TSTST 2026 Solutions

United States of America — TST Selection Test

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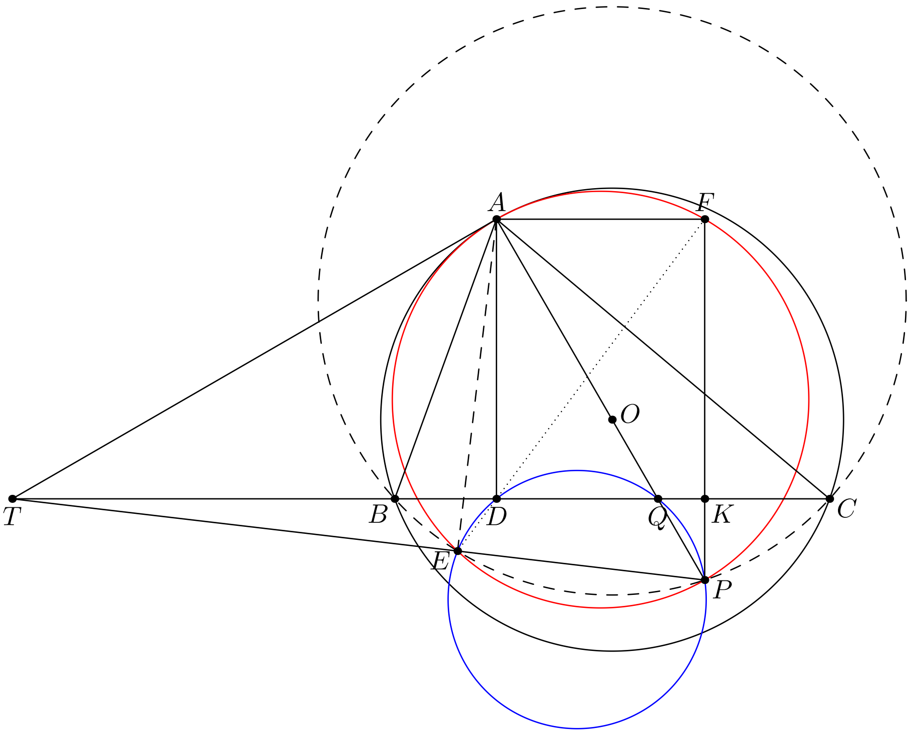
§1 Solutions to Day 1

§1.1 Solution to TSTST 1, by Ankit Bisain

Let ABC be an acute scalene triangle. Let O be the circumcenter of ABC, D be the foot of the altitude from A to BC, and P be a point on line AO not lying on the circumcircle of ABC. The circle with diameter AP intersects the circumcircle of BCP at $E \neq P$, and the line through A parallel to BC at $F \neq A$. Prove that D, E, and F are collinear.

¶ Solution 1



Let AP meet BC at Q. Note that (ABC), (AP), and (AQ) are tangent at A; let their common tangent at A meet BC at T. By the radical axis theorem on (AP), (ABC), and (PBC), we have that T lies on PE. Since

$$TD \cdot TQ = TA^2 = TE \cdot TP,$$

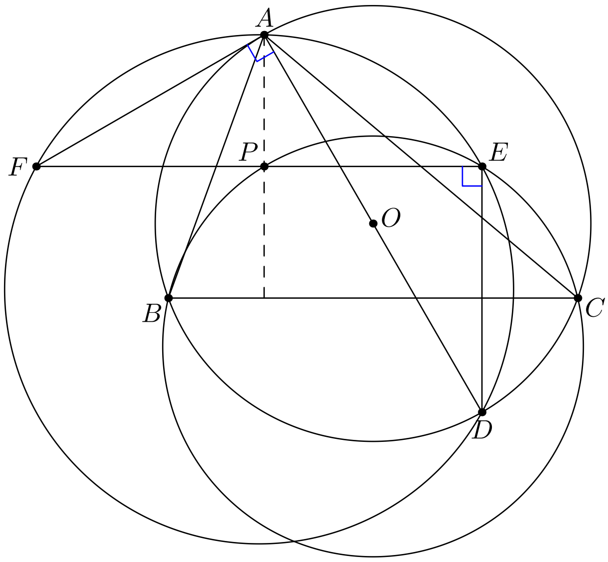
$QDEP$ is cyclic. Now, we have

$$\measuredangle QDE = \measuredangle QPE = \measuredangle APE = \measuredangle AFE.$$

Since QD and AF are parallel, we have that DEF are collinear, as desired.

¶ Solution 2 As in the previous solution, note that applying radical axis on (AP), (ABC), and (PBC) implies that PE, BC, and the tangent to (ABC) at A meet at a point T. Let K be the foot of the altitude from P to BC, so $AKPT$ is cyclic with diameter PT. Then D, E, and F are the feet from A to lines KT, PT, and KP respectively, so they lie on the Simson line of A with respect to KPT.

¶ Solution 3 ($\sqrt{bc}$-inversion)



Consider an inversion at A with radius $\sqrt{AB \cdot AC}$, followed by a reflection over the internal angle bisector of angle BAC. Let X' denote the image of a point X under this map. (The diagram shows these points after inversion with no prime symbol.) We have that $B' = C$, $C' = B$, and D' is the antipode of A on the circumcircle of ABC. Next, note that P' is an arbitrary point on A-altitude of triangle ABC and the circle with diameter AP maps to the line through P' parallel to BC. The circumcircle of BCP maps to the circumcircle of BCP', and the line through A parallel to BC maps to the line through A tangent to the circumcircle of ABC. This implies that $AF' \perp AD'$.

Now, since $BC \parallel P'E'$ and B, C, P', and E' lie on a circle, E' is the reflection of P' over the perpendicular bisector of BC. Since the reflection of D' over the perpendicular bisector of BC lies on the A-altitude, we have that $E'D' \perp E'P' = E'F'$. So, D', E', F', and A all lie on the circle with diameter $D'F'$. Undoing the inversion implies that D, E, and F are collinear.

§1.2 Solution to TSTST 2, by Linus Tang

Given a graph G and a positive integer ℓ, an *ℓ-coloring* of G is a function $\sigma\colon V(G) \to \{1, 2, 3, \dots, \ell\}$ such that for any edge uv in G, $\sigma(u) \neq \sigma(v)$. Two ℓ-colorings σ and σ' are *close* if $\sigma(u) \neq \sigma'(u)$ for at most one $u \in V(G)$.

Find all pairs of positive integers (k, m) with $k \le m$ such that for all graphs G and pairs (τ, τ') of k-colorings of G, there exists a sequence $\pi_0, \pi_1, \dots, \pi_n$ of m-colorings of G such that $\pi_0 = \tau$, $\pi_n = \tau'$, and π_i and π_{i+1} are close for all $0 \le i < n$.

Note: $V(G)$ denotes the vertices of G. Since $k \le m$, τ and τ' are also m-colorings of G.

The answer is all pairs of positive integers (k, m) with $m \ge 2k - 1$.

Assume $k > 1$, since the case $k = 1$ is trivial.

Let G_k denote the graph on vertex set $[k]^2$ with an edge between (i, j) and (i', j') if and only if $i \neq i'$ and $j \neq j'$. Let τ_k and τ'_k be colorings of G_k defined by $\tau_k((i, j)) = i$ and $\tau'_k((i, j)) = j$. We also define vertex subsets $R_i = \{(i, 1), \dots, (i, k)\}$ ("rows") and $C_j = \{(1, j), \dots, (k, j)\}$ ("columns") for each $i, j \in [k]$.

¶ Lower bound We first show that $m \ge 2k - 1$ is required to connect the colorings τ_k and τ'_k of G_k. For any coloring π of G_k, we say that row R_i is *satisfied* in π if $\pi(u) = \pi(v)$ for some distinct $u, v \in R_i$. Note that the repeated color can only appear in R_i.

Note that τ has k satisfied row while τ' has 0 satisfied rows. So, in a sequence of colorings connecting τ_k and τ'_k, there exists a pair of close colorings σ and σ' such that σ has k satisfied rows and σ' does not. Since σ and σ' are close, σ' has exactly $k - 1$ satisfied rows. So, σ' has at least $k - 1$ colors used to satisfy each of the satisfied rows and k other colors present in the non-satisfied row. This gives at least $2k - 1$ total colors, as desired.

¶ Construction Now we give a sequence of single-vertex color changes to transform τ into τ' for any given G with $m \ge 2k - 1$. Let $V_{i,j}$ be the set of vertices with color i in τ and j in τ'. Note that G cannot have any edges within $V_{i,j}$. We proceed with the following steps.

- For each $1 \le i < j \le k$, color each vertex in $V_{i,j}$ with $k + i$ (in an arbitrary order).
- For each $1 \le j \le k$ in increasing order, repeat the following:
  - For each $1 \le i \le k$, color each vertex in $V_{i,j}$ with j (in an arbitrary order).

This process produces the coloring τ' as desired and only uses colors in $[2k - 1]$. We now check that each intermediate coloring is valid. In the first step, note that the vertices receiving color $k + i$ all have color i in τ, so no edge in G receives $k + i$ at both endpoints. Now, suppose that an edge has color $j \in [k]$ at both endpoints during the second step. This edge must be between $V_{i,j}$ and $V_{j,\ell}$ for some $i, \ell \in [k] \setminus \{j\}$. If $\ell < k$, then $V_{j,\ell}$ would already be colored with color ℓ. If $\ell > k$, then $V_{j,\ell}$ would be colored with color $k + j$. So this is impossible, completing the proof.

> **Remark.** The construction essentially solves the problem for the "worst case" of (G_k, τ_k, τ'_k).

Remark. The problem can be generalized to have τ be a k_1-coloring and τ' be a k_2-coloring, in which case Bob needs $k_1 + k_2 - 1$ colors to guarantee that the task is possible. The solution to the generalization is very similar to the original.

§1.3 Solution to TSTST 3, by Daniel Zhu

Suppose $f_1, f_2, \dots, f_{1000} \in \mathbb{C}[x]$ are single-variable polynomials with complex coefficients such that there do not exist distinct $i, j \in \{1, 2, \dots, 1000\}$ and $a \in \mathbb{C}$ with $f_i = af_j$. Given that the degree of f_1 is 10, find the minimum possible total degree of

$$\sum_{k=1}^{1000} f_k(x_1) f_k(x_2) \cdots f_k(x_{2026}).$$

Note: For example, the total degree of the polynomial $g(x, y) = 20x^3y^4 + 26x^6$ is 7 because $7 = 3 + 4 > 6$.

Let $d = 10$, $m = 1000$, and $n = 2026$. The answer is $d(n - m + 1) = 10270$.

For a construction, let $\omega = e^{\frac{2\pi i}{m}}$ and let $f_k(x) = \omega^{k/n}(x^d + \omega^k)$. Then

$$\prod_{j=1}^{n} f_k(x_j) = \sum_{S \subseteq [n]} \omega^{k(n+1-|S|)} \prod_{i \in S} x_i^d,$$

so summing over all k zeroes out all terms where $|S| > n - m + 1$.

To show the bound, we present three solutions.

¶ Solution 1 (Author) For notational convenience assume $\deg f_m = d$ instead. Let $A = \{1 \le k \le m - 1 \mid \deg f_k \ge d\}$. For each $k \in A$, since f_k is not a constant multiple of f_m, there exists some $z_k \in \mathbb{C}$ that is a root of greater multiplicity in f_k than in f_m. Let r_k (possibly zero) be the multiplicity of z_k in f_m.

Let the polynomial we care about be $g(x_1, x_2, \dots, x_n)$. Consider the expression

$$h = \left(\prod_{\ell \in A} \frac{\partial^{r_\ell}}{\partial x_\ell^{r_\ell}} \right) g \Bigg|_{x_\ell = z_\ell \text{ for } \ell \in A} = \sum_{k=1}^{m} \left(\prod_{\ell \in A} f_k^{(r_\ell)}(z_\ell) \cdot \prod_{\ell \notin A} f_k(x_\ell) \right).$$

We examine the sum on the right-hand side.

- If $k \in A$, by construction we have $f_k^{(r_k)}(z_k) = 0$, so the term is 0.
- If $k = m$, we have $f_m^{(r_\ell)}(z_\ell) \neq 0$ for all $\ell \in A$, so this is $c \prod_{\ell \notin A} f_m(x_\ell)$ for some nonzero c. In particular, there is a nonzero $\prod_{\ell \notin A} x_\ell^d$ term.
- For all other k, we have that $\deg f_k < d$, so there is no $\prod_{\ell \notin A} x_\ell^d$ term.

Hence h has a $\prod_{\ell \notin A} x_\ell^d$ term and has total degree at least $d(n - |A|) \ge d(n - m + 1)$. But h is just a bunch of derivatives applied to g, so g must have total degree at least $d(n - m + 1)$ as well.

¶ Solution 2 (Author) The key claim is that

$$\sum_{k=1}^{m} c_k f_k(x_1) f_k(x_2) \cdots f_k(x_{m-1}) \neq 0$$

for any choice of complex numbers c_k with $c_1 \neq 0$. This will prove the original problem as it implies that there exists a nonzero term of the form $x_1^{e_1} x_2^{e_2} \cdots x_{m-1}^{e_{m-1}} x_m^d x_{m+1}^d \cdots x_n^d$.

Suppose not and consider the functions $g_k(x) = f_k'(x)/f_k(x)$; note that this is a rational function that uniquely determines f_k up to a constant. Thus, all the g_k are distinct, meaning that we may find some complex number z such that all the $g_k(z)$ are defined and distinct.

Let $c_k' = f_k(z)^{m-1} \cdot c_k$; then, modulo $(x_1^2, \ldots, x_{m-1}^2)$, we have

$$\begin{aligned} 0 &= \sum_{k=1}^{m} c_k f_k(x_1 + z) f_k(x_2 + z) \cdots f_k(x_{m-1} + z) \\ &\equiv \sum_{k=1}^{m} c_k' (1 + g_k(z)x_1)(1 + g_k(z)x_2) \cdots (1 + g_k(z)x_{m-1}). \end{aligned}$$

Looking at the $x_1 x_2 \cdots x_e$-coefficient, we conclude that

$$\sum_{k=1}^{m} c_k' g_k(z)^e = 0 \text{ for all } 0 \le e \le m-1.$$

But this means that

$$\sum_{k=1}^{m} c_k' P(g_k(z)) = 0 \text{ for all } P \text{ with } \deg P \le m-1,$$

which is impossible upon letting $P(t) = \prod_{k>1}(t - g_k(z))$, since $c_1' \neq 0$.

¶ Solution 3 (Krishna) This solution is similar to Solution 2.

For each k, let c_k be a (nonzero) constant such that $f_k = c_k^{1/n} g_k$ for some monic g_k. Then we want to show that

$$G(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} c_k g_k(x_1) \ldots g_k(x_n)$$

has total degree at least $d(n - m + 1)$, and we know that $g_i \neq g_j$ for any $i \neq j$.

Since $g_i \neq g_j$ for any $i \neq j$, there is some z such that the values $g_k(z)$ are distinct for all k. By shifting $x_i \to x_i + z$ we may assume without loss of generality that $z = 0$. Now, let $a_k = g_k(0)$. Let M be the maximum of the degrees of g_k over all k, and note that $M \ge d$. Next, let $c_k' = c_k$ if $\deg g_k = M$ and $c_k' = 0$ otherwise. Note that there exists ℓ such that $c_\ell' \neq 0$.

The key idea is to focus only on the monomials where the exponent of each variable is either M or 0. For $1 \le j \le n$, consider the monomial $x_1^M x_2^M \ldots x_j^M$. Let A_j be the coefficient of this monomial in G. By expanding each term, we have

$$A_j = \sum_{k=1}^{m} c_k' a_k^{n-j}.$$

Suppose for sake of contradiction that the total degree of G is less than $M(n - m + 1)$. Then we must have $A_j = 0$ for $j \ge n - m + 1$. Now, we finish as in Solution 2, obtaining that $\sum_{k=1}^{m} c_k' P(a_k) = 0$ for all polynomials P with $\deg P \le m - 1$. Letting $P(t) = \prod_{k \neq \ell}(t - a_k)$ gives a contradiction.

Remark (Author)**.** This problem is secretly a generalization of 2018 ISL A6, though I think it's sufficiently obfuscated that figuring out how is already a tricky exercise. None of the posted solutions for that problem appear to work for this.

Remark (Author)**.** The key claim of Solution 2 has nothing to do with polynomials at all and is most naturally expressed in the language of symmetric tensors. In particular, it is equivalent to showing that for some nonzero vector v, we cannot write $v^{\otimes r}$ as the linear combination of fewer than $r+1$ tensors $v_k^{\otimes r}$ where v_k is not a scalar multiple of v. Put this way, there's in fact a one-line solution: write $v^{\otimes r} = \sum_{k=1}^{r} c_k v_k^{\otimes r}$ and contract both sides with $u_1 \otimes u_2 \otimes \cdots \otimes u_r$ where $\langle u_k, v_k \rangle = 0$ and $\langle u_k, v \rangle \neq 0$. In fact, the solutions given here are essentially special cases of this argument.

§2 Solutions to Day 2

§2.1 Solution to TSTST 4, by Carlos Rodriguez

A *beehive* is a polygon with all internal angles being multiples of $60°$. Find the least positive integer k for which every 2026-sided beehive can be expressed as the union of at most k convex beehives.

Note: A polygon contains its interior and boundary. Its boundary must be a single loop that does not intersect itself. Internal angles of $180°$ are not allowed.

We will show that for a n-sided beehive with $n = 3k + 1$ the answer is $2k - 1$. For $n = 2026$, this gives 1349.

Claim — A n-beehive has at most $\lfloor 2n/3 \rfloor - 2$ internal angles exceeding $180°$.

Proof. Suppose a beehive has m internal angles exceeding $180°$. This implies that

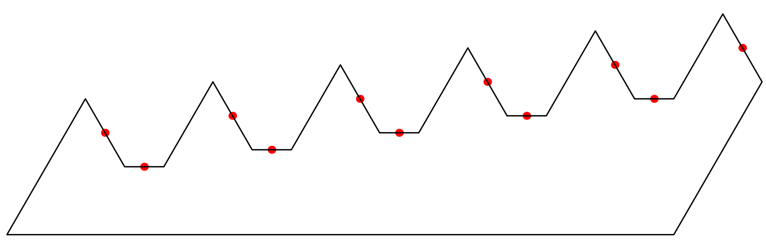
$$180(n-2) = \sum \text{angles in beehive} \geq 240m + 60(n-m)$$

$$\implies m \leq \frac{2n}{3} - 2$$

yielding the desired bound. □

Thus, consider the following operation. Choose a concave internal angle, take one of the two sides, and cut the beehive along that line until it hits another cut or the boundary of the beehive. Note that this splits the concave angle into a convex angle which is a multiple of $60°$ and a $180°$ angle, and the angle formed with the intersection that terminates the cut (whether that may be a side or a vertex) also forms two convex angles which are multiples of $60°$. Thus, this operation removes at least one concave angle from the beehive partitioning, and it increases the number of beehives in the partitioning by exactly one. Since there were initially at most $\lfloor 2n/3 \rfloor - 2$ concave angles, arbitrarily repeating this operation will result in at most $\lfloor 2n/3 \rfloor - 1$ convex beehives as desired.

Now we show that this is optimal. Consider the following generalizable diagram for the case $n = 3k + 1$ with $2k - 1$ red points (they are midpoints). No two red points can be on the boundary of the same convex figure contained in the original beehive since the line segment connecting them is not entirely contained in the original beehive.



Remark. Similar constructions can show that the answer is $\lfloor 2n/3 \rfloor - 1$ in general.

§2.2 Solution to TSTST 5, by Ankit Bisain

Let $\Omega(k)$ denote the number of prime factors of k, counted with multiplicity. For example, $\Omega(20) = 3$, because $20 = 2 \cdot 2 \cdot 5$. Prove that for all positive integers n, there exists a positive integer k strictly between n^2 and $(n+1)^2$ such that $\Omega(k)$ is odd.

¶ Boilerplate (Pitchayut Saengrungkongka) Suppose the problem is not true. Observe that $\Omega(xy) = \Omega(x) + \Omega(y)$ for all x and y.

Say that $x \sim y$ if $\Omega(x)$ and $\Omega(y)$ have the same parity, or equivalently $\Omega(xy)$ is even. By assumption, $x \sim y$ whenever $n^2 \le xy \le (n+1)^2$. Thus let $a < b$ be positive integers with $b - a$ maximal such that

- For all $a < x \le b$, $x \sim b$, and
- $n^2 \le (a+1)b \le (n+1)^2$.

We have that $(n, n+1)$ satisfies the two conditions above, so a maximal pair exists. Since $b - a$ is maximal, the pairs $(a-1, b)$ and $(a, b+1)$ cannot satisfy the conditions. So we must have $ab \le n^2 - 1$ and $(a+1)(b+1) \ge (n+1)^2 + 1$.

¶ Solution 1 (Ankit Bisain and Luke Robitaille)

> **Claim** — In the interval $(a, b]$, there is a number of the form x^2 and another number of the form $2y^2$.

Proof. We would like to find integers x and y in the intervals $\left(\sqrt{a}, \sqrt{b}\right]$ and $\left(\sqrt{a/2}, \sqrt{b/2}\right]$, respectively. So, it suffices to show that $\sqrt{b} - \sqrt{a} \ge 1$ and $\sqrt{b/2} - \sqrt{a/2} \ge 1$. We have that

$$a + b = (a+1)(b+1) - ab - 1 \ge (n+1)^2 + 1 - (n^2 - 1) - 1 = 2n + 2.$$

Thus $(a + b - 2)^2 \ge 4n^2 > 4ab$. Taking the square root gives $a + b - 2 > 2\sqrt{ab}$, or $(\sqrt{b} - \sqrt{a})^2 > 2$, as desired. □

Thus $\Omega(x^2) \sim \Omega(2y^2)$, which is a contradiction.

¶ Solution 2 (Pitchayut Saengrungkongka) We strong induct on n, with base case $n = 1$ easy. Now suppose $n \ge 2$.

Note that $n+1 \sim n \sim n+2 \sim n-1$, so $\Omega(n^2 - 1) = \Omega((n-1)(n+1))$ is even. Thus we can relax the second condition in the boilerplate to just $n^2 - 1 \le (a+1)b \le (n+1)^2$.

> **Claim** — $a + b \ge 2n + 4$.

Proof. We now have that $ab \le n^2 - 2$, which gives

$$a + b = (a+1)(b+1) - ab - 1 \ge (n+1)^2 + 1 - (n^2 - 2) - 1 = 2n + 3.$$

Suppose FTSOC that equality holds. Then

$$(a - b)^2 = (a + b)^2 - 4ab = (2n+3)^2 - 4(n^2 - 2) = 12n + 17 \equiv 2 \bmod 3,$$

giving a contradiction. Thus $a + b \ge 2n + 4$. □

Claim — In the interval $(a, b]$, there are two consecutive squares with the larger one less than $(n+1)^2$.

Proof. We first prove there are at least two squares. We have that $(a+b-4)^2 \geq 4n^2 > 4ab$. Taking the square root gives $a + b - 4 > 2\sqrt{ab}$, or $(\sqrt{b} - \sqrt{a})^2 > 4$. Thus $\sqrt{b} - \sqrt{a} > 2$, proving the claim.

Since $ab < n^2 - 1$, we have $b < (n+1)^2$, so the larger square cannot be at least $(n+1)^2$. □

Let c^2 and $(c+1)^2$ be the two squares from the previous claim, with $c < n$. So, by the inductive hypothesis, there exists $x \in (c^2, (c+1)^2)$ with $\Omega(x)$ odd. On the other hand $\Omega(c^2)$ is even, which contradicts $x \sim b \sim c^2$.

Remark (Krishna)**.** This writeup starts with a, b such that $b - a$ maximal because without it, Solution 2 is a bit of a pain to write up. Here is an example to show what is happening in both solutions.

When $n = 31$, just from the fact that $x \sim y$ if $n^2 \leq xy \leq (n+1)^2$, we observe the following chain.

$$32 \sim 31 \sim 33 \sim 30 \sim 34 \sim 29 \sim 35 \sim 28 \sim 36 \sim 27 \sim 37 \sim 26 \sim 38$$

Although $25 \cdot 38 < 31^2$, we can do a 'jump' by noticing that $31^2 \leq 26 \cdot 39 \leq 32^2$. Hence we can continue with the chain

$$26 \sim 39 \sim 25 \sim 40$$

and Solution 1 finishes from here because $\sqrt{40} - \sqrt{24} \geq \sqrt{2}$.

Solution 2 requires 3 jumps. Because $40 \cdot 24 < 31^2$ and $41 \cdot 25 > 32^2$, a second jump is naively not possible, but because we assume $n - 1 \sim n + 1$, $\Omega(n^2 - 1)$ is even and we actually just need $n^2 - 1 \leq xy \leq (n+1)^2$, which allows us to extend the previous chain with $40 \sim 24$. We can further extend it with $24 \sim 41$ and then do 2 more jumps, to get the following chains.

$$24 \sim 42 \sim 23 \sim 43, 23 \sim 44 \sim 22 \sim 45$$

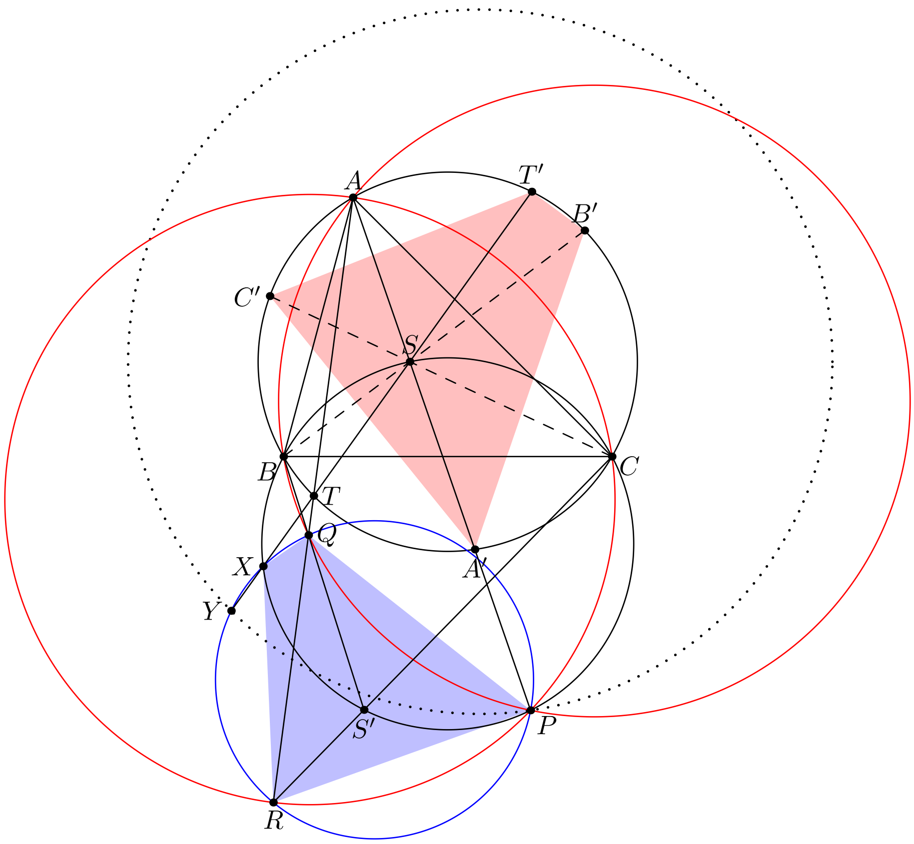
Solution 2 finishes from here because $\sqrt{45} - \sqrt{21} \geq 2$.

§2.3 Solution to TSTST 6, by Frank Han

Let ABC be a triangle with circumcircle Γ and let S be a point inside triangle ABC. Let $P \neq S$ be the second intersection of line AS with the circumcircle of BCS.

Let $T \neq A$ be a variable point on Γ, and let $Q \neq A$ and $R \neq A$ be the second intersections of line AT with the circumcircles of ABP and ACP, respectively. Let ω be the circumcircle of PQR. Show that, as T varies along Γ, the intersection(s) of line ST with ω lie on two fixed circles (excluding cases where ω is undefined).

¶ Solution 1, without inverting (Krishna)



Let $S' = BQ \cap CR$. We claim that S' is on the circumcircle of BCS. To check this, we compute $\angle S'CS$:

$$\measuredangle S'CS = \measuredangle RCS = \measuredangle RCP + \measuredangle PCS = \measuredangle RAP + \measuredangle PBS = \measuredangle TAP + \measuredangle PBS.$$

Similarly, we have $\measuredangle S'BS = \measuredangle TAP + \measuredangle PBS$, so $\measuredangle S'BS = \measuredangle S'CS$ as desired.

Now, for $Z \in \{A, B, C, T\}$, let Z' be the second intersection of ZS with the circumcircle of ABC. We claim that $\triangle A'B'S \sim \triangle PQS'$. To check this, we compute two angles. First, we have

$$\measuredangle A'B'S = \measuredangle A'B'B = \measuredangle A'AB = \measuredangle PAB = \measuredangle PQB = \measuredangle PQS'.$$

Next, we have

$$\measuredangle A'SB' = \measuredangle PSB = \measuredangle PS'B = \measuredangle PS'Q.$$

So $\triangle A'B'S \sim \triangle PQS'$. Similarly, we have $\triangle A'C'S \sim \triangle PRS'$. So, the shapes $A'B'C'S$ and $PQRS'$ are similar.

Let $X \neq P$ be the second intersection of (PQR) and (BCS). We claim that the shapes $A'B'C'ST'$ and $PQRS'X$ are similar. Since $PQRX$ and $A'B'C'T$ are cyclic, it suffices to check that $\measuredangle S'XP = \measuredangle ST'A'$. Using the calculation of $\measuredangle S'CS$ from above, we have

$$\measuredangle S'XP = \measuredangle S'CP = \measuredangle S'CS - \measuredangle PCS = \measuredangle TAP = \measuredangle TAA' = \measuredangle TT'A' = \measuredangle ST'A'.$$

From this similarity, we have $\measuredangle XS'P = \measuredangle T'S'A$. This imples that S, T, and X are collinear, since

$$\measuredangle XSP = \measuredangle XS'P = \measuredangle T'SA' = \measuredangle TSP.$$

Now, let Y be the second intersection of line ST with the circumcircle of PQR. Then we have

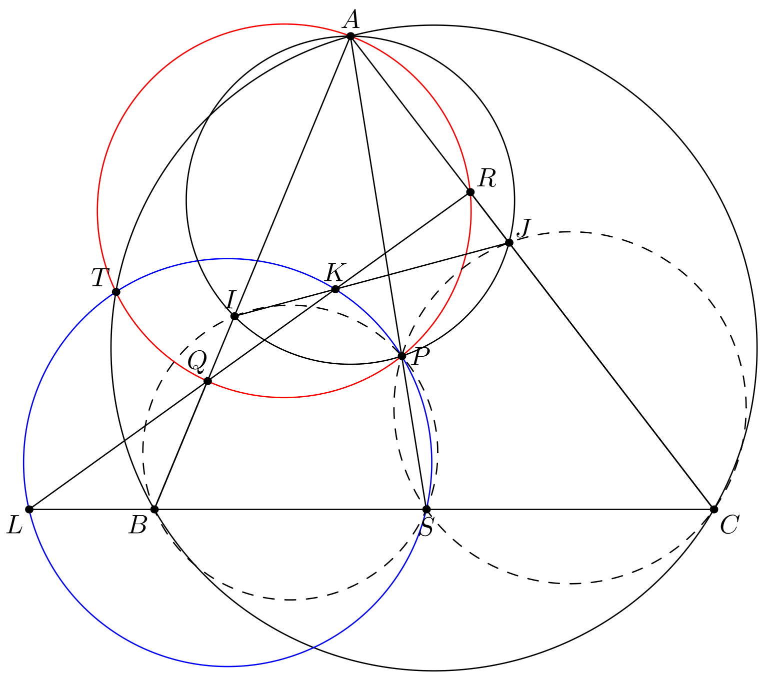
$$\measuredangle PRY = \measuredangle PXY = \measuredangle PXS = \measuredangle PBS.$$

Thus $\measuredangle PRY$ is fixed as T varies. Similarly, $\measuredangle PQY$ is fixed. So $PQRS'Y$ is similar to a fixed shape. Since P is fixed and R varies along the (fixed) circumcircle of ACP, Y also varies along a fixed circle (it is the image of the circumcircle of ACP under the fixed spiral similarity centered at P that takes R to Y).

¶ Solution 2, inverting about P (Rey Li) After inverting, we get the following problem: (For the sake of simplicity, we will refer to the inverted image of point Q as Q.)

Inverted problem

We are given a triangle $\triangle ABC$, point S on side BC, and a point P on line AS. An arbitrary circle passing through A and P intersects (ABC) at T, line AB at Q, and line AC at R. Prove that as the arbitrary circle varies, the intersections of line QR with circle (PST) lie on two fixed lines/circles.



Observe that by taking line AT very close to line AP in the uninverted diagram, the two intersections of (PQR) with ST in the uninverted diagram become arbitrarily close to P, so it makes sense for the two fixed circles in the uninverted diagram to pass through P. This would mean that the intersections in the inverted diagram lie on two fixed lines.

We will first prove that one of the lines is BC. It suffices to show that $QR \cap BC \in (PST)$. If we define $L = QR \cap BC$, then T is the Miquel Point of complete quadrilateral $ABCQRL$ so it lies on circles (ABC), (QBL), (CLR), and (AQR). Hence

$$\measuredangle SLT = \measuredangle BQT = \measuredangle APT = \measuredangle SPT,$$

so L is on (PST).

Now, we will use the first intersection point L to find the fixed line that the second intersection point is on. Define $I = AB \cap (PBS)$ and $J = AC \cap (PCS)$; we will prove that the second intersection point lies on line IJ. We can angle chase to show that $AIPJ$ is cyclic:

$$\measuredangle AIP = \measuredangle BSP = \measuredangle CSP = \measuredangle AJP.$$

Now, let $K = QR \cap IJ$. Then $P = (AQR) \cap (AIJ)$ is the Miquel point of complete quadrilateral $AIJQKR$, so it lies on (KIQ) and (KJR). Then $K \in (PSTL)$ because

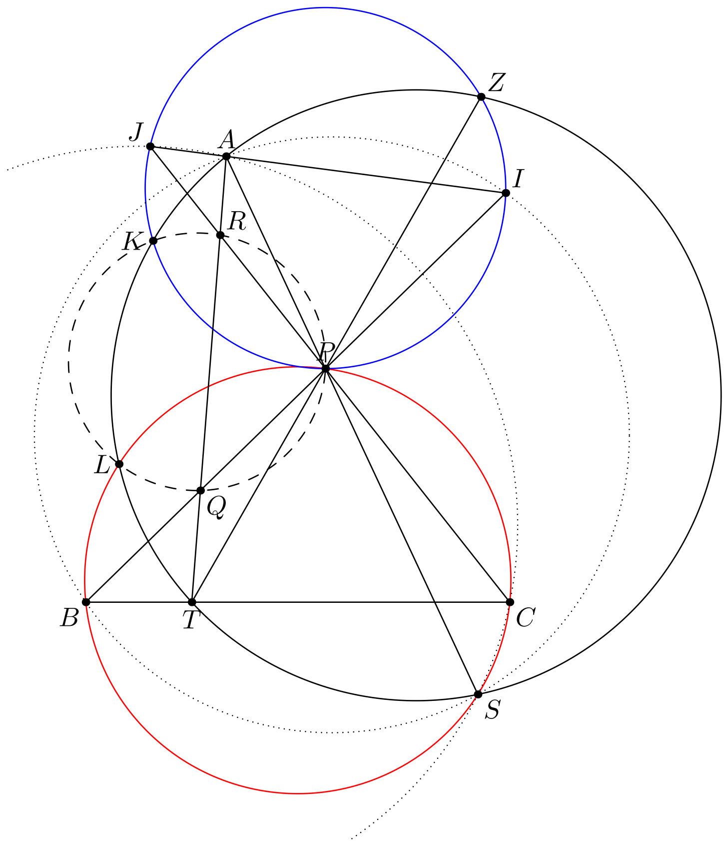
$$\measuredangle PKL = \measuredangle PIB = \measuredangle PSB = \measuredangle PSL.$$

Thus, K is the second intersection point (other than L) of QR and (PST), and it lies on fixed line IJ, as desired.

¶ Solution 3, inverting about A (Vivian Loh) First, invert about A. The motivation is that (ABP) and (ACP) become lines, and T varies on a line rather than a circle. We only work with the inverted diagram, so refer to the image of a point as itself. We get the following problem:

Inverted problem

Given triangle ABC, let points P and S be such that A, P, S are collinear and $(BSCP)$ is cyclic. Let T be an arbitrary point on line BC. Let line AT intersect lines BP and CP and Q and R respectively. Prove that the intersection point(s) of (PQR) and (AST) lie on two fixed circles/lines as T ranges.



We first show that one of the fixed circles is (BPC). Define L to be the Miquel Point of complete quadrilateral $PQRBCT$; then $L = (PQR) \cap (BPC)$, and we wish to show that L lies on (AST). This is just angle chasing: since L lies on (PQR), (BPC), (QTB), and (CRT), we have that

$$\measuredangle ATL = \measuredangle QTL = \measuredangle QBL = \measuredangle PBL = \measuredangle PSL = \measuredangle ASL.$$

Thus L is one of the intersection points of (PQR) with (AST), and (BPC) is one of the fixed circles.

Now we find the other fixed circle. Define J as $CP \cap (ACS)$, $I = BP \cap (ABS)$, $Z = TP \cap (AST)$. By Power of a Point at P,

$$PT \cdot PZ = PA \cdot PS = PC \cdot PJ = PB \cdot PI \implies IBCJ, BTIZ, CTJZ \text{ cyclic.}$$

Furthermore, I, A, and J are collinear since $\measuredangle IAS = \measuredangle PBS = \measuredangle PCS = \measuredangle JAS$, and Z lies on (AIJ) because $\measuredangle PIZ = \measuredangle BIZ = \measuredangle BTZ = \measuredangle CTZ = \measuredangle CJZ = \measuredangle PJZ$.

We'll show that (IPJ) is the second fixed circle. Let K be the Miquel point of complete quadrilateral $PQRAIJ$. Then we can angle chase:

$$\measuredangle TAK = \measuredangle RAK = \measuredangle RJK = \measuredangle PIK = \measuredangle PZK = \measuredangle TZK.$$

Therefore K lies on $(ASTZ)$.

> **Remark.** Points I, J, K, and L correspond to the points with the same labels in the previous solution. Point Z in this solution corresponds to $AT \cap IJ \cap (PST)$ in the previous solution (which was not used). Similarly, it is possible to directly angle chase $\measuredangle AKL = \measuredangle ASL$ in this solution without involving Z.

Remark. There is an involution of point labels in this diagram swapping the following pairs:

$$A \longleftrightarrow T$$
$$Q \longleftrightarrow R$$
$$I \longleftrightarrow B$$
$$J \longleftrightarrow C$$
$$S \longleftrightarrow Z$$
$$K \longleftrightarrow L$$
$$P \longleftrightarrow P$$

that preserves all relevant collinearity and cyclicity relations. With this symmetry in mind, the two parts of the proof (that L lies on fixed circle (BPC) and K lies on fixed circle (IPJ)) are completely analogous.

§3 Solutions to Day 3

§3.1 Solution to TSTST 7, by Pitchayut Saengrungkongka

Determine all ordered pairs of positive integers (a, b) such that $n+1$ divides $\binom{an+b}{n}$ for all positive integers n.

The answer is all pairs of the form $(k+2, k)$ or (k, k) for some positive integer k.

¶ Proof that these work

- For $(a, b) = (k, k)$ note the identity
$$\binom{a(n+1)}{n+1} = \frac{(a-1)(n+1)+1}{n+1}\binom{a(n+1)}{n},$$
which implies that $n+1$ divides $\binom{a(n+1)}{n}$.
- For $(a, b) = (k+2, k)$ note the identity
$$\binom{a(n+1)-2}{n+1} = \frac{(a-1)(n+1)-1}{n+1}\binom{a(n+1)-2}{n},$$
which implies that $n+1$ divides $\binom{a(n+1)-2}{n} = \binom{an+b}{n}$.

¶ Proof that no other pairs work Assume that $|a-b-1| \neq 1$. Let p be a prime divisor of $a-b-1$ and take $n = p-1$. Thus, p divides

$$\binom{a(p-1)+b}{p-1} = \frac{(a(p-1)+b)(a(p-1)+b-1)\cdots(a(p-1)+b-(p-2))}{(p-1)!}.$$

Since $p \mid a-b-1$, the factors in the numerator are congruent to $p-1$, $p-2$, $\dots$, 1 modulo p. Thus p does not divide $\binom{a(p-1)+b}{p-1}$, as desired.

§3.2 Solution to TSTST 8, by Merlijn Staps

Let A be a finite set of positive real numbers. Define a_k to be the number of distinct positive real numbers that can be expressed as a sum of at most k not necessarily distinct elements of A. Prove that $2025a_{2026} \geq 2026a_{2025}$.

We establish some common notation for both solutions. Let $n = 2025$. Let V be the set of positive real numbers that can be written as the sum of at most n members of A and let W be the set of positive real numbers that can be written as the sum of at most $n+1$ members of A. By definition $a_n = |V|$ and $a_{n+1} = |W|$, so we wish to show $|W| \geq \frac{n+1}{n}|V|$.

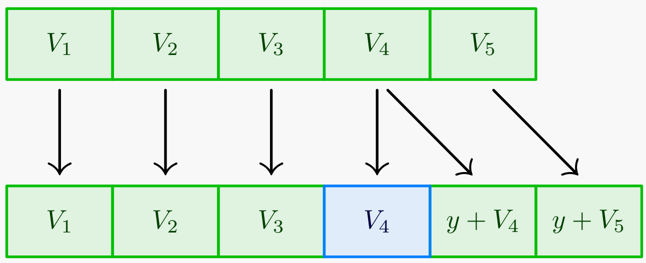
Let y denote the largest element of A, so that $V \subseteq (0, ny]$ and $W \subseteq (0, (n+1)y]$. Let $W' = V \cup (y+V) \subseteq W$. Both solutions below actually show $|W'| \geq \frac{n+1}{n}|V|$, which suffices since $W' \subseteq W$.

¶ Solution 1 (Author, Ritwin Narra) It suffices to show there are at least $\frac{n+1}{n}|V|$ elements in W'. We will construct this many. Partition $(0, (n+1)y]$ into $n+1$ blocks $B_1, \dots, B_{n+1}$ of length y, and let $V_i = V \cap B_i$. For any $j \in \{1, \dots, n\}$ note that

$$V_j' := V_1 \cup \dots \cup V_j \cup (y + V_j) \cup \dots \cup (y + V_n) \subseteq W',$$

since those $n+1$ sets lie in $n+1$ different blocks B_i.

In particular we have $|V_j'| = |V| + |V_j| \leq |W'|$. The largest V_j has at least $\frac{1}{n}|V|$ elements, so we are done.

> **Remark.** Here's a picture of what's going on, in the $n = 5$, $j = 4$ case.
>
> 
>

¶ Solution 2 (Pitchayut Saengrungkongka) The idea is to consider each residue class modulo y. Specifically, we make the following two claims.

> **Claim** — For any $r \in \mathbb{R}$, we have $|(r + y\mathbb{Z}) \cap V| \leq n$.

Proof. Since $V \subseteq (0, ny]$, we have

$$|(r + y\mathbb{Z}) \cap V| \leq |(r + y\mathbb{Z}) \cap (0, ny]| = n.$$

□

Claim — For any $r \in \mathbb{R}$ with $(r + y\mathbb{Z}) \cap V$ nonempty, we have

$$|(r + y\mathbb{Z}) \cap W'| \geq |(r + y\mathbb{Z}) \cap V| + 1$$

Proof. Since $V \subseteq W'$, it suffices to find an element in $(r + y\mathbb{Z}) \cap (W' \setminus V)$. Let x be the maximum element of $(r + y\mathbb{Z}) \cap V$. Then $x + y$ is in $r + y\mathbb{Z}$ and W', but $x + y$ is not in V by maximality of x. □

From those two claims, it follows that (for *all* r)

$$|(r + y\mathbb{Z}) \cap W'| \geq \frac{n+1}{n} |(r + y\mathbb{Z}) \cap V|.$$

Summing across all $r \in [0, y)$ gives $|W'| \geq \frac{n+1}{n}|V|$.

Remark. If $A = \{r, 2r, \dots, mr\}$ for some positive integer m and positive real number r, then $a_n = nm$ for all n so we have equality.

§3.3 Solution to TSTST 9, by Ankit Bisain

In the plane, we say that a set of points $A = \{A_1, A_2, \dots, A_k\}$ *demolishes* a set of points $B = \{B_1, B_2, \dots, B_\ell\}$ if, for all points P,

$$PA_1 + PA_2 + \cdots + PA_k \geq PB_1 + PB_2 + \cdots + PB_\ell.$$

Let S be a set of n points in the plane. Show that we can partition S into two sets A and B such that A demolishes B and $|A| \leq |B| + 100\sqrt{n}$.

We will refer to points in A as "red" and points in B as "blue".

The key idea is to use the triangle inequality in multiple ways. Having a pair of red points that are far apart guarantees a significant contribution to the LHS. Also, having each blue point pair with a nearby red point makes their contributions to the LHS and RHS roughly equal. So, we will (roughly) greedily picking close (red,blue) pairs and far (red,red) pairs. The key inequality in the analysis is that in a set of n points, the greatest distance between two points is at least $C\sqrt{n}$ times the least distance between two points, for some constant C.

We will prove the following claim, from which the problem follows by induction with $|A| \leq |B| + 5\lceil \sqrt{n} \rceil$.

Claim — Let n and k be positive integers with $k^2 < n \leq (k+1)^2$, and let T be a set of n points in the plane. Then we can color $n - k^2$ points in T either red or blue such that the red points demolish the blue points and there are at most 5 more red points than blue points.

We prove the claim using the following algorithm:

1. While the number of uncolored points is greater than $k^2 + 5$, take the two closest uncolored points, and color one red and one blue.

2. While the number of uncolored points is between $k^2 + 2$ and $k^2 + 5$ inclusive, take the two furthest apart uncolored points, and color them both red.

3. If the number of uncolored points is $k^2 + 1$, color an arbitrary point red.

For $n \leq k^2+5$, it is clear that this works. Suppose $n > k^2+5$. Let $(R_1, B_1), \dots, (R_t, B_t)$ be the pairs of red and blue points obtained from the first step (in this order). Let $(W, X), (Y, Z)$ be the pairs of red points obtained from the second step (in this order). Note that we have $k^2 + 2t + 4 \leq n \leq (k+1)^2$, so $t \leq (k-1)$.

We will now prove the key inequality we use to relate the distances between these pairs:

Lemma

In a set U of k^2 points in the plane, the maximum distance D and minimum distance d between points in U satisfy

$$2D \geq (k-1)d.$$

Proof. Consider drawing a circle of radius $d/2$ at each point in U. By the minimality of d, these circles are disjoint. Next, pick an arbitrary fixed point $X \in U$. By maximality of D, U is contained in a circle of radius D centered at X. So, each of the former circles is contained in a circle of radius $D + d/2$ centered at X. Now, by computing areas we obtain

$$\pi\left(D+\frac{d}{2}\right)^2 \geq k^2 \cdot \left(\pi(d/2)^2\right) \iff 2D \geq (k-1)d.$$

$\square$

We will apply the lemma to the uncolored points and define D and d accordingly. Note that by construction we have

$$WX \geq YZ \geq D \geq d \geq R_tB_t \geq \cdots \geq R_1B_1.$$

So, the lemma gives

$$WX + YZ \geq 2D \geq (k-1)d \geq td \geq R_1B_1 + R_2B_2 + \cdots + R_tB_t.$$

Now, we use this with the triangle inequality to show that the red points demolish the blue points. For any point P in the plane, we have

$$\begin{aligned}
PB_1 + PB_2 + \cdots + PB_t &\leq (PR_1 + R_1B_1) + (PR_2 + R_2B_2) + \cdots + (PR_t + R_tB_t) \\
&= (PR_1 + PR_2 + \cdots + PR_t) + (R_1B_1 + R_2B_2 + \cdots + R_tB_t) \\
&\leq (PR_1 + PR_2 + \cdots + PR_t) + (WX + YZ) \\
&\leq (PR_1 + PR_2 + \cdots + PR_t) + (PW + PX + PY + PZ),
\end{aligned}$$

as desired.