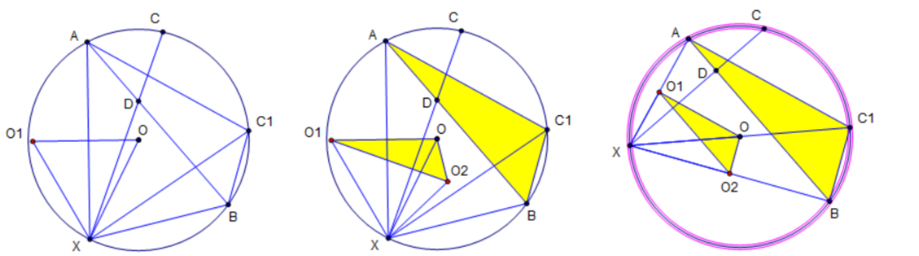
Solutions to USOMO 2020

USA Online Mathematical Olympiad

June 19 – 20, 2020

1 Solution to USOMO 1

Note: In the diagram below, we have $\angle C$ obtuse to make things easier to read, but the exact same proof still applies when $\angle C$ is acute as specified in the problem.



Let C_1 be the reflection of C across the perpendicular bisector of segment AB. Then, C_1 lies on circle ω, and ABC_1C is a isosceles trapezoid with $AC = BC_1$. We have

$$\angle XAD = \angle XAB = \angle XC_1B$$

and

$$\angle AXD = \angle AXC = \angle BXC_1,$$

from which it follows that $\triangle ADX \sim \triangle C_1BX$.

Because O_1 and O are the corresponding circumcenters of the triangles, we can conclude that quadrilateral $XDAO_1$ is a similar to quadrilateral XBC_1O. It follows that $\triangle XAO_1 \sim \triangle XC_1O$, implying that $\triangle XO_1O \sim \triangle XAC_1$ with ratio of similarity being XO/XC_1.

In exactly the same way (using B instead of A), we also have that $\triangle XO_2O \sim \triangle XBC_1$ with ratio of similarity also being XO/XC_1.

Therefore, quadrilateral XO_1OO_2 is similar to quadrilateral XAC_1B with ratio XO/XC_1. It follows that $\triangle O_1OO_2 \sim \triangle AC_1B$ or $\triangle O_1OO_2 \sim \triangle BCA$ (as $\triangle AC_1B \cong \triangle BCA$) with ratio XO/XC_1.

Since the area of ABC is fixed, it suffices to minimize the ratio XO/XC_1. But XO is a radius of ω, while XC_1 is a chord. Thus, $XO/XC_1 \geq \frac{1}{2}$ with equality if and only if

$$XC_1 \text{ is a diameter of } \omega \iff \angle C_1CX = 90 \iff CX \perp AB.$$

Thus, the unique point X which minimizes the area of OO_1O_2 is the one where $CX \perp AB$. Note that for this point X, the area is $\frac{1}{4}$ the area of ABC.

This problem was proposed by Zuming Feng.

2 Solution to USOMO 2

The answer is 3030. First, we describe an arrangement using exactly 3030 beams. We view the cube as being divided into 2020^3 unit cells and label them by their coordinates (i.e. the possible labels are (x, y, z) for $1 \leq x, y, z \leq 2020$). Let $B_x(a, b)$ denote the beam oriented along the x-axis with a and b as its y- and z-coordinates. More formally, we may write

$$B_x(a, b) = \{(1, a, b), (2, a, b), \ldots, (2020, a, b)\}.$$

Similarly, we define

$$B_y(a, b) = \{(b, 1, a), (b, 2, a), \ldots, (b, 2020, a)\}$$

$$B_z(a, b) = \{(a, b, 1), (a, b, 2), \ldots, (a, b, 2020)\}.$$

Then, the 3030 beams

$$B_x(1, 2), B_x(3, 4), \ldots, B_x(2019, 2020),$$

$$B_y(1, 2), B_y(3, 4), \ldots, B_y(2019, 2020),$$

$$B_z(1, 2), B_z(3, 4), \ldots, B_z(2019, 2020)$$

form a valid arrangement. Note that any two beams $B_x(2n-1, 2n)$ and $B_y(2m-1, 2m), 1 \leq m, n \leq 1010$ have different parities of the z-coordinate and thus do not intersect. Similarly, no two other beams from different sets intersect. Any two beams from the same set are parallel.

Consider beam $B = B_x(2n-1, 2n)$.

1. B touches the interiors of faces of beams $B_y(2n-1, 2n)$ and $B_z(2n-1, 2n)$.

2. If $n = 1010$, then B touches the face of the cube. Otherwise, B touches the beam $B_y(2n+1, 2n+2)$.

3. if $n = 1$, then B touches the face of the cube. Otherwise, B touches the interior of the face of the beam $B_z(2n-3, 2n-2)$.

Therefore, every face of every beam in B_x touches either a face of a cube or an interior of the face of another beam and hence every beam in B_x satisfies given condition. Similarly, every beam in B_y and B_z satisfies given condition.

To show this is optimal, first note that if there are no beams oriented along the x- or y-axis, then all beams are oriented along the z-axis. But then in order to satisfy the required property, we need $2020^2 > 3030$ beams. Thus, we may assume at least one beam is oriented along the x- or y-axis, so that it has one short dimension along the z-axis.

Note that for any beam that is short in the z direction, its bottom face must touch either the bottom of the cube or another beam that is also short in the z direction. Similarly, its top face must touch the top of the cube or another beam short in the z direction. It follows that we must have at least 2020 beams that are short in the z direction.

By a symmetrical argument, the same is true for the x and y directions. Since each beam has only 2 short dimensions, it follows that we must have at least $3 \cdot 2020/2 = 3030$ beams.

This problem was proposed by Alex Zhai.

3 Solution to USOMO 3

Look at the set B of all b such that b is a residue modulo p and $4-b$ is also a residue mod p. Any such b corresponds to a set of solutions to $x^2+y^2=4$, where negating x or y gives the same set, but interchanging them does not. Let S be the set of (x,y) such that $x^2+y^2=4$ and S includes only one of (x,y) and $(-x,-y)$. Then

$$\begin{aligned} P = \prod_{b\in B} b &= \prod_{(x,y)\in S} x^2 \\ &= \prod_{(x,y)\in S} (4-y^2) \\ &= \prod_{(x,y)\in S} (2-y)(2+y). \end{aligned}$$

Letting $2-y=c$ and $2+y=d$ (or vice versa since $-y$ is the same solution), we see that $P=\prod_{\{c,d\}} cd$ where the product runs over all unordered pairs $\{c,d\}$ other than $\{2,2\}$ with $c+d=4$ and c and d either both residues or both non-residues modulo p. Switching to ordered pairs, rather than unordered and then adding in the missing case $(2,2)$, we see that $2P=\prod_{(c,d)} c$, where the product runs over all ordered pairs (c,d) with $c+d=4$ and c and d either both residues or both non-residues modulo p. Putting this together, we get

$$2\prod_{b\in B} b = \prod_{a\in A} a\cdot\prod_{b\in B} b,$$

and hence

$$\prod_{a\in A} a \equiv 2 \pmod p.$$

Alternate solution from Carl Schildkraut. Work in $\mathbb{F}_p$. Define for all $0\le k\le |A|$

$$q_k=\sum_{a\in A} a^k, \quad s_k=\sum_{\substack{S\subset A\\ |S|=k}}\prod_{a\in S} a.$$

Note that $q_0=|A|$, $s_0=1$, and we want to show that $s_{|A|}=2$. We also have the Newton sums

$$ks_k=\sum_{i=1}^{k}(-1)^{i-1}s_{k-i}q_i$$

for all $0\le k\le |A|$. Now, we find the sums q_k. Recall the following lemma, stated here without proof:

Lemma. *For any positive integer m,*

$$\sum_{x\in\mathbb{F}_p} x^m = \begin{cases} -1 & \frac{m}{p-1}\in\mathbb{N} \\ 0 & \textit{otherwise.}\end{cases}$$

We first note that the quantity

$$\frac{1}{4}\left(1-\left(\frac{x}{p}\right)\right)\left(1-\left(\frac{4-x}{p}\right)\right)$$

is 1 if $x \in A$; if $x \notin A$ then either $\left(\frac{x}{p}\right)$ or $\left(\frac{4-x}{p}\right)$ is 0 or 1; if one is 0, then the other is 1 (as 4 is a square), so this is always 0. Thus

$$1_A(x) = \frac{1}{4}\left(1 - \left(\frac{x}{p}\right)\right)\left(1 - \left(\frac{4-x}{p}\right)\right).$$

We now have

$$\begin{aligned} q_k &= \sum_{a \in A} a^k \\ &= \sum_{x \in \mathbb{F}_p} 1_A(x) x^k \\ &= \frac{1}{4}\sum_{x \in \mathbb{F}_p} x^k \left(1 - \left(\frac{x}{p}\right)\right)\left(1 - \left(\frac{4-x}{p}\right)\right) \\ &= \frac{1}{4}\sum_{x \in \mathbb{F}_p} x^k \left(x^{\frac{p-1}{2}} - 1\right)\left((4-x)^{\frac{p-1}{2}} - 1\right). \end{aligned}$$

For $k = 0$ we have that the only term with a power $\geq p-1$ is the term corresponding to taking $x^{\frac{p-1}{2}}$ and the $(-1)^{\frac{p-1}{2}} x^{\frac{p-1}{2}}$ term from the expansion of $(4-x)^{\frac{p-1}{2}}$, giving that

$$|A| = q_0 \equiv -\frac{(-1)^{\frac{p-1}{2}}}{4} \implies \boxed{|A| = \frac{p - (-1)^{\frac{p-1}{2}}}{4}}.$$

We now split into cases.

Case 1. $p \equiv 3 \bmod 4$.

Here $|A| = \frac{p+1}{4}$. We claim that $a \in A \iff \frac{16}{a} \in A$. Indeed,

$$\left(\frac{4 - \frac{16}{x}}{p}\right) = \left(\frac{\frac{4x-16}{x}}{p}\right) = \left(\frac{-4}{p}\right)\left(\frac{4-x}{p}\right)\left(\frac{\frac{1}{x}}{p}\right) = -\left(\frac{x}{p}\right)\left(\frac{4-x}{p}\right),$$

so if a and $4-a$ are both nonresidues then $4 - 16/a$ is a nonresidue as well (as $a \to 16/a$ is an involution, this shows both directions). Now, if $p \equiv 7 \bmod 8$, $|A|$ is even, and so the terms are all paired by $a \to 16/a$ and the product is

$$16^{\frac{|A|}{2}} = (-4)^{|A|}.$$

If $p \equiv 3 \bmod 8$, then there exists one unpaired term which must be a fixed point of the involution; as $4 \notin A$ this must be -4, and thus the product is

$$16^{\frac{|A|-1}{2}}(-4) = (-4)^|A|.$$

Either way, the product is

$$(-4)^{|A|} = (-4)^{\frac{p+1}{4}} = (-1)^{\frac{p+1}{4}} 2^{\frac{p+1}{2}}.$$

If $p \equiv 3 \bmod 8$, then $(-1)^{\frac{p+1}{4}} = -1$ but 2 is not a quadratic residue, so $2^{\frac{p-1}{2}} \equiv -1$ as well, and our product is $(-1)\cdot(-1)\cdot 2 = 2$. If $p \equiv 7 \bmod 8$, then $(-1)^{\frac{p+1}{4}} = 1$ and 2 is a quadratic residue, so our product is $1 \cdot 1 \cdot 2 = 2$. Either way, we are done.

Case 2. $p \equiv 1 \bmod 4$.

We have

$$|A| = \frac{p-1}{4} < \frac{p-1}{2}$$

Thus we only care about q_k for $k < \frac{p-1}{2}$. Now

$$\begin{aligned}
q_k &= \frac{1}{4} \sum_{x \in \mathbb{F}_p} x^k \left(x^{\frac{p-1}{2}} - 1 \right) \left((4-x)^{\frac{p-1}{2}} - 1 \right) \\
&= \sum_{x \in \mathbb{F}_p} x^k x^{\frac{p-1}{2}} \sum_{i=0}^{\frac{p-1}{2}} 4^i (-x)^{\frac{p-1}{2} - i} \binom{\frac{p-1}{2}}{i} \\
&= \frac{1}{4} \sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i} 4^i (-1)^{\frac{p-1}{2} - i} \sum_{x \in \mathbb{F}_p} x^{p-1+k-i} \\
&= -\frac{1}{4} \binom{\frac{p-1}{2}}{k} (-1)^{\frac{p-1}{2} - k} 4^k \\
&= \binom{\frac{p-1}{2}}{k} (-4)^{k-1}.
\end{aligned}$$

We now write a recurrence for s_k. We have

$$\begin{aligned}
ks_k &= \sum_{i=1}^{k} (-1)^{i-1} s_{k-i} q_i \\
&= \sum_{i=1}^{k} (-1)^{i-1} s_{k-i} \binom{\frac{p-1}{2}}{i} (-4)^{i-1} \\
&= \sum_{i=1}^{k} 4^{i-1} s_{k-i} \binom{\frac{p-1}{2}}{i}.
\end{aligned}$$

Let $t_k = s_k / 4^k$. Now, dividing by 4^k,

$$kt_k = \sum_{i=1}^{k} 4^{-k} 4^{i-1} t_{k-i} 4^{k-i} \binom{\frac{p-1}{2}}{i} \implies 4kt_k = \sum_{i=1}^{k} t_{k-i} \binom{\frac{p-1}{2}}{i}.$$

This holds as an identity in $\mathbb{F}_p$ for all $k \leq |A|$. Rewrite this as

$$(4k+1)\, t_k = \sum_{i=0}^{k} t_{k-i} \binom{\frac{p-1}{2}}{i}.$$

Define a sequence u_k so that $u_0 = 1$ and

$$(4k+1) u_k = \sum_{i=0}^{k} u_{k-i} \binom{-\frac{1}{2}}{i}$$

for all $k \geq 1$. Note that $t_k \equiv u_k \pmod{p}$ for all $k \leq |A|$. Let $f(x) = \sum_{x=0}^{\infty} u_k x^k$. We have, by summing the above and using that the right side is a convolution,

$$f(x) + 4xf'(x) = f(x) \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^k = f(x)(1+x)^{-\frac{1}{2}}.$$

$$4\frac{f'(x)}{f(x)} = \frac{(1+x)^{-1/2}-1}{x}.$$

We want to solve this differential equation given the initial condition $f(0)=1$. Write it as

$$\frac{4df}{f} = \frac{(1+x)^{-1/2}-1}{x}dx.$$

We claim the right side is the derivative of $-2\log(1+\sqrt{1+x})$. Indeed,

$$\begin{aligned}
\frac{d}{dx}\bigg(-2\log\left(1+\sqrt{1+x}\right)\bigg) &= -2\frac{1}{1+\sqrt{1+x}}\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}} \\
&= -\frac{1}{\sqrt{1+x}(1+\sqrt{1+x})} \\
&= -\frac{1}{\sqrt{1+x}}\frac{\sqrt{1+x}-1}{(1+x)-1} \\
&= \frac{1-\sqrt{1+x}}{x\sqrt{1+x}} = \frac{(1+x)^{-1/2}-1}{x},
\end{aligned}$$

as desired. Now, we have

$$4\log f = -2\log(1+\sqrt{1+x})+C \implies f(x) = A\left(1+\sqrt{1+x}\right)^{-\frac{1}{2}};$$

as $f(0)=0$ we want

$$f(x) = \left(\frac{1+\sqrt{1+x}}{2}\right)^{-\frac{1}{2}}.$$

Define $v_k = 4^k u_k$ so $u_k \equiv v_k$ for all $0 \le k \le |A|$. The generating function g of v_k satisfies

$$g(x) = f(4x) = \left(\frac{1+\sqrt{1+4x}}{2}\right)^{-\frac{1}{2}} = \left(\frac{2}{1+\sqrt{1+4x}}\right)^{\frac{1}{2}} = \left(\frac{1-\sqrt{1+4x}}{2x}\right)^{\frac{1}{2}}.$$

Recalling that the generating function $c(x)$ of the Catalan numbers is $\frac{1-\sqrt{1-4x}}{2x}$, we have that

$$g(x) = c(-x)^{\frac{1}{2}}.$$

We claim that

$$g(-x) = \sum_{i=0}^{\infty}\frac{C_{2i}}{2^{2i}}x^i.$$

Indeed,

$$\begin{aligned}
\left[\sum_{i=0}^{\infty}\frac{C_{2i}}{2^{2i}}x^i\right]^2 &= \left(\frac{1}{2}\sum_{i=0}^{\infty}\frac{C_i}{2^i}x^{i/2}(1+(-1)^i)\right)^2 \\
&= \frac{1}{4}\left(c\left(\sqrt{x/4}\right)+c\left(-\sqrt{x/4}\right)\right)^2 \\
&= \frac{1}{4}\left(\frac{1-\sqrt{1-2\sqrt{x}}}{\sqrt{x}}+\frac{1-\sqrt{1+2\sqrt{x}}}{-\sqrt{x}}\right)^2 \\
&= \frac{1}{4x}\left(\sqrt{1+2\sqrt{x}}-\sqrt{1-2\sqrt{x}}\right)^2 \\
&= \frac{1}{4x}\left(2-2\sqrt{1-4x}\right) = c(x).
\end{aligned}$$

Thus

$$\sum_{i=0}^{\infty} \frac{C_{2i}}{2^{2i}} (-1)^i x^i = \sum_{i=0}^{\infty} v_i x^i,$$

which implies that

$$s_k \equiv \frac{C_{2k}}{2^{2k}} (-1)^k \pmod p$$

for every $0 \le k \le \frac{p-1}{4}$. We have that

$$s_{\frac{p-1}{4}} = \frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}} \frac{1}{\frac{p-1}{2}+1} \binom{p-1}{\frac{p-1}{2}}.$$

Now,

$$\binom{p-1}{k} \equiv \frac{(p-1)(p-2)\cdots(p-k)}{k!} \equiv \prod_{i=1}^{k} \frac{p-i}{i} = (-1)^k,$$

which is 1 if $p \equiv 1 \bmod 4$. Also, if $p \equiv 1 \bmod 8$, 2 is a quadratic residue, and so $2^{\frac{p-1}{2}} \equiv 1 \bmod p$; otherwise, $2^{\frac{p-1}{2}} \equiv -1 \bmod p$. Either way,

$$2^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{4}} \bmod p,$$

so

$$s_{\frac{p-1}{4}} \equiv \frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}} \frac{1}{\frac{p-1}{2}+1} \binom{p-1}{\frac{p-1}{2}} \equiv 1 \cdot 2 \cdot 1 = 2,$$

finishing the proof.

This problem was proposed by Richard Stong and Toni Bluher. The result itself is from a recent paper of Toni Bluher, but her proof is much more sophisticated (and proves much more).

4 Solution to USOMO 4

The answer is 197. Say a pair of integers (k, ℓ) with $1 \le k < \ell \le 100$ is *friendly* if $|a_k b_\ell - a_\ell b_k| = 1$. An example achieving 197 friendly pairs is $(0, 1)$, $(1, 1)$, $(1, 2)$, ..., $(1, 99)$.

Now we prove that 197 is the best possible. In general, we will prove that for $N \ge 2$ points there are no more than $2N - 3$ friendly pairs.

We proceed by induction on N, the base case $N = 2$ being obvious. Now consider $N + 1$ points $P_1, \dots, P_{N+1}$. Let $O = (0, 0)$ be the origin; then friendly pairs (P_k, P_ℓ) correspond to triangles OP_kP_ℓ with area $\frac{1}{2}$.

Suppose P_{N+1} has the largest sum of coordinates. If the coordinates of P_{N+1} are not relatively prime, then this point does not belong to any friendly pairs and the result is obvious.

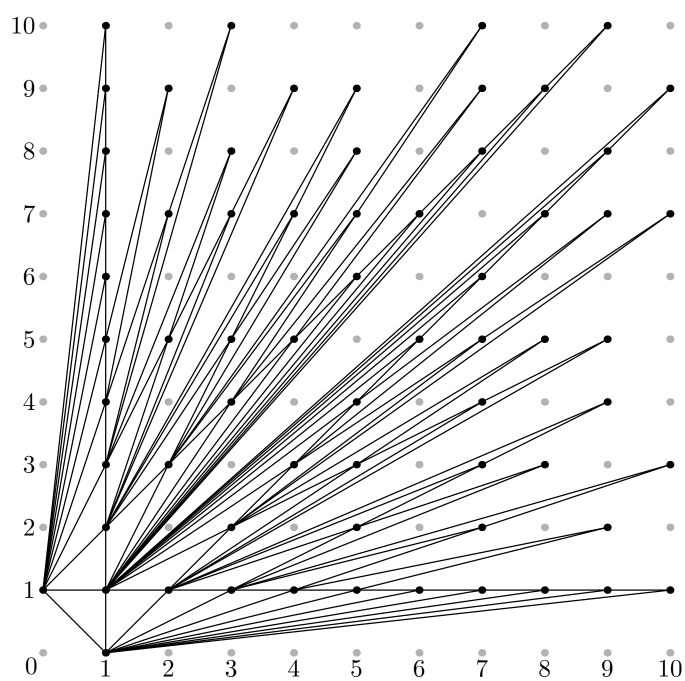
Otherwise, it easily follows that no $\overline{P_iP_j}$ is parallel to $\overline{OP_{N+1}}$. Consequently, there are at most two points P_i satisfying $[OP_iP_{N+1}] = \frac{1}{2}$ (one on each side of $\overline{OP_{N+1}}$).

To summarize, the point P_{N+1} belongs to at most two friendly pairs and the remaining N points form at most $2N - 3$ friendly pairs by inductive hypothesis, and so together the points form at most $2N - 1$ friendly pairs, as desired.

Remark. Consider the graph G_∞ on $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ with a segment joining (a, b) and (c, d) if $|ad - bc| = 1$. This graph is planar in the sense that no two drawn segments intersect (even at endpoints).

We show this planarity actually implies the estimate $2N - 3$. In fact, we claim each finite subgraph G of G_∞ is outerplanar. A graph is *outerplanar* if it is planar and can be drawn (without crossing edges) so every vertex is adjacent to the outer face. To see this, consider an extended graph $\overline{G}$ obtained by adjoining a vertex ∞ at infinity to G, and draw edges from each $(x, y) \in V(G)$ to ∞ by drawing a ray from it with slope $\frac{y}{x}$.

For fun, here is an image of part of G_∞:



This problem was proposed by Ankan Bhattacharya.

5 Solution to USOMO 5

We show that the answer is $2^{2019} - 2020$, and in general is $2^{n-1} - n$ if 2020 is replaced by $n \geq 2$. We will prove the result in this generality.

Let A denote the set of n points.

Solution by Alex Zhai. For any subset $S \subseteq A$ whose x-coordinates are all distinct, we denote by P_S the lowest degree polynomial satisfying $P_S(i) = f(i)$ for all $i \in S$. Thus, S is overdetermined if and only if $\deg(P_S) \leq |S| - 2$. (Evidently, if S has two points with the same x-coordinate, then S is not overdetermined; we do not define P_S in this case). Note also that $\deg(P_S) \leq |S| - 1$ for any such $S \subseteq A$, since we can always find a polynomial of degree at most $|S| - 1$ passing through $|S|$ prescribed points.

We first construct an example with at least $2^{n-1} - n$ overdetermined subsets. Choose $A = \{(1,1), (2,1), (3,1), \dots, (n,2)\}$. For any $S \subseteq A - \{(n,2)\}$, we have $P_S(t) \equiv 1$, so $\deg(P_S) = 0$. Thus, any subset of $A - \{(n,2)\}$ with at least two elements is overdetermined, and there are $2^{n-1} - n$ such subsets.

We must also verify that A is not overdetermined. To see this, suppose for sake of contradiction that $\deg(P_A) \leq n - 2$. Then, P_A is a degree-$(n-2)$ polynomial taking the value 1 at $n - 1$ inputs (namely, $1, 2, \dots, n-1$), so it must be identically equal to 1. But then, $P_A(n) = 1 \neq 2$, a contradiction.p

We next prove that it is not possible to have more than $2^{n-1} - n$ overdetermined subsets of A under the constraints of the problem.

First, we dispense of the case where A has any repeated x-coordinates. Suppose $p = (x, y) \in A$ and $q = (x, y') \in A$ where $y \neq y'$. Then no overdetermined set may contain both p and q. Moreover, if S is any subset of $A - \{p, q\}$, then at most one of $S \cup \{p\}$ and $S \cup \{q\}$ is overdetermined; indeed if $P_{S\cup\{p\}}$ and $P_{S\cup\{q\}}$ both have degree at most $|S| - 1$, and they agree on S, they must coincide, hence $y = P_{S\cup\{p\}}(x) = P_{S\cup\{q\}}(x) = y'$. Thus, there are at most $2^{n-2} - 1$ overdetermined subsets of A containing one of p and q (one for each nonempty subset of $A - \{p, q\}$). Also, $A - \{p, q\}$ itself has at most $2^{n-2} - (n-2) - 1$ subsets with at least two elements. This gives a total count of $[2^{n-2} - (n-2) - 1] + [2^{n-2} - 1] = 2^{n-1} - n$ as claimed.

We now assume henceforth that A has no repeated x-coordinates. Let us call a subset S of A *ideal* if it is not overdetermined. Additionally, given subsets S and T of A satisfying $S \subseteq T$ and $|T \setminus S| = 1$, we will say that S is a *child* of T and T is a *parent* of S.

Lemma 5.1. *Any ideal subset S of A has at most one overdetermined child.*

Proof. Assume the contrary, and let $S_1 = S \setminus \{(x_1, y_1)\}$ and $S_2 = S \setminus \{(x_2, y_2)\}$ be two overdetermined children of S. Then, P_{S_1} and P_{S_2} are polynomials of degree at most $|S| - 3$, and their graphs pass through all points of $|S_1 \cap S_2|$ which is a set of of size $|S| - 2$. It follows that $P_{S_1} = P_{S_1}$. But then $P_{S_1}(x) = y$ for all $(x, y) \in A$, which implies S is overdetermined, contradiction. □

Lemma 5.2. *Any overdetermined subset $S \subseteq A$ has at least one ideal parent.*

Proof. Suppose on the contrary that every parent of S is overdetermined. Then, for every $(x, y) \in A \setminus S$, the polynomials P_S and $P_{S\cup\{(x,y)\}}$ both have degree at most $|S| - 1$ and agree at all points in S, which implies that $P_S = P_{S\cup\{(x,y)\}}$.

But this means $P_S(x) = y$ for all $(x, y) \in A$, and since this applies to each $(x, y) \in A \backslash S$, it follows that $P_S = P_A$. This implies A is overdetermined, which is a contradiction. $\square$

Now, pair each overdetermined subset O with an ideal parent $p(O)$, choosing one arbitrarily if there is more than one. (Note also that Lemma 5.2 guarantees at least one ideal parent exists.) By Lemma 5.1, no two overdetermined sets are paired to the same ideal parent, so these pairs are all disjoint.

Finally, exactly one set from each pair $(O, p(O))$ has odd cardinality, and neither O nor $P(O)$ can have cardinality 1. Thus, the total number of these pairs (and hence the total number of overdetermined sets) is at most

$$\sum_{\substack{3 \leq k \leq n \\ k \text{ odd}}} \binom{n}{k} = 2^{n-1} - \binom{n}{1} = 2^{n-1} - n,$$

as desired.

Solution by downward induction We proceed as in the previous solution up to the proof of Lemma 5.1. However, rather than using Lemma 5.2, we will use a counting argument to finish the proof.

Let O_k and I_k be the number of overdetermined and ideal subsets S of $[n]$ of size k, respectively. Note that $O_0 = O_1 = O_n = 0$ and $I_0 = I_n = 1$, $I_1 = n$. Also note that $I_k + O_k = \binom{n}{k}$.

Lemma 5.3. *We have $O_k \leq \binom{n-1}{k}$, for $2 \leq k \leq n$.*

Proof. We will prove this by downwards induction on k. Since $O_n = 0 = \binom{n-1}{n}$ this establishes the base case of $k = n$.

Assume that $O_k \leq \binom{n-1}{k}$, for some k with $3 \leq k \leq n$. We will show that $O_{k-1} \leq \binom{n-1}{k-1}$. Let P be the number of parent-child pairs (T, S) of subsets of $[n]$ with $|T| = |S| + 1 = k$, and with S overdetermined. First note that $P = O_{k-1} \cdot (n - (k-1))$, by choosing S and then T. Next, note that P is at most $I_k \cdot 1 + O_k \cdot k$, by choosing T and then S. Thus we have

$$\begin{aligned}
O_{k-1} \cdot (n-k+1) = P &\leq I_k + kO_k \\
&= \binom{n}{k} - O_k + kO_k \\
&= \binom{n-1}{k-1} + \binom{n-1}{k} + (k-1)O_k. \\
&\leq \binom{n-1}{k-1} + \binom{n-1}{k} + (k-1)\binom{n-1}{k} \\
&= \binom{n-1}{k-1} + k\binom{n-1}{k} \\
&= \binom{n-1}{k-1} + (n-k)\binom{n-1}{k-1} \\
&= (n-k+1)\binom{n-1}{k-1}
\end{aligned}$$

Thus $O_{k-1} \leq \binom{n-1}{k-1}$ as claimed. $\square$

Now we have

$$\begin{aligned}
&O_0 + O_1 + O_2 + O_3 + \cdots + O_{n-1} + O_n \\
&\le 0 + 0 + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-1} + 0 \\
&= 2^{n-1} - n.
\end{aligned}$$

Furthermore, note that we can obtain equality with $f(i) = 1$ for $1 \le i \le n-1$ and $f(n) = 2$. Thus the answer is indeed $k_n = 2^{n-1} - n$.

This problem was proposed by Carl Schildkraut. The second solution is joint with Milan Haiman.

6 Solution to USOMO 6

We present two approaches.

Solution by David Speyer and Kiran Kedlaya We start with the following lemma.

Lemma. *Let $z_1, \dots, z_m$ and c be real numbers satisfying*

$$\sum_{j=1}^{n} z_j = 0, \qquad \sum_{j=1}^{m} z_j^2 = mc^2.$$

Then $\max_j\{z_j\} - \min_j\{z_j\} \ge 2c$.

Proof. Let $b := \frac{1}{2}(\max_j\{z_j\} + \min_j\{z_j\})$. We then have

$$\sum_{j=1}^{m}(z_j - b)^2 = \sum_{j=1}^{m} z_j^2 - 2b\sum_{j=1}^{m} z_j + \sum_{j=1}^{m} b^2 \ge mc^2.$$

The two largest terms on the left-hand side are the equal quantities $(\max_j\{z_j\} - b)^2 = (\min_j\{z_j\} - b)^2$, so each of these must exceed c^2. We deduce that

$$\max_j\{z_j\} - b = b - \min\{z_j\} \ge c$$

and thus $\max_j\{z_j\} - \min_j\{z_j\} \ge 2c$, as desired. $\square$

We apply the lemma as follows. Let S_n be the set of permutations of $\{1, \dots, n\}$, enumerated in some order as $\sigma_1, \dots, \sigma_{n!}$. For $j = 1, \dots, n!$, set

$$z_j := \sum_{i=1}^{n} x_i y_{\sigma_j(i)}.$$

To calculate $\sum_{j=1}^{n!} z_j$, we observe that each pair (i, k) with $i, k \in \{1, \dots, n\}$ occurs as $(i, \sigma(i))$ for exactly $(n-1)!$ values of $\sigma \in S_n$. This yields

$$\sum_{j=1}^{n!} z_j = (n-1)! \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i = 0.$$

Similarly, to calculate $\sum_{j=1}^{n!} z_j^2$, we count how often each pair (i,k,i',k') with $i,k,i',k' \in \{1,\dots,n\}$ occurs as $(i,\sigma(i),k,\sigma(k))$ for $\sigma \in S_n$. This count is $(n-1)!$ if $i=i', k=k'$; it is $(n-2)!$ if $i \neq i'$, $k \neq k'$; and it is zero otherwise. We thus have

$$\sum_{j=1}^{n!} z_j^2 = (n-1)! \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 + (n-2)! \sum_{i \neq i'} x_i x_{i'} \sum_{i \neq i'} y_i y_{i'}.$$

By writing

$$\sum_{i \neq i'} x_i x_{i'} = \left(\sum_{i=1}^n x_i\right)^2 - \sum_{i=1}^n x_i^2 = 0^2 - 1 = -1,$$

we deduce that

$$\sum_{j=1}^{n!} z_j^2 = (n-1)! + (n-2)! = n(n-2)!.$$

By the rearrangement inequality, $\max_j\{z_j\} = \sum_{i=1}^n x_i y_i$ and $\min_j\{z_j\} = \sum_{i=1}^n x_i y_{n+1-i}$. We may thus apply the lemma to deduce that $\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i y_{n+1-i} \geq 2c$ where

$$c^2 = \frac{1}{n!}\sum_{i=1}^{n!} z_i^2 = \frac{n(n-2)!}{n!} = \frac{1}{n-1}.$$

This yields the desired lower bound.

Remark. For n even, the inequality is sharp for

$$x_1 = \cdots = x_{n/2} = \frac{1}{\sqrt{n}}, \quad x_{n/2+1} = \cdots = x_n = -\frac{1}{\sqrt{n}}$$
$$y_1 = \frac{n-1}{\sqrt{n(n-1)}}, \quad y_2 = \cdots = y_n = -\frac{1}{\sqrt{n(n-1)}}.$$

Solution from Alex Zhai Consider a dual problem where the conditions are

$$x_n \geq x_{n-1} \geq \cdots \geq x_1$$
$$y_n \geq y_{n-1} \geq \cdots \geq y_1$$
$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0$$
$$\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i y_{n+1-i} = \frac{2}{\sqrt{n-1}}$$

and we want to show that $\sum_{i=1}^n x_i^2 \sum y_i^2 \leq 1$.

If we fix y and optimize x, we are maximizing a convex objective under a convex constraint, so the optimum occurs when x hits at least n equality constraints, i.e. the x_i take at most two distinct values. Thus, we may assume x takes this form. An analogous argument shows that we can assume y takes this form as well.

Let v_k be the vector whose first k entries are $\sqrt{\frac{n-k}{k}}$ and last $n-k$ entries are $-\sqrt{\frac{k}{n-k}}$. Then, the above discussion basically says x and y are multiples of v_k and v_ℓ for some k and ℓ. After some rescaling, our problem reduces to showing that

$$\langle v_k, v_\ell\rangle - \langle v_k, v_{n-\ell}\rangle \geq \frac{2\|v_k\|\|v_\ell\|}{\sqrt{n-1}} = \frac{2n}{\sqrt{n-1}}.$$

By direct calculation, it turns out that

$$\langle v_k, v_\ell \rangle = \sqrt{\frac{k(n-\ell)}{\ell(n-k)}},$$

and so the inequality simplifies to

$$\sqrt{\frac{k}{n-k}} \left(\sqrt{\frac{n-\ell}{\ell}} + \sqrt{\frac{\ell}{n-\ell}} \right) \geq \frac{2n}{\sqrt{n-1}},$$

which follows from simple AM-GM and $k \geq 1$.

This problem was proposed by David Speyer and Kiran Kedlaya.