS and AT are isogonal with respect to $\angle BAC$. Hence, A, T, T' are collinear. Since $\triangle ABC$ is scalene, combining this with the previous paragraph gives $T = T'$, or S and T are isogonal conjugates. □

A second different proof of the claim. Let γ_B be the circle through B and C tangent to AB, and define γ_C similarly. We claim that $X \in \gamma_B$ and $Y \in \gamma_C$.

Let E be the second intersection of γ_B with AC, and let F be the intersection of BE and AD. Consider the transformation τ that is the composition of a reflection across AD and a homothety at A with ratio $\frac{AB}{AC}$. Note that τ maps $Y \mapsto X$, $B \mapsto E$, and $D \mapsto F$. Thus, X lies on $\tau(\odot(ABD)) = \odot(AEF)$, so X is the second intersection of $\odot(AEF)$ and $\odot(ADC)$, i.e. it is the Miquel point of quadrilateral $CDFE$. Thus, X lies on $\odot(BEC) = \gamma_B$, as claimed.

Finally, the main claim follows from $\angle BCX = \angle ABX = \angle ACY$. □

Here are two ways to finish after the claim.

Finish with isosceles trapezoid. We compute

$$\angle AXB = 360^\circ - \angle BXC - \angle CXA = 360^\circ - \angle BEC - \angle CDA = 180^\circ - \frac{\angle A}{2}.$$

Thus, if BX meets $\odot(ACD)$ again at C', then

$$\angle ADC' = \angle AXC' = \frac{\angle A}{2} = \angle CAD,$$

so $ADCC'$ is an isosceles trapezoid. Similarly, if CY meets $\odot(ABD)$ again at B', then $ADBB'$ is an isosceles trapezoid. Hence, $BCC'B'$ is an isosceles trapezoid whose diagonals meet at S, so we have that $SA = SD$.

However, by the claim, S and T are isogonal conjugates in ABC. Thus,

$$\angle ADS = \angle SAD = \angle DAT,$$

which completes the proof.

Finish with angle chasing (Pitchayut Saengrungkongka). By the claim, let $\angle DAX = \angle DAY = \angle ABX = \angle ACY = \theta$. Then,

$$\begin{aligned} \angle XSY &= \angle A + 2\theta \\ \angle XDY &= \angle BDY + \angle CDX - 180^\circ = 180^\circ - (\angle BAY + \angle CAX) \\ &= 180^\circ - (\angle A + 2\theta), \end{aligned}$$

so $XSYD$ is cyclic. Moreover,

$$\angle XTY = 180^\circ - \angle YBC - \angle XCB = 180^\circ - 2\theta = 180^\circ - \angle XAY,$$

so $AXTY$ is also cyclic. Finally,

$$\begin{aligned} \measuredangle(AT, AX) &= \measuredangle TYX = \measuredangle BYD + \measuredangle DYX \\ &= \measuredangle BAD + \measuredangle DSX = \measuredangle BXA + \measuredangle DSX \\ &= \measuredangle(DS, AX). \end{aligned}$$

¶ Solution 2 (Ruben Carpenter). Let M be the midpoint of BC, and S', T' the reflections of S, T over M. In what follows, DDIT is short for the dual of Desargues' involution theorem.

> **Claim —** S' lies on AT.

Proof. From $\triangle ABX \bar{\sim} \triangle AYC$, reflection along the angle bisector of $\angle ABC$ is an involution on the pencil of lines through A with pairs (AB, AC), (AX, AY) and $(A\infty_{BS}, A\infty_{CS})$. By DDIT from A onto complete quadrilateral $XYSTBC$, AS is sent to AT. By DDIT from A onto $BC\infty_{BX}\infty_{CY}SS'$, (AS, AS') is also an involutive pair, so $AS' \equiv AT$. □

Claim — T' lies on DS.

Proof. Using the circles $(ACDX)$, $(ABDY)$ we obtain

$$\angle BDT = \angle DAY = \angle DAC - \angle YAC = \angle BAD - \angle BAX = \angle XAD = \angle TCB,$$

so T lies on the perpendicular bisector of BC. Furthermore

$$\angle CDY = \angle BAY = \angle XAC = \angle XDB,$$

so by DDIT from D onto $XSYTBC$ shows $\angle CDT = \angle SDB$. The conclusion follows. □

Finally, since $STS'T'$ is a parallelogram we immediately have $DS \parallel AT$.

Remark. This problem was discovered by taking a degenerate case of IMO 2018/6 in which three of the vertices of the quadrilateral are collinear. Nevertheless, this origin is very obscured in the problem statement and does not seem to help with the solution at all. The original statement asked to show that the midpoints of AD, BC, and XY are collinear. However, it was found that this is a straightforward consequence of a high-powered result about isogonal conjugates, so the statement was changed.

§3.3 TSTST 2024/9

🔗 *Available online at* https://aops.com/community/p31007051.

Problem statement

Let $n \geq 2$ be a fixed integer. The cells of an $n \times n$ table are filled with the integers from 1 to n^2 with each number appearing exactly once. Let N be the number of unordered quadruples of cells on this board which form an axis-aligned rectangle, with the two smaller integers being on opposite vertices of this rectangle. Find the largest possible value of N.

The largest possible value of N is $\frac{1}{12}n^2(n^2-1)$. Call these rectangles *wobbly*. We defer the construction until the proof is complete, since the proof suggests the construction.

¶ Proof of bound. Call a triple of integers (a, b, c) an *elbow* if a and b are in the same row, b and c are in the same column, and $a < b > c$. Observe that the wobbly rectangles are exactly the ones with 2 elbows.

Claim — Every axis-aligned rectangle has at least 1 elbow.

Proof. The smallest integer in any rectangle is the center of an elbow. □

Remark. In fact, it is true that any rectangle must have exactly 1 or 2 elbows. However, this fact is not needed in the proof.

Let E be the number of elbows and M be the number of non-wobbly rectangles. Then, the number of elbows is at least $M + 2N = \binom{n}{2}^2 + N$, so

$$E \geq \binom{n}{2}^2 + N \iff N \leq E - \binom{n}{2}^2.$$

To this end, we will provide an upper bound on E. For each cell c, define:

- $f(c)$ to be the number of cells in c's row which are smaller than c, and
- $g(c)$ to be the number of cells in c's column which are smaller than c.

Then, the number of elbows centered at c equals $f(c)g(c)$. Thus, the number of elbows satisfies

$$\begin{aligned} E &= \sum_c f(c)g(c) \\ &\leq \sum_c \frac{1}{2}\left[f(c)^2 + g(c)^2\right] \\ &= n(0^2 + \cdots + (n-1)^2) \\ &= \frac{n^2(n-1)(2n-1)}{6}. \end{aligned}$$

It follows that the number of wobbly rectangles satisfies

$$N \leq E - \binom{n}{2}^2$$

$$\leq \frac{n^2(n-1)(2n-1)}{6} - \frac{n^2(n-1)^2}{4}$$
$$= \frac{n^2(n^2-1)}{12}$$

as desired.

¶ Construction. From the proof above, equality holds if and only if $f(c) = g(c)$ everywhere.

Select any $n \times n$ Latin square on symbols $0, \ldots, n-1$, and replace the n copies of symbol k with the integers $kn+1, \ldots, kn+n$ in some order. This construction is valid, because for each cell c, $f(c)$ and $g(c)$ both equal the symbol originally placed in c.

Here is an example for $n = 6$. The left grid is the Latin square, and the right grid is one possible table that can be derived from it.

0	1	2	3	4	5
5	0	1	2	3	4
4	5	0	1	2	3
3	4	5	0	1	2
2	3	4	5	0	1
1	2	3	4	5	0

1	7	13	19	25	31
32	2	8	14	20	26
27	33	3	9	15	21
22	28	34	4	10	16
17	23	29	35	5	11
12	18	24	30	36	6

Remark (Structure of optimal tables)**.** In fact, any optimal table must lead to a Latin square in a similar way, by reversing the above idea: replace each cell c with $f(c) = g(c)$. This suggests how to construct all optimal tables: choose any Latin square on $0, \ldots, n-1$, and fill in the cells so that the relative order of cells in every row and column is preserved.

Remark. If the grid is filled out in a random order, the expected number of wobbly rectangles is $\frac{1}{12}n^2(n-1)^2$. This is asymptotically equal to the maximum value.